# Mean-field approximation for large-population beauty-contest games

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*Abstract*— **We study a class of Keynesian beauty contest games where a large number of heterogeneous players attempt to estimate a common parameter based on their own observations. The players are rewarded for producing an estimate close to a certain multiplicative factor of the average decision, this factor being specific to each player. This model is motivated by scenarios arising in commodity or financial markets, where investment decisions are sometimes partly based on following a trend. We provide a method to compute Nash equilibria within the class of affine strategies. We then develop a mean-field approximation, in the limit of an infinite number of players, which has the advantage that computing the best-response strategies only requires the knowledge of the parameter distribution of the players, rather than their actual parameters. We show that the mean-field strategies lead to an $\varepsilon$-Nash equilibrium for a system with a finite number of players. We conclude by analyzing the impact on individual behavior of changes in aggregate population behavior.**

## I. Introduction

"Beauty contest" games, first introduced by Keynes in 1936 [1], are strategic games where each player attempts to make a choice that is close to a certain aggregate choice of the group, e.g., an average selection. Such models have been used for example to study trading decisions in financial markets or the social value of information [2]–[4]. In this paper, we consider a type of beauty contest game where a large number of players with heterogeneous characteristics estimate the value of an underlying parameter based on their local observation, while at the same time trying to remain close to a scaled version of the average estimate produced by all the players in the system. The payoff function for our model relates to the general form of $\rho$-beauty contest games first proposed in [5], but with a player-specific scaling weight $\rho$ multiplying the average estimate. An interpretation of this weight could be as a degree of "bullishness" about an asset in the context of a financial trading decision [4] or of "polarization" when evaluating a political issue [6].

The existence and characteristics of symmetric Nash equilibria for beauty-contest games were studied in [7], which did not however include the parameter estimation aspect of the model that we consider here. In our setting, the players trade-off the accuracy of their estimate with its "popularity". A closely related model was studied in [8], where players were connected via a graph and wished to remain close to the average estimate of their neighbors. It was shown that a myopic policy leads to a Nash equilibrium where players reach a consensus with their neighbors. Similar conclusions were drawn in [6], where the authors study how the opinion alignment between neighbors leads to network polarization

Beauty contest games with $\rho = 1$ were also considered in [9]–[11], where the authors analyzed the impact of public information on social welfare when homogeneous players have access to both private and public information. In [9], the uniqueness of the equilibrium in linear strategies was analyzed and it was shown that in the absence of private information, an increase in public information leads to an increase in the social welfare. The authors draw a connection with the "island economy" model [12]–[14] and show the similarity between the equilibrium attained and that of the features of such models. In [10], the authors show that in beauty contest games, mixing private noise with public information is often socially beneficial. In [11], the model of [9] was generalized to analyze the impact of privacy considerations on equilibrium strategies.

Computing Nash equilibria for the games mentioned above becomes more difficult as the number of players in the system increases and can, in many cases, only be done numerically, which complicates the analysis of the solution. Moreover, in our setting it requires that players know perfectly the parameters of every other player, e.g., their scaling weight. To address these limitations, we study a mean-field approximation for the beauty-contest game, assuming an infinity of players. Indeed, the beauty-contest term may be viewed as a special case of "mean-field coupling" between the players considered in the mean-field games (MFG) literature [15], [16]. We outline the conditions for the existence of a mean-field Nash equilibrium and show that the mean-field Nash strategies only require the knowledge of the parameter distribution in the population. Furthermore, we show that these strategies lead to an $\varepsilon-$Nash equilibrium when applied by the players in a finite game. This is consistent with typical results obtained in the large literature on MFG, see, e.g., [17], [18] for recent surveys.

Beauty-contest games with a continuum of players have also been analyzed in [9], [11], but our analysis is different. These papers consider the limit of the Nash equilibrium strategy as the number of players go to infinity but do not consider the rationalizability of using infinite population strategies in finite population games. In contrast, we explicitly show that the infinite-population mean-field equilibrium strategy is an $\varepsilon$-Bayesian Nash equilibrium in the finite population game. It is also worth highlighting that [9], [11] consider only the setup with homogeneous players and assume an improper uniform prior on the unknown value

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of the underlying parameter. In contrast, we model a game with heterogeneous players and consider a Gaussian prior on the unknown value of the parameter. Therefore, the results of [9], [11] are not directly applicable to our model. Another related paper is [19], which does not make explicit reference to beauty-contest games but discusses from the MFG point of view a simplified version of our problem with homogeneous players, $\rho = 1$ and no common observation. In contrast, one of our main motivations is to study the effect of varying $\rho$ in the population, see Section IV. Finally, beauty contest games can also be analyzed as particular examples of aggregative games [20], which are closely related to mean field games, see [21] for a discussion of this connection.

The rest of the paper is organized as follows. In Sec. II, we present the model of the general beauty-contest game and characterize a Bayesian Nash equilibrium of the game. In Sec. III, we present a mean-field approximation of the Bayesian Nash equilibrium. In Sec. IV, we investigate the impact of variation of aggregate population behavior on the individual behavior. Finally, we conclude in Sec. V. All proofs are provided in the appendix.

