Static teams with common information

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Abstract: We consider a static team problem in which agents observe correlated Gaussian observations and seek to minimize a quadratic cost. It is assumed that the observations can be split into two parts: common observations that are observed by all agents and local observations that are observed by individual agents. It is shown that the optimal strategies are affine and the corresponding gains can be determined by solving appropriate systems of linear equations. Two structures of optimal strategies are identified. The first may be viewed as a common-information based solution; the second may be viewed as a hierarchical control based solution. A decentralized estimation example is presented to illustrate the results.

Keywords: Stochastic control, Static teams, Multi-Agent Systems, Common Information.

1. INTRODUCTION

Decentralized decision making or *team* problems arise in a variety of applications including networked control systems, sensor networks, communication networks, transportation networks, and economics. In such problems, there are multiple decision makers or *agents* that have access to different information but aim to coordinate their actions to minimize a common cost function. A team problem is called *static* if the observations of agents are not affected by the actions of other agents; otherwise, the problem is called *dynamic*.

Static team problems were first investigated by Marschak (1955); Radner (1962); Marschak and Radner (1972), who identified necessary and sufficient conditions to determine team optimal strategies. Subsequently, there has been significant work on both static and dynamic teams. Sandell and Athans (1974) extended these results to vector valued observations; Krainak and Marcus (1982); Krainak et al. (1982) extended them to the exponential cost criteria.

Ho and Chu (1972); Chu (1972) investigated linear quadratic dynamic teams with partially nested information structure and showed that they can be converted to static teams with an appropriate change of variables. Sandell and Athans (1974); Yoshikawa (1975) provided an explicit solution to the linear quadratic dynamic team with one-step delay sharing. Casalino et al. (1984) generalized these results to general partially nested teams with a common past. Yüksel (2009) generalized the results of Ho and Chu to stochastically nested information structures. Recently, Mahajan and Nayyar (2015) identified sufficient statistics for best linear controllers for linear quadratic dynamic teams with partial history sharing.

In addition to the above results for linear quadratic teams, other variations of decentralized control problems have also been considered in the literature. These include models with non-linear Markovian dynamics, $\mathcal{H}_2/\mathcal{H}_\infty$ models with sparsity constraints, amongst others. We refer the reader to Mahajan et al. (2012) for details.

In this paper, we investigate the following question. Before the system starts running, suppose it is possible to build an observation channel and broadcast its measurements to all agents. What is the value of such common information? We investigate this question under the assumptions that the observations are jointly Gaussian and the cost is quadratic.

To answer this question, let J^* denote the optimal performance when the additional common observation channel is available and let J° denote the optimal performance when such a channel is not available. Then, the value of the additional common information is $J^\circ - J^*$ (i.e., it is beneficial to build the common observation channel if it costs less than $J^\circ - J^*$).

We develop two methods to efficiently compute the optimal strategy and the optimal performance for teams with common information. The first approach is inspired by the common information approach of Nayyar et al. (2013). We view the system from the point of view of a virtual agent that has access to the common observations. This virtual agent solves a static team problem where the means and covariances of the observations are the conditional means and covariances given the common information.

The second approach is a hierarchical control approach in which agents choose a part of their action based on their local observations and a virtual coordinator provides a linear correction to these local actions. The hierarchical control approach has interesting implications for the implementation of the optimal solution. Instead of transmitting the measurements of the common observation channel to all agents, a coordinator may only transmit a small correction term to each agent. When the common observation is high-dimensional (e.g., a video), communicating corrective terms instead of the common observations may lead to substantially smaller communication overhead.

The rest of the paper is organized as follows. In Sec. 2, we describe the model and the main results. In Sec. 3, we present an example of decentralized estimation. We prove the main results in Sec. 4 and conclude in Sec. 5.

Notation

Given a matrix A, A_{ij} denotes its (i,j)-th element, $A^\intercal$ denotes its transpose, $\text{vec}(A)$ denotes the column vector of A formed by stacking the columns of A on top of each other. Given matrices A and B with the same number of columns, $\text{rows}(A,B)$ denotes the matrix obtained by stacking matrix B under A. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. $\mathbb{R}$ denotes the set of real numbers.