## II. General Beauty-Contest Games

### A. Specification of the Game

Consider a general-sum Bayesian game where $N := \{1, \dots, n\}$ denotes the set of players and other components are described below.

*a) Nature:* Let $\omega = (\theta, v_0, v_1, \dots, v_n) \in \mathbb{R}^{n+1}$ denote the state of nature. We assume that $\omega$ is a vector of independent Gaussian random variables where $\theta \sim \mathcal{N}(0, 1)$ and $v_0 \sim \mathcal{N}(0, \sigma_0^2)$ and $v_i \sim \mathcal{N}(0, \sigma^2)$, for all $i \in N$.

*b) Players:* There are three parameters $(\alpha_i, \lambda_i, \rho_i)$ associated with player $i$, $i \in N$, where $\alpha_i \in [0, 1]$, $\lambda_i \in [0, 1]$, and $\rho_i \in \mathbb{R}$. There is also a global parameter $\alpha_0 \in [0, 1]$. These parameters affect the player's observations and cost. For ease of notation, we use $\phi_i = (\alpha_i, \lambda_i, \rho_i)$ to denote the parameters of player $i$ and $\phi = (\alpha_0, \phi_1, \dots, \phi_n)$ to denote the parameters of all players.

Player $i$, $i \in N$, gets a signal $(y_0, y_i) \in \mathbb{R}^2$ given by

$$y_0 = \alpha_0 \theta + v_0 \quad \text{and} \quad y_i = \alpha_i \theta + v_i. \tag{1}$$

The signal $y_0$ is common to all players while the signal $y_i$ is only observed by player $i$. Player $i$ chooses an action $u_i \in \mathbb{R}$. Let $u_{-i}$ denote the profile of actions taken by all players except player $i$ and let

$$\bar{u} = \frac{1}{n} \sum_{i \in N} u_i$$

denote the average of the actions chosen by all players.

The cost incurred by player $i$ is given by

$$c_i(\theta, u_i, u_{-i}; \phi_i) = (1 - \lambda_i)(\theta - u_i)^2 + \lambda_i (u_i - \rho_i \bar{u})^2. \tag{2}$$

This cost is a weighted sum of two terms: how close is the action taken by the player to the nature's state $\theta$ and how close is the action to $\rho_i$ times the average of the actions of the players. When $\lambda_i = 1$ and $0 < \rho_i \leq 1$ is identical for all players, the cost reduces to that of the standard beauty-contest game, where the objective for each player is to guess a fraction of the average of all players. As described in the introduction, various scenarios can be modelled by assuming general $(\lambda_i, \rho_i)$, which could also depend on the players.

*c) Information structure:* For our initial analysis, we assume that the parameters $\phi$ of the players are common knowledge to all players. We will relax this assumption later. The information set of player $i$ is therefore

$$I_i = \{y_0, y_i, \phi\}.$$

Let $g_i$ denote the (measurable) strategy used by player $i$ to choose its action, i.e.,

$$u_i = g_i(y_0, y_i, \phi).$$

Given a strategy profile $g = (g_1, \dots, g_n)$, the expected cost incurred by player $i$, $i \in N$, is given by

$$J_i(g_i, g_{-i}; \phi) = \mathbb{E}_\omega[c_i(\theta, g_1(y_0, y_1, \phi), .., g_n(y_0, y_n, \phi); \phi_i)]. \tag{3}$$

A strategy profile $g$ is called a *Bayesian Nash equilibrium* (BNE) if it satisfies the following property: for any player $i \in N$ and any strategy $h_i$ for player $i$, we have

$$J_i(g_i, g_{-i}; \phi) \leq J_i(h_i, g_{-i}; \phi). \tag{4}$$

We are interested in the following problem.

**Problem 1** *Given the parameters $\phi$ of the players and the noise variances $\sigma_0$ and $\sigma$, identify a Bayesian Nash equilibrium strategy $g$.*

### B. Characterization of Bayesian Nash Equilibrium

We show that for almost all values of the parameters $\phi$, the beauty-contest game described above has a BNE in pure strategies. To characterize this BNE, we define the constants

$$H_i = \frac{\alpha_0 \sigma^2}{\alpha_0^2 \sigma^2 + \alpha_i^2 \sigma_0^2 + \sigma_0^2 \sigma^2}, \quad K_i = \frac{\alpha_i \sigma_0^2}{\alpha_0^2 \sigma^2 + \alpha_i^2 \sigma_0^2 + \sigma_0^2 \sigma^2},$$
$$\Lambda_i = (1 - \lambda_i) + \lambda_i \big(1 - \tfrac{\rho_i}{n}\big)^2, \quad \overline{\Lambda}_i = \lambda_i \tfrac{\rho_i}{n} \big(1 - \tfrac{\rho_i}{n}\big),$$

vectors $\eta, \kappa \in \mathbb{R}^n$ as

$$\eta = \mathrm{vec}((1 - \lambda_1) H_1, \dots, (1 - \lambda_n) H_n),$$
$$\kappa = \mathrm{vec}((1 - \lambda_1) K_1, \dots, (1 - \lambda_n) K_n),$$

and matrices $A, B, \bar{B} \in \mathbb{R}^{n \times n}$ as

$$A_{ij} = \begin{cases} \Lambda_i & \text{if } i = j, \\ -\overline{\Lambda}_i & \text{if } i \neq j, \end{cases}$$

$$B_{ij} = \begin{cases} \Lambda_i & \text{if } i = j, \\ -\overline{\Lambda}_i K_i \alpha_j & \text{if } i \neq j, \end{cases} \quad \bar{B}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -\overline{\Lambda}_i H_i \alpha_j & \text{if } i \neq j. \end{cases}$$