2. STATIC TEAMS WITH COMMON INFORMATION

2.1 Model and Problem Formulation

Assume the system consists of n agents that are indexed by the set $N = \{1,\dots,n\}$. We use N_0 to denote the set $\{0,1,\dots,n\}$.

Let $(x, y_0, y_1, \dots, y_n)$ be jointly Gaussian random variables where $x \in \mathbb{R}^{d_x}$ and $y_i \in \mathbb{R}^{d_y^i}$ for $i \in N_0$. Let

$$\mathbb{E}[x] = \bar{x}, \quad \mathbb{E}[y_i] = \bar{y}_i, \text{ for } i \in N_0$$
$$\text{cov}(x, y_i) = \Theta_i, \quad \text{cov}(y_i, y_j) = \Sigma_{ij}, \text{ for } i \in N_0.$$

Agent i, $i \in N$, observes (y_0, y_i) and chooses $u_i \in \mathbb{R}^{d_u^i}$ according to a decision rule g_i, i.e., $u_i = g_i(y_0, y_i)$. The performance is measured by the following cost function:

$$c(x, u_1, \dots, u_n) = \sum_{i\in N}\sum_{j\in N} u_i^\intercal R_{ij} u_j + 2\sum_{i\in N} u_i^\intercal P_i x, \quad (1)$$

where $\{R_{ij}\}_{i,j}$ and $\{P_i\}_{i\in N}$ are matrices of appropriate dimensions. For ease of notation, define $u = \text{vec}(u_1,\dots,u_n)$, $P = \text{rows}(P_1,\dots,P_n)$, $\Theta = \text{rows}(\Theta_1,\dots,\Theta_n)$,

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{bmatrix}.$$

Then, the cost (1) may be written succinctly as

$$c(x,u) = u^\intercal R u + 2u^\intercal P x. \quad (2)$$

Assume the following:

(A1) The matrix R is symmetric and positive definite.
(A2) The parameters N, P, R, Θ, and Σ are common knowledge to all agents.

We call the following optimization problem as *static team with common information.*

Problem 1. Assuming (A1) and (A2) and given the joint distribution of $(x, y_0, y_1, \dots, y_n)$ and the cost matrices P and R, choose decision rules $g^* = (g_1^*, \dots, g_n^*)$ such that

$$J^* := J(g^*) = \min_g J(g)$$

where $J(g) = \mathbb{E}^g[c(x,u)]$ and $c(\cdot,\cdot)$ is defined in (1).

2.2 Main results

Define

$$\hat{x}_0 = \mathbb{E}[x \,|\, y_0] = \bar{x} + \Theta_0\Sigma_{00}^{-1}(y_0 - \bar{y}_0),$$

and for $i, j \in N$,

$$\hat{y}_i = \mathbb{E}[y_i \,|\, y_0] = \bar{y}_i + \Sigma_{i0}\Sigma_{00}^{-1}(y_0 - \bar{y}_0),$$
$$\hat{\Theta}_i = \text{cov}(x, y_i \,|\, y_0) = \Theta_i - \Theta_0\Sigma_{00}^{-1}\Sigma_{0i},$$
$$\hat{\Sigma}_{ij} = \text{cov}(y_i, y_j \,|\, y_0) = \Sigma_{ij} - \Sigma_{i0}\Sigma_{00}^{-1}\Sigma_{0j}.$$

Theorem 1. In Problem 1, we have the following:

(1) The optimal control laws are given by
$$u_i = L_i(y_i - \hat{y}_i) + H_i\hat{x}_0, \quad \forall i \in N. \quad (3)$$
(2) Alternatively, by substituting the value of $\hat{y}_i$ and $\hat{x}_0$ in (3), the optimal control laws may be written as
$$u_i = L_i(y_i - \bar{y}_i) + G_i(y_0 - \bar{y}_0) + H_i\bar{x}, \quad \forall i \in N, \quad (4)$$
where $G_i = \big(-L_i\Sigma_{i0} + H_i\Theta_0\big)\Sigma_{00}^{-1}$.
(3) The corresponding gains are computed as follows. Define $H = \text{rows}(H_1,\dots,H_n)$, $L = \text{vec}(L_1,\dots,L_n)$, $\hat{\eta} = \text{vec}(P_1\hat{\Theta}_1,\dots,P_n\hat{\Theta}_n)$, and $\hat{\Gamma} = [\hat{\Gamma}_{ij}]_{i,j\in N}$ where $\hat{\Gamma}_{ij} = \hat{\Sigma}_{ij} \otimes R_{ij}$. Then,
$$H = -R^{-1}P \quad \text{and} \quad L = -\hat{\Gamma}^{-1}\hat{\eta}. \quad (5)$$
(4) The optimal cost is given by
$$J^* = -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta} - \hat{x}_0^\intercal P^\intercal R^{-1} P\hat{x}_0$$
$$= -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta} - \bar{x}^\intercal P^\intercal R^{-1}P\bar{x} - \text{Tr}(\Theta_0\Sigma_{00}^{-1}\Theta_0^\intercal P^\intercal R^{-1}P).$$

Theorem 1 provides two structures of the optimal control laws. The first structure, which is given by (3), may be viewed as a common-information based solution. In particular, consider a static team problem with n agents where the true state of the world is $\mathring{x}$ and agent i, $i \in N$, observes $\mathring{y}_i$ where $(\mathring{x}, \mathring{y}_1, \dots, \mathring{y}_n)$ are jointly Gaussian such that $\mathbb{E}[\mathring{x}] = \mathbb{E}[x \,|\, y_0]$, $\mathbb{E}[\mathring{y}_i] = \mathbb{E}[y_i \,|\, y_0]$, $\text{cov}(\mathring{x}, \mathring{y}_i) = \text{cov}(x, y_i \,|\, y_0)$, and $\text{cov}(\mathring{y}_i, \mathring{y}_j) = \text{cov}(y_i, y_j \,|\, y_0)$. From Radner (1962), the optimal solution of such a static team is given by (3) and the gains $\{L_i\}_{i\in N}$ and $\{H_i\}_{i\in N}$ are computed as specified in Theorem 1.

The above argument is not a proof of Theorem 1 because we have not proved that the system viewed from the point of view of the common information is *equivalent* to the original system. A formal proof of Theorem 1 is given in Sec. 4. Nonetheless, viewing the system from the point of view of common information provides a way to interpret the results of Theorem 1.

The second structure of the optimal controller, which is given by (4), may be viewed as a hierarchical control based solution. In particular, consider a hierarchical control system in which n "lower-level" agents observe y_i and choose a "local control action", $\tilde{u}_i$ and a "higher-level" coordinator that observes y_0 and sends "global correction signals" v_i to agent i. Agent i then chooses its control action as

$$u_i = \tilde{u}_i + v_i.$$

Theorem 1 states that the optimal local control is

$$\tilde{u}_i = L_i(y_i - \bar{y}_i) + H_i\bar{x}$$

while the optimal correction signal is

$$v_i = G_i(y_0 - \bar{y}_0).$$

Note that the gains $\{L_i\}_{i\in N}$, $\{H_i\}_{i\in N}$ of the local control and the gains $\{G_i\}_{i\in N}$ of the global correction depend on the statistical information of all the agents.

Another way to view the result of Theorem 1 is as follows. For $i, j \in N$, define

$$\hat{x}_i = \mathbb{E}[x \,|\, y_i] = \bar{x} + \Theta_i\Sigma_{ii}^{-1}(y_i - \bar{y}_i),$$
$$\breve{x}_i = \mathbb{E}[x \,|\, y_i - \hat{y}_i] = \bar{x} + \hat{\Theta}_i\hat{\Sigma}_{ii}^{-1}(y_i - \hat{y}_i),$$
$$\breve{\Theta}_i = \text{cov}(x, \breve{x}_i) = \hat{\Theta}_i\hat{\Sigma}_{ii}^{-1}\hat{\Theta}_i^\intercal,$$
$$\breve{\Sigma}_{ij} = \text{cov}(\breve{x}_i, \breve{x}_j) = \hat{\Theta}_i\hat{\Sigma}_{ii}^{-1}\hat{\Sigma}_{ij}\hat{\Sigma}_{jj}^{-1}\hat{\Theta}_j^\intercal.$$