**Theorem 1** *Suppose the following system of equations has a solution:*

$$A\mathbf{a} + \bar{B}\mathbf{b} = \eta, \quad B\mathbf{b} = \kappa \tag{5}$$

*where $\mathbf{a} = \mathrm{vec}(a_1, \dots, a_n)$ and $\mathbf{b} = \mathrm{vec}(b_1, \dots, b_n)$. Then the strategy profile*

$$g_i(y_0, y_i, \phi) = a_i y_0 + b_i y_i, \quad \forall i \in N, \tag{6}$$

*is a BNE of the beauty-contest game. Moreover, if $A$ and $B$ are invertible, then the strategy profile* (6) *is the unique BNE within the class of affine strategies.* □

### C. Some Special Cases

*a) No common observation:* The setting where the players have no common observation can be captured by choosing $\alpha_0 = 0$. In this case $H_i = 0$ and $K_i = \alpha_i/(\alpha_i^2 + \sigma^2)$. This implies that $\bar{B} = 0$ and $\eta = 0$. Hence, $\mathbf{a} = 0$ is a solution and $\mathbf{b}$ is given by the solution of

$$B\mathbf{b} = \kappa.$$

The corresponding BNE is $g_i(y_i; \phi) = b_i y_i$. Moreover, if $A$ and $B$ are invertible, then this is the unique BNE among the class of affine strategies.

*b) No private observation:* The setting where the players have no private observation can be captured by choosing $\alpha_i = 0$. In this case, $K_i = 0$ and $H_i = \alpha_0/(\alpha_0^2 + \sigma_0^2)$. This implies that $\kappa = 0$ and $B$ is diagonal. Hence, $\mathbf{b} = 0$ and $\mathbf{a}$ is given by the solution of

$$A\mathbf{a} = \eta.$$

The corresponding BNE is $g_i(y_0; \phi) = a_i y_0$. Moreover, if $A$ is invertible, then this is the unique BNE among the class of affine strategies.

*c) Homogeneous players:* Consider the setting where all players are homogeneous, i.e., $\phi_i = (\lambda_i, \alpha_i, \rho_i) = (\lambda, \alpha, \rho)$ for all $i \in N$. Then $H_i$, $K_i$, $\Lambda_i$, and $\overline{\Lambda}_i$ do not depend on $i$, so we drop the subscript $i$. In this setting, the system of equations (5) has a symmetric solution where $a_i$ and $b_i$ do not depend on $i$. We denote these by $a$ and $b$ and they are given by

$$b = \frac{(1-\lambda)K}{\Lambda - (n-1)\overline{\Lambda}K\alpha}, \quad a = \frac{(1-\lambda)H + (n-1)\overline{\Lambda}H\alpha b}{\Lambda - (n-1)\overline{\Lambda}}. \tag{7}$$

The solution does not exist if $\Lambda = (n-1)\overline{\Lambda}K\alpha$ or $\Lambda = (n-1)\overline{\Lambda}$. Note however that in the special case $\rho = 1$ and $\lambda < 1$ a straightforward calculation shows that these conditions cannot be satisfied and hence in this case a symmetric solution always exists.

It can be verified that $b < a$ when $\Lambda < (n-1)\overline{\Lambda}K\alpha$ (and, therefore, also less than $(n-1)\overline{\Lambda}$ because $K\alpha < 1$), and both the private and the common observations have same channel gain $\alpha$ and variance $\sigma$. Thus, the players put more weight on the common signal than on their private signal. This is consistent with the observations made in [9] for a slightly different model.

In the special case when players have no common observation, $H = 0$ and therefore $a = 0$. In the special case when the players have no private observation, $K = 0$ and therefore $b = 0$, which implies $a = (1-\lambda)H/(\Lambda - (n-1)\overline{\Lambda})$.

## III. Mean-field Approximation

There are two shortcomings of the BNE characterized in Theorem 1. First, the equilibrium is derived under the assumption that the parameters of all players are common knowledge. This is a strong assumption and is unlikely to hold in games with a large number of players. Second, computing the equilibrium strategy requires solving two systems of $n$ linear equations, which can get computationally expensive when the game has a large number of players. To circumvent both these limitations, we characterize the mean-field limit of the beauty-contest game and show that the mean-field equilibrium is an $\varepsilon$-Nash equilibrium for the finite player game.