Corollary 2. In Problem 1:

(1) The optimal strategy may be written as

$$u_i = F_i(\breve{x}_i - \bar{x}) + H_i \hat{x}_0, \quad \forall i \in N. \tag{6}$$

(2) Alternatively, by substituting the value of $\breve{x}_i$ in (6), the optimal control laws may be written as

$$u_i = \hat{F}_i(\hat{x}_i - \bar{x}) + \hat{G}_i(\hat{x}_0 - \bar{x}) + H_i \bar{x}, \quad \forall i \in N, \tag{7}$$

where $\hat{F}_i = F_i \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Sigma_{ii} \Theta_i^{-1}$ and $\hat{G}_i = H_i - F_i \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Sigma_{i0} \Theta_0^{-1}$.

(3) The corresponding gains are computed as follows. H_i is given by (5). Define $F = \text{vec}(F_1, \cdots, F_n)$, $\breve{\eta} = \text{vec}(P_1 \breve{\Theta}_1, \ldots, P_n \breve{\Theta}_n)$, and $\breve{\Gamma} = [\breve{\Gamma}_{ij}]_{i,j \in N}$ where $\breve{\Gamma}_{ij} = \breve{\Sigma}_{ij} \otimes R_{ij}$. Then,

$$F = -\breve{\Gamma}^{-1} \breve{\eta}. \tag{8}$$

(4) Furthermore, the optimal cost is given by

$$J^* = -\breve{\eta}^\intercal \breve{\Gamma}^{-1} \breve{\eta} - \hat{x}_0^\intercal P^\intercal R^{-1} P \hat{x}_0.$$

The result is obtained by substituting $y_i - \hat{y}_i = \hat{\Sigma}_{ii} \hat{\Theta}_i^{-1}(\breve{x}_i - \bar{x})$ in (3).

2.3 Comparison with Radner's results

When the common information is absent (i.e., y_0 is independent of $(x, y_1, \ldots, y_n)$), Problem 1 is the same as the static team problem investigated by Radner (1962), who showed the following:

Theorem 3. (Radner (1962)). In Problem 1, if y_0 is independent of $(x, y_1, \ldots, y_n)$ (i.e., $\Theta_0 = 0$ and for $i \in N$, $\Sigma_{i0} = 0$), then the optimal control laws are given by

$$u_i = L_i^\circ(y_i - \bar{y}_i) + H_i^\circ \bar{x} \tag{9}$$

where the gains $\{L_i^\circ\}_{i \in N}$ and $\{H_i^\circ\}_{i \in N}$ are computed as follows. Define $H^\circ = \text{rows}(H_1^\circ, \ldots, H_n^\circ)$, $L^\circ = \text{vec}(L_1^\circ, \ldots L_n^\circ)$, $\eta^\circ = \text{vec}(P_1 \Theta_1, \ldots, P_n \Theta_n)$, and $\Gamma^\circ = [\Gamma_{ij}^\circ]_{i,j \in N}$ where $\Gamma_{ij}^\circ = \Sigma_{ij} \otimes R_{ij}$. Then,

$$H^\circ = -R^{-1} P \quad \text{and} \quad L^\circ = -(\Gamma^\circ)^{-1} \eta^\circ. \tag{10}$$

Furthermore, the optimal cost is given by

$$J^\circ = -\eta^\circ (\Gamma^\circ)^{-1} \eta^\circ - \bar{x} P^\intercal R^{-1} P \bar{x}.$$

Note that the gains $\{H_i^\circ\}_{i \in N}$ are exactly the same $\{H\}_{i \in N}$ defined in Theorem 1.

It is possible to directly use Theorem 3 to solve Problem 1. The observation of each agent is (y_0, y_i). Therefore, the optimal control law is of the form

$$u_i = L_i^\circ \begin{bmatrix} y_0 - \bar{y}_0 \\ y_i - \bar{y}_i \end{bmatrix} + H_i \bar{x}.$$

Such a naive solution requires more calculations than the solution given in Theorem 1. In particular, define $d_u = \sum_{i \in N} d_u^i$, $d_l = \sum_{i \in N} d_u^i \times d_y^i$, and $d_l^\circ = \sum_{i \in N} d_u^i \times (d_y^0 + d_y^i)$. Observe that $d_l^\circ = d_l + N(d_u \times d_y^0)$. To obtain the gains $\{L_i^\circ\}_{i \in N}$ as argued above, we need to solve a system of d_l° linear equations; to obtain the gains $\{L_i\}_{i \in N}$ using the method of Theorem 1, we need to solve a system of d_l linear equations. Thus, the method of Theorem 1 is more efficient than the naive method described above.

It is also possible to rewrite the result of Theorem 3 in the form of Corollary 2 as follows.

Corollary 4. In Problem 1, if y_0 is independent of $(x, y_1, \ldots, y_n)$(i.e., $\Theta_0 = 0$ and for $i \in N$, $\Sigma_{i0} = 0$), then the optimal control laws are given by

$$u_i = F_i^\circ(\hat{x}_i - \bar{x}) + H_i^\circ \bar{x} \tag{11}$$

where the gains $\{F_i^\circ\}_{i \in N}$ and $\{H_i^\circ\}_{i \in N}$ are computed as follows. Define $H^\circ = \text{rows}(H_1^\circ, \ldots, H_n^\circ)$, $F^\circ = \text{vec}(F_1^\circ, \ldots F_n^\circ)$, $\hat{\eta}^\circ = \text{vec}(P_1 \Theta_1^\circ, \ldots, P_n \Theta_n^\circ)$, and $\hat{\Gamma}^\circ = [\hat{\Gamma}_{ij}^\circ]_{i,j \in N}$ where $\hat{\Gamma}_{ij}^\circ = \Sigma_{ij}^\circ \otimes R_{ij}$, $\Theta_i^\circ = \Theta_i \Sigma_{ii}^{-1} \Theta_i^\intercal$, and $\Sigma_{ij}^\circ = \Theta_i \Sigma_{ii}^{-1} \Sigma_{ij} \Sigma_{jj}^{-1} \Theta_j^\intercal$. Then,

$$H^\circ = -R^{-1} P \quad \text{and} \quad F^\circ = -(\hat{\Gamma}^\circ)^{-1} \hat{\eta}^\circ. \tag{12}$$

Furthermore, the optimal cost is given by

$$J^\circ = -\hat{\eta}^\circ (\hat{\Gamma}^\circ)^{-1} \hat{\eta}^\circ - \bar{x}^\intercal P^\intercal R^{-1} P \bar{x}.$$

As before, directly using Corollary 4 to solve Problem 1 requires more calculations that the solution given by Corollary 2. In particular, define $d_m = \sum_{i \in N} d_y^i$ and $d_m^0 = \sum_{i \in N} (d_y^i + d_y^0)$. Observe that $d_m^0 = d_m + N d_y^0$. Note that matrix Σ° is $d_m^0 \times d_m^0$ while matrix $\breve{\Sigma}$ is $d_m \times d_m$.

3. AN ILLUSTRATIVE EXAMPLE: DECENTRALIZED ESTIMATION

As an illustrative example of the static team model with common information, we consider a decentralized estimation problem. Suppose there is a random variable $x \in \mathbb{R}^{d_x}$ of interest. There are n agents. Agent i, $i \in N$, observes (y_0, y_i) where for $k \in N_0$, $y_k = C_k x + w_k$, $y_k, w_k \in \mathbb{R}^{d_y^k}$ and C_k is a matrix of appropriate dimension. We assume that $(x, w_0, \ldots, w_n)$ are independent; $x \sim \mathcal{N}(0, \Sigma_x)$ and $w_i \sim \mathcal{N}(0, \Sigma_w^i)$, $i \in N_0$.