Suppose that rather than being specified arbitrarily, the parameters $\phi$ are modeled as realizations of a random allocation. In particular, it is assumed that $\alpha_0$ is chosen randomly according to some distribution and each of the parameters $\alpha_i$, $\lambda_i$, $\rho_i$ are independent and identically distributed across players and also independent of $\alpha_0$. Recall that $\alpha_i$ and $\lambda_i$ have support $[0, 1]$, For simplicity, we assume that $\rho_i$ also has finite support. In our analysis, the exact distribution of these variables does not matter, just their means.

Given a strategy profile $g = (g_1, \dots, g_n)$, we define the expected cost of a generic player by

$$\bar{J}_i(g_i, g_{-i}) = \mathbb{E}_\phi[J_i(g_i, g_{-i}; \phi)].$$

In this setting, we will approximate the BNE of Theorem 1 by its mean-field approximation. A strategy profile $g = (g_1, \dots, g_n)$ is called *$\varepsilon$-Bayesian Nash equilibrium* ($\varepsilon$-BNE) if it satisfies the following property: for any player $i \in N$ and any strategy $h_i$ for player $i$, we have

$$\bar{J}_i(g_i, g_{-i}) \le \bar{J}_i(h_i, g_{-i}) + \varepsilon. \tag{8}$$

### A. Characterization of Mean-Field Equilibrium

Let $\bar{\lambda}$, $\bar{\rho}$, $\bar{H}$ denote the mean of $\lambda_i$, $\rho_i$, $H_i$ and let $\bar{L} = \mathbb{E}_{\alpha_0,\alpha_i}[K_i \alpha_i]$. All these quantities exist because the random variables have finite support.

**Theorem 2** *Suppose $\bar{\lambda}\bar{\rho}\bar{L} \neq 1$ and $\bar{\lambda}\bar{\rho} \neq 1$, so that*

$$\bar{M} := \frac{(1-\bar{\lambda})\bar{L}}{1 - \bar{\lambda}\bar{\rho}\bar{L}}, \qquad \bar{a} := \frac{\big[(1-\bar{\lambda}) + \bar{\lambda}\bar{\rho}\bar{M}\big]\bar{H}}{1 - \bar{\lambda}\bar{\rho}}$$

*are finite. Then the strategy profile $\bar{g} = (\bar{g}_1, \dots, \bar{g}_n)$ where*

$$\bar{g}_i(y_0, y_i, \phi_i) = \big[(1-\lambda_i) + \lambda_i \rho_i \bar{M}\big](H_i y_0 + K_i y_i) + \rho_i \lambda_i \bar{a} y_0$$

*is an $\varepsilon$-BNE for the $n$-player beauty contest game, where $\varepsilon \in \mathcal{O}(1/\sqrt{n})$.* □

**Corollary 1** *In the special case of no common observations, let $\hat{L} = \mathbb{E}_{\alpha_i}[\alpha_i^2/(\alpha_i^2 + \sigma^2)]$ and $\widehat{M} = (1-\bar{\lambda})\hat{L}/(1 - \bar{\lambda}\bar{\rho}\hat{L})$. If $\widehat{M}$ is finite, then the strategy profile $\bar{g} = (\bar{g}_1, \dots, \bar{g}_n)$, where*

$$\bar{g}_i(y_i, \phi_i) = \big[(1-\lambda_i) + \lambda_i \rho_i \widehat{M}\big] K_i y_i$$

*is an $\varepsilon$-BNE for the $n$-player beauty contest game, with $\varepsilon \in \mathcal{O}\big(1/\sqrt{n}\big)$.* □

**Remark 1** The mean-field strategy identified in Theorem 2 exists only when $\bar{M}$ and $\bar{a}$ are finite, i.e., if $\bar{\lambda}\bar{\rho} \notin \{1, 1/\bar{L}\}$. In the absence of common observation, the mean-field strategy identified in Corollary 1 exists only when $\widehat{M}$ is finite, i.e., if $\bar{\lambda}\bar{\rho} \neq 1/\bar{L}$. These conditions are different from those identified in Sec. II-B. Thus, there can be finite players games such that the BNE of Theorem 1 does not exist but the mean-field limit of Theorem 2 exists, and vice versa. □

## IV. Effect of the aggregate population behavior on individual behavior

We now consider the system from the point of view of a specific player, say player $i$. Observe that the expected cost of player $i$ playing a strategy $u_i^\infty = \bar{g}(y_i, \phi_i, y_0)$ is given by

$$\begin{aligned} &C_i(u_i^\infty, \bar{g}_{-i} \mid y_0, y_i; \phi) \\ &\quad := \mathbb{E}_\omega[c_i(\theta, u_i^\infty, \bar{g}_{-i}(y_0, y_{-i}); \phi) \mid y_0, y_i] \\ &\quad = (u_i^\infty)^2 - 2\Big[(1-\lambda_i)(K_i y_i + H_i y_0) \\ &\qquad + \rho_i \lambda_i \big[(K_i y_i + H_i y_0)\bar{M} + y_0 \bar{a}\big]\Big] u_i^\infty + S, \end{aligned} \tag{9}$$

where $S$ is independent of $u_i^\infty$. We consider the setting where the "bullisness" $\bar{\rho}$ or "social degree" $\bar{\lambda}$ of the population changes and the player adapts its individual $\rho_i$ and $\lambda_i$ such that its expected cost (9) does not change. These results illustrate the impact of aggregate population behavior on individual behavior.