After observing (y_0, y_i), agent i generates an estimate $u_i \in \mathbb{R}^{d_x}$ of x. The estimation error depends on how close the estimates are to the true state x and how close are the estimates of "neighboring" agents.

To make the notion of neighbors precise, suppose there is an undirected graph G where nodes are indexed by N. For each node $i \in N$, let N_i denote the set of neighbors of i. There are "weight" matrices M_{ii} associated with each node and "weight" matrices M_{ij} associated with each edge (Note that $M_{ij} = M_{ji}$). It is assumed that all weight matrices are positive definite. Then, the estimation error is measured by

$$c(x, u_1, \ldots, u_n) = \sum_{i \in N} (x - u_i)^\intercal M_{ii} (x - u_i) + \sum_{i \in N} \sum_{j \in N_i, j > i} (u_i - u_j)^\intercal M_{ij} (u_i - u_j). \tag{13}$$

See Figure 1 for an example.

The above model is a special case of the static team model described in Sec 2.1. In particular, $\bar{x} = 0$ and for $i, j \in N_0$, $\bar{y}_i = 0$ and

$$\Theta_i = \text{cov}(x, y_i) = \Sigma_x C_i^\intercal$$

and

$$\Sigma_{ij} = \text{cov}(y_i, y_j) = \begin{cases} C_i \Sigma_x C_i^\intercal + \Sigma_w^i, & \text{if } i = j \\ C_i \Sigma_x C_j^\intercal, & \text{if } i \neq j \end{cases}$$

Note that Σ_{ij} can be succinctly written as $C_i \Sigma_x C_j^\intercal + \Sigma_w^i \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Then,

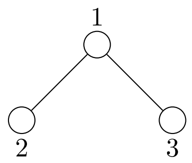

Fig. 1. An example of a distributed estimation problem in which $N = \{1, 2, 3\}$, $N_1 = \{2, 3\}$, $N_2 = N_3 = \{1\}$. Suppose $d_x = d_y^i = 1$ and $M_{11} = M_{22} = M_{33} = M_{12} = M_{21} = M_{13} = M_{31} = 1$. Then, the cost (13) is $(x-u_1)^2+(x-u_2)^2+(x-u_3)^2+(u_1-u_2)^2+(u_1-u_3)^2$. Hence, the objective is to minimize the sum of (i) the square error between the estimates generated by the agents and the true x; and (ii) the square error between between the estimates of agents 1 and 2 and agents 1 and 3.

$$\hat{x}_0 = \mathbb{E}[x \mid y_0] = \Sigma_x C_0^\intercal (C_0 \Sigma_x C_0^\intercal + \Sigma_w^0)^{-1} y_0,$$

and for $i, j \in N$,

$$\begin{aligned}
\hat{y}_i &= \mathbb{E}[y_i \mid y_0] = C_i \Sigma_x C_0^\intercal (C_0 \Sigma_x C_0^\intercal + \Sigma_w^0)^{-1} y_0, \\
\hat{\Theta}_i &= \operatorname{cov}(x, y_i \mid y_0) = \Sigma_x C_i^\intercal \\
&\qquad - \Sigma_x C_0^\intercal (C_0 \Sigma_x C_0^\intercal + \Sigma_w^0)^{-1} C_0 \Sigma_x C_i^\intercal, \\
\hat{\Sigma}_{ij} &= \operatorname{cov}(y_i, y_j | y_0) = C_i \Sigma_x C_j^\intercal + \Sigma_w^i \delta_{ij} \\
&\qquad - C_i \Sigma_x C_0^\intercal (C_0 \Sigma_x C_0^\intercal + \Sigma_w^0)^{-1} C_0 \Sigma_x C_j^\intercal, \\
\hat{x}_i &= \mathbb{E}[x \mid y_i] = \Sigma_x C_i^\intercal (C_i \Sigma_x C_i^\intercal)^{-1} y_i, \\
\breve{x}_i &= \mathbb{E}[x \mid y_i - \hat{y}_i] = \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i), \\
\breve{\Theta}_i &= \operatorname{cov}(x, \breve{x}_i) = \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \hat{\Theta}_i^\intercal, \\
\breve{\Sigma}_{ij} &= \operatorname{cov}(\breve{x}_i, \breve{x}_j) = \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \hat{\Sigma}_{ij} \hat{\Sigma}_{jj}^{-1} \hat{\Theta}_j^\intercal.
\end{aligned}$$