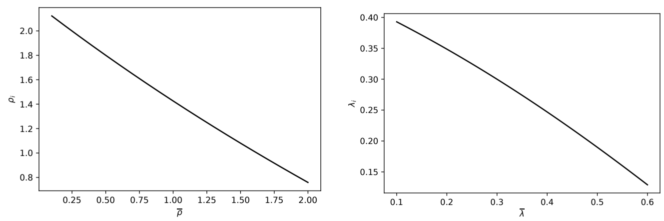
We consider a system where the channel noise $\sigma_0 = \sigma = 1$, the channel gains $\alpha_0 = 0.5$, $\alpha_j \sim \mathtt{unif}[0,1]$, $j \in N$, (and, therefore, $\bar{H} = 0.288$ and $\bar{L} = 0.184$), the average social degree $\bar{\lambda} = 0.3$, and the average bullishness $\bar{\rho} = 1.25$. We take a generic player $i$ whose parameters $\phi_i^\circ = (\alpha_i^\circ, \lambda_i^\circ, \rho_i^\circ) = (0.5, 0.3, 1.25)$ and investigate two scenarios:

- *Scenario 1:* We vary $\bar{\rho}$ from $0.1$ to $2.0$ and compute the value of $\rho_i$ such that the expected cost of player $i$ remains the same as when its parameters were $\phi_i^\circ$. Recall that the mean-field equilibrium exists as long as $\bar{\rho} \notin \{1/\bar{\lambda}, 1/\bar{\lambda}\bar{L}\} = \{3.333, 18.116\}$. Thus, the mean-field equilibrium always exists in the specified range of variation of $\bar{\rho}$.
- *Scenario 2:* We vary $\bar{\lambda}$ from $0.1$ to $0.6$ and compute the value of $\lambda_i$ such that that the expected cost of player $i$ remains the same as when its parameters were $\phi_i^\circ$. The computed values of $\lambda_i$ were in the range $[0,1]$. Recall that the mean-field equilibrium exists as long as $\bar{\lambda} \notin \{1/\bar{\rho}, 1/\bar{\rho}\bar{L}\} = \{0.8, 4.348\}$. Thus, the mean-field equilibrium always exists in the specified range of variation of $\bar{\lambda}$.

The results in Fig. 1 show that if the player wants to maintain the same cost when $\bar{\rho}$ or $\bar{\lambda}$ are increased, then he should decrease his $\rho_i$ or $\lambda_i$.

## V. Conclusion

We considered general beauty-contest games where players want to estimate a common parameter based on their observation and, at the same time, be close to a multiplicative factor of the average decision. We showed that the BNE in affine strategies is characterized by the solution of a linear system of equations. We then characterized the mean-field approximation of the finite population BNE, which has the advantage that the computing the equilibrium strategy of the




(a) Scenario 1 (b) Scenario 2

Fig. 1: Variation of individual parameters with the aggregate population parameters to keep the same expected cost.

players requires the knowledge of the parameter distribution of the players, rather than their actual parameters. We show that the mean-field approximation is an $\varepsilon$-BNE for the system with a finite number of players.

The results were presented for scalar variables for convenience. They generalize to vector-valued variables in an obvious manner. The model presented in this paper focused on the static setting. Investigating the dynamic setting where the decisions made my the agents evolve over time is an interesting and important future direction.

## Appendix

### A. Proof of Theorem 1

We start by defining the interim expected cost of a player. Given a player $i \in N$, an observation $(y_0, y_i) \in \mathbb{R}^2$ and an action $u_i \in \mathbb{R}$ of player $i$, and a strategy profile $g_{-i}$ of all players other than $i$, the interim cost of player $i$ is given by

$$\begin{aligned} &C_i(u_i, g_{-i} \mid y_0, y_i; \phi) \\ &\qquad := \mathbb{E}_\omega[c_i(\theta, u_i, g_{-i}(y_0, y_{-i}); \phi) \mid y_0, y_i]. \end{aligned}$$

The definition of BNE given in (4) is sometimes called the ex-ante definition. An equivalent, interim definition, is as follows: a strategy profile $g = (g_1, \dots, g_n)$ is a Bayesian Nash equilibrium (BNE) if for any players $i \in N$ and any observation $(y_0, y_i) \in \mathbb{R}^2$ of player $i$, we have that for all $u_i \in \mathbb{R}$,

$$C_i(g_i(y_0, y_i), g_{-i} \mid y_0, y_i; \phi) \le C_i(u_i, g_{-i} \mid y_0, y_i; \phi).$$

We use the above definition of BNE to identify the BNE in the class of affine strategies. First observe that $c_i(\theta, u_i, u_{-i})$ is convex in $u_i$. Since convexity is preserved under expectation, $C_i(u_i, g_{-i} \mid y_0, y_i; \phi)$ is also convex in $u_i$. Now, to identify a BNE within the class of affine strategies, we pick a arbitrary player, say $i$, and assume that all player $j \neq i$ are playing an affine strategy

$$g_j(y_0, y_j; \phi) = a_j y_0 + b_j y_j + d_j, \tag{10}$$

where $a_j$, $b_j$ and $d_j$ are constants (possibly depending on $n$ and $\phi$). Now, consider the best response at player $i$. Since the interim cost $C_i(u_i, g_{-i} \mid y_0, y_i; \phi)$ is convex in $u_i$, the best response $u_i^*$ is the solution of the first order optimality conditions

$$\frac{\partial}{\partial u_i} C_i(u_i, g_{-i} \mid y_0, y_i; \phi) = 0.$$

**Lemma 1** *Given a player $i \in N$, if all players $j \neq i$ are playing an affine strategy given by* (10)*, then*

$$\frac{\partial}{\partial u_i} C_i(u_i, g_{-i} \mid y_0, y_i; \phi) = 2\Lambda_i u_i - 2\bar{\Lambda}_i \big[\sum_{j\neq i} a_j y_0 + d_j\big] - 2\big[(1-\lambda_i) + \bar{\Lambda}_i \sum_{j\neq i} b_j \alpha_j\big](H_i y_0 + K_i y_i), \tag{11}$$

*where $H_i$, $K_i$, $\Lambda_i$, $\bar{\Lambda}_i$ are as defined before Theorem 1. Therefore, the interim best response of player $i$ is given by*

$$u_i = \Lambda_i^{-1}\big[(1-\lambda_i) + \overline{\Lambda}_i \sum_{j\neq i} b_j\alpha_j\big](H_i y_0 + K_i y_i) + \Lambda_i^{-1}\overline{\Lambda}_i\big[\sum_{j\neq i} a_j y_0 + \sum_{j\neq i} d_j\big]. \tag{12}$$

□

PROOF First observe that

$$\frac{\partial}{\partial u_i}\mathbb{E}[(\theta - u_i)^2 \mid y_0, y_i] = -2\mathbb{E}[\theta | y_0, y_i] + 2u_i = -2H_i y_0 - 2K_i y_i + 2u_i. \tag{13}$$

Now, note that $\partial \bar{u}/\partial u_i = 1/n$. Therefore,

$$\frac{\partial}{\partial u_i}\mathbb{E}[(u_i - \rho_i\bar{u})^2 \mid y_0, y_i] = 2\big(1 - \tfrac{\rho_i}{n}\big)(u_i - \rho_i \mathbb{E}[\bar{u} \mid y_0, y_i]). \tag{14}$$

Finally, for any player $j \neq i$ who is playing a strategy of the form (10), $\mathbb{E}[u_j \mid y_0, y_i] = a_j y_0 + b_j\alpha_j(H_i y_0 + K_i y_i) + d_j$. Therefore,

$$\mathbb{E}[\bar{u} \mid y_i] = \frac{1}{n}\big[u_i + y_0 \sum_{j\neq i} a_j + (H_i y_0 + K_i y_i)\sum_{j\neq i} b_j\alpha_j + \sum_{j\neq i} d_j\big]. \tag{15}$$

By combining (13)–(15), we get that (11). Setting $\partial C_i/\partial u_i = 0$, we get (12). ■

Observe that the best response of player $i$ given by (12) is affine in $(y_0, y_i)$. Thus, the class of affine strategies is closed under best response. Now, there exist a BNE in affine strategies if for all $i \in N$,

$$a_i = \Lambda_i^{-1}\Big[\big[(1-\lambda_i) + \overline{\Lambda}_i \sum_{j\neq i} b_j\alpha_j\big]H_i + \overline{\Lambda}_i \sum_{j\neq i} a_j\Big], \tag{16}$$

$$b_i = \Lambda_i^{-1}\big[(1-\lambda_i) + \overline{\Lambda}_i \sum_{j\neq i} b_j\alpha_j\big]K_i, \tag{17}$$

and

$$d_i = \Lambda_i^{-1}\overline{\Lambda}_i \sum_{j\neq i} d_j, \tag{18}$$

Note that (18) can be written more compactly as

$$A\mathbf{d} = 0 \tag{19}$$

and (16) and (17) can be written as compactly as (5). The vector $\mathbf{d} = 0$ is always a solution of (19). It is the unique solution when $A$ is invertible. Thus, any $\mathbf{a} = \text{vec}(a_1, \dots, a_n)$ and $\mathbf{b} = \text{vec}(b_1, \dots, b_n)$ satisfying (5) is a BNE and if (5) has a unique solution and $A$ is invertible, then the corresponding solution is the unique solution within the class of affine strategies.

## *B. Proof of Theorem 2*

In the solution obtained in Theorem 1, the coefficients $a_i$ and $b_i$ of player $i$ depended on the parameters $\phi$ of the entire population. In the limit of large number of players, we assume that $a_i$ and $b_i$ converge to limiting functions $a(\phi_i)$ and $b(\phi_i)$ and we identify these functions. Then we define

$$g_i(y_0, y_i; \phi_i) = a(\phi_i)y_0 + b(\phi_i)y_i$$

as the mean-field strategy and show that it is an $\varepsilon$-BNE.