By expanding (13) and comparing it with (1), we get that $P = \operatorname{rows}(-M_{11}, \dots, -M_{nn})$ and $R = [R_{ij}]_{i,j \in N}$, where

$$R_{ij} = \begin{cases} M_{ii} + \sum_{j \in N_i} M_{ij}, & \text{if } i = j \\ -M_{ij}, & \text{if } j \in N_i \\ 0, & \text{otherwise.} \end{cases}$$

Note that R is equivalent to the Laplacian of the cost graph (with additional self loops of weight M_{ii} at each node i). For example, for the example of Fig. 1, $P = \operatorname{rows}(-1, -1, -1)$ and $R = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

Now, by Theorem 1 and Corollary 2, the optimal estimation strategy for the above model is given by

$$u_i = L_i(y_i - \hat{y}_i) + H_i \hat{x}_0 \quad \text{or} \quad u_i = F_i \breve{x}_i + H_i \hat{x}_0$$

where the gains $\{H_i\}_{i \in N}$, $\{F_i\}_{i \in N}$ and $\{L_i\}_{i \in N}$ are computed as in Theorem 1 and Corollary 2. Or equivalently, the optimal estimation strategy is given by

$$u_i = L_i y_i + G_i y_0 \quad \text{or} \quad u_i = \hat{F}_i \hat{x}_i + \hat{G}_i \hat{x}_0$$

where $G_i = (-L_i \Sigma_{i0} + H_i \Theta_0) \Sigma_{00}^{-1}$, $\hat{F}_i = F_i \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Sigma_{ii} \Theta_i^{-1}$ and $\hat{G}_i = H_i - F_i \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} \Sigma_{i0} \Theta_0^{-1}$.

Furthermore, the optimal cost is given by

$$J^* = -\hat{\eta}^\intercal \hat{\Gamma}^{-1} \hat{\eta} - \operatorname{Tr}(\Theta_0 \Sigma_{00}^{-1} \Theta_0^\intercal P^\intercal R^{-1} P) + \sum_{i \in N} \operatorname{Tr}(R_{ii} \Sigma_x),$$

where $\hat{\eta}$ and $\hat{\Gamma}$ are defined as in Theorem 1.

Now, we numerically compute the optimal strategies and optimal performance for some specific cases. We consider a system with $n = 4$ nodes for three different cost functions, whose graphs are shown in Fig. 2.

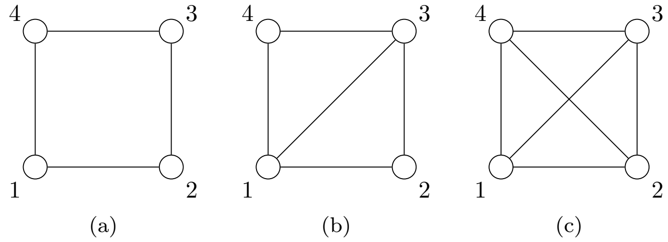

Fig. 2. Three different cost functions for the decentralized estimation problem with 4 agents.

We assume that

- $d_x = d_y^i = d_u^i = 1$.
- For $i \in N$, $C_i = 1$, $\Sigma_x = 1$, $\Sigma_w^i = \sigma^2$.
- $\Sigma_w^0 = \sigma_0^2$.
- For $i \in N$, $M_{ii} = 1$ and for $i, j \in N$, whenever the edge (i, j) exists, $M_{ij} = 1$.

Thus, we have two parameters: the variance σ^2 of local observations, and the variance σ_0^2 of the common observations. We will evaluate the performance of the system for different choice of these parameters.