First observe that $\lim_{n\to\infty} \Lambda_i = 1$. Moreover, since the parameters are chosen independently across players, the strong law of large numbers gives that

$$\lim_{n\to\infty} \frac{1}{n-1}\sum_{j\neq i} b(\alpha_j, \lambda_j, \rho_j)\alpha_j = \mathbb{E}[b(\alpha, \lambda, \rho)\alpha] =: \bar{M},$$

where the convergence is in the almost sure sense. Note that the strong law of large numbers holds here because all random variables have finite support. Thus, in the limit as $n \to \infty$, (17) simplifies to

$$b(\alpha_i, \lambda_i, \rho_i) = \big[(1-\lambda_i) + \lambda_i\rho_i\bar{M}\big]K_i, \tag{20}$$

where $K_i$ is the same as in Theorem 1. Since $\bar{M} = \mathbb{E}[b(\alpha, \lambda, \rho)\alpha]$, $\bar{M}$ must satisfy the following fixed point equation:

$$\bar{M} = \mathbb{E}\Big[\big[(1-\lambda_i) + \lambda_i\rho_i\bar{M}\big]K_i\alpha_i\Big] = \big[(1-\bar{\lambda}) + \bar{\lambda}\bar{\rho}\bar{M}\big]\bar{L}, \tag{21}$$

where the last equality follows because $(\alpha, \lambda, \rho)$ are independent. Solving (21), we get

$$\bar{M} = \frac{(1-\bar{\lambda})\bar{L}}{1 - \bar{\lambda}\bar{\rho}\bar{L}}. \tag{22}$$

Subsisting (22) in (20) gives us the limiting function $b(\phi_i)$.

Similarly, in the limit as $n \to \infty$, Eq. (16) simplifies to

$$a(\alpha_i, \lambda_i, \rho_i) = \big[(1-\lambda_i) + \lambda_i\rho_i\bar{M}\big]H_i + \lambda_i\rho_i\bar{a}. \tag{23}$$

Since $\bar{a} = \mathbb{E}[a(\alpha, \lambda, \rho)]$, $\bar{a}$ must satisfy the following fixed point equation:

$$\bar{a} = \mathbb{E}\big[[(1-\lambda_i) + \lambda_i\rho_i\bar{M}]H_i + \lambda_i\rho_i\bar{a}\big] = \big[(1-\bar{\lambda}) + \bar{\lambda}\bar{\rho}\bar{M}\big]\bar{H} + \bar{\lambda}\bar{\rho}\bar{a} \tag{24}$$

where the last equality follows because $(\alpha, \rho, \lambda)$ are independent. Solving for (24) we get

$$\bar{a} = \frac{\big[(1-\bar{\lambda}) + \bar{\lambda}\bar{\rho}\bar{M}\big]\bar{H}}{1 - \bar{\lambda}\bar{\rho}}. \tag{25}$$

Substituting (25) in (23) gives us the limiting function $a(\phi_i)$. Combining this with the function $b(\phi_i)$ identified in (20) and (22), gives the mean-field strategy $\bar{g}$ specified in Theorem 2.

Now to establish the $\varepsilon$-BNE property for this strategy, we arbitrarily pick a player, say $i$, and assume that all players other than $i$ are playing strategy $\bar{g}_j$. From (16)-(17), the best response strategy $g_i^*$ of player $i$ is $g_i^*(y_i) = a_i^* y_0 + b_i^* y_i$ where the gains are given by

$$a_i^* = \Lambda_i^{-1}\big[\big[(1-\lambda_i) + \overline{\Lambda}_i \sum_{j\neq i} b(\phi_j)\alpha_j\big]H_i + \bar{\Lambda}_i \sum_{j\neq i} a(\phi_j)\big],$$
$$b_i^* = \Lambda_i^{-1}\big[(1-\lambda_i) + \overline{\Lambda}_i \sum_{j\neq i} b(\phi_j)\alpha_j\big]K_i.$$

Let $\bar{u}_{-i}$ denote $\big(\sum_{j\neq i} u_j\big)/(n-1)$. Then $u_i - \rho_i \bar{u} = \big(1-\frac{\rho_i}{n}\big)u_i - \rho_i\big(1-\frac{1}{n}\big)\bar{u}_{-i}$. Now observe that

$$\begin{aligned}
&\bar{J}_i(\bar{g}_i, \bar{g}_{-i}) - \bar{J}_i(g_i^*, \bar{g}_{-i}) \\
&= \mathbb{E}_{\omega,\phi_i}[(1-\lambda_i)\big((\theta - a(\phi_i)y_0 - b(\phi_i)y_i)^2\big)] \\
&\qquad - (\theta - a_i^* y_0 - b_i^* y_i)^2 \\
&+ \mathbb{E}_{\omega,\phi_i}\Big[\lambda_i\Big(\big(\big(1-\tfrac{\rho_i}{n}\big)(a(\phi_i)y_0 + b(\phi_i)y_i) - \tfrac{\rho_i(n-1)}{n}\bar{u}_{-i}\big)^2 \\
&\qquad - \big(\big(1-\tfrac{\rho_i}{n}\big)(a_i^* y_0 + b_i^* y_i) - \tfrac{\rho_i(n-1)}{n}\bar{u}_{-i}\big)^2\Big)\Big].
\end{aligned} \tag{26}$$

Consider now the first term of (26):

$$\begin{aligned}
&\mathbb{E}_{\omega,\phi_i}\big[(1-\lambda_i)\big((\theta - a(\phi_i)y_0 - b(\phi_i)y_i)^2\big)\big] \\
&\qquad - (\theta - a_i^* y_0 - b_i^* y_i)^2 \\
&= \mathbb{E}_{\omega,\phi_i}\big[(1-\lambda_i)(2\theta - (a(\phi_i)+a_i^*)y_0 - (b(\phi_i)+b_i^*)y_i) \\
&\qquad \times \big((a(\phi_i)-a_i^*)y_0 + (b(\phi_i)-b_i^*)y_i\big)\big] \\
&\le k_1 \mathbb{E}_{\omega,\phi_i}[|a(\phi_i)-a_i^*|] + k_2 \mathbb{E}_{\omega,\phi_i}[|b(\phi_i)-b_i^*|]
\end{aligned} \tag{27}$$

where $k_1$ and $k_2$ are constants. In the last inequality we used the fact that $|a(\phi_i)+a_i^*|$ and $|b(\phi_i)+b_i^*|$ are bounded. Similarly, we can bound the second term of (26) as

$$\begin{aligned}
&\mathbb{E}_{\omega,\phi_i}\Big[\lambda_i\Big(\big(\big(1-\tfrac{\rho_i}{n}\big)(a(\phi_i)y_0 + b(\phi_i)y_i) - \rho_i\big(1-\tfrac{1}{n}\big)\bar{u}_{-i}\big)^2 \\
&\qquad - \big(\big(1-\tfrac{\rho_i}{n}\big)(a_i^* y_0 + b_i^* y_i) - \rho_i\big(1-\tfrac{1}{n}\big)\bar{u}_{-i}\big)^2\Big)\Big] \\
&\le k_3 \mathbb{E}_{\omega,\phi_i}[|a(\phi_i)-a_i^*|] + k_4 \mathbb{E}_{\omega,\phi_i}[|b(\phi_i)-b_i^*|]
\end{aligned} \tag{28}$$

where $k_3$ and $k_4$ are constants.

Now, observe that $\Lambda_i = (1-\lambda_i) + \lambda_i(1-\rho_i/n)^2 \in 1 + \mathcal{O}(1/n)$. Thus, $\Lambda_i^{-1} = 1 + \mathcal{O}(1/n)$. Thus,

$$\begin{aligned}
b_i^* &\in \big[(1-\lambda_i) + \lambda_i \tfrac{\rho_i}{n}\big(1-\tfrac{\rho_i}{n}\big)\sum_{j\neq i} b(\phi_j)\alpha_j\big]K_i + \mathcal{O}\big(\tfrac{1}{n}\big). \\
&\in \underbrace{\big[(1-\lambda_i) + \lambda_i \tfrac{\rho_i}{n-1}\sum_{j\neq i} b(\phi_j)\alpha_j\big]K_i}_{=:\hat{b}_i} + \mathcal{O}\big(\tfrac{1}{n}\big)
\end{aligned} \tag{29}$$

Since $b(\phi_j)\alpha_j \in \mathbb{R}$, from [22, Theorem 1], we have that

$$\mathbb{E}_{\omega,\phi_i}\left[\left|\frac{1}{n-1}\sum_{j\neq i} b(\phi_j)\alpha_j - \bar{M}\right|\right] \le \frac{k_5}{\sqrt{n-1}},$$

where $k_5$ is a constant which depends on the support of the random variables. With $b(\phi_i)$ and $\hat{b}_i$ defined in (20) and (29), we have

$$\mathbb{E}_{\omega,\phi_i}[|b(\phi_i) - \hat{b}_i|] = \mathbb{E}_{\omega,\phi_i}\Big[\frac{k_5\lambda_i\rho_i K_i}{\sqrt{n-1}}\Big] \in \mathcal{O}\big(\tfrac{1}{\sqrt{n}}\big). \tag{30}$$

Substituting (30) in (29), we get that

$$\mathbb{E}_{\omega,\phi_i}[|b(\phi_i) - b_i^*|] \in \mathcal{O}\big(\tfrac{1}{\sqrt{n}}\big). \tag{31}$$

By a similar argument, we can show that

$$\mathbb{E}_{\omega,\phi_i}[|a(\phi_i) - a_i^*|] \in \mathcal{O}\big(\tfrac{1}{\sqrt{n}}\big). \tag{32}$$

Substituting (31) and (32) in (26), we get that

$$\bar{J}_i(\bar{g}_i, \bar{g}_{-i}) - \bar{J}_i(g_i^*, \bar{g}_{-i}) \in \mathcal{O}\big(\tfrac{1}{\sqrt{n}}\big),$$

Thus, the mean-field strategy is an $\varepsilon$-BNE.

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