We consider two cases: without common information (i.e., when y_0 is not available to the nodes) and with common information. For the case without common information, the optimal estimate is

$$u_i = L_i^\circ y_i \quad \text{or} \quad u_i = F_i^\circ \hat{x}_i$$

while for the case with common information, the optimal estimate is

$$u_i = L_i y_i + G_i y_0 \quad \text{or} \quad u_i = \hat{F}_i \hat{x}_i + \hat{G}_i \hat{x}_0.$$

The corresponding optimal performances are denoted by J° and J^*.

When $\sigma^2 = \sigma_0^2 = 1$, the optimal solution is as follows:

Cost (a)

- For the case without common information $J^\circ = 3$ and
$$u_i = \tfrac{1}{4} y_i, \quad \text{or} \quad u_i = \tfrac{1}{2} \hat{x}_i, \quad i \in N.$$
- For the case with common information $J^* = \frac{12}{7} \approx 1.7143$ and
$$u_i = \tfrac{1}{7} y_i + \tfrac{3}{7} y_0 \quad \text{or} \quad u_i = \tfrac{2}{7} \hat{x}_i + \tfrac{6}{7} \hat{x}_0, \quad i \in N.$$
- Thus, the value of the common information channel is $J^\circ - J^* = \frac{9}{7} \approx 1.2857$.

Cost (b)

- For the case without common information $J^\circ = \frac{59}{19} \approx 3.1053$ and
$$\begin{aligned} u_i &= \tfrac{4}{19} y_i \quad \text{or} \quad u_i = \tfrac{8}{19} \hat{x}_i, \quad i \in \{1, 3\}, \\ u_i &= \tfrac{9}{38} y_i \quad \text{or} \quad u_i = \tfrac{9}{19} \hat{x}_i, \quad i \in \{2, 4\}. \end{aligned}$$
- For the case with common information $J^* = \frac{166}{95} \approx 1.7474$ and
$$\begin{aligned} u_i &= \tfrac{11}{95} y_i + \tfrac{42}{95} y_0 \text{ or } u_i = \tfrac{22}{95} \hat{x}_i + \tfrac{84}{95} \hat{x}_0, \quad i \in \{1, 3\}, \\ u_i &= \tfrac{13}{95} y_i + \tfrac{41}{95} y_0 \text{ or } u_i = \tfrac{26}{95} \hat{x}_i + \tfrac{82}{95} \hat{x}_0, \quad i \in \{2, 4\}. \end{aligned}$$

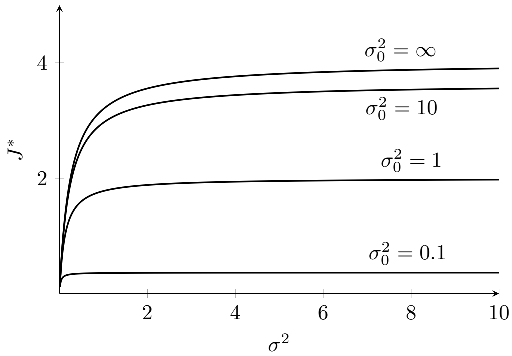

Fig. 3. Plot of J^* (and J°) as a function of σ^2 for different values of σ_0^2. Note that J° may be thought of the limiting case as $\sigma_0^2 \to \infty$.

- Thus, the value of the common information channel is $J^\circ - J^* = \frac{129}{95} \approx 1.3579$.

Cost (c)

- For the case without common information $J^\circ = \frac{16}{5} = 3.2$ and
$$u_i = \tfrac{1}{5} y_i \quad \text{or} \quad u_i = \tfrac{2}{5}\hat{x}_i, \quad i \in N.$$
- For the case with common information $J^* = \frac{16}{9} \approx 1.7778$ and
$$u_i = \tfrac{1}{9} y_i + \tfrac{4}{9} y_0 \quad \text{or} \quad u_i = \tfrac{2}{9}\hat{x}_i + \tfrac{8}{9}\hat{x}_0, \quad i \in N.$$
- Thus, the value of the common information channel is $J^\circ - J^* = \frac{64}{45} \approx 1.4222$.

Next, we plot J^* (and J°) as a function of σ^2 for different values of σ_0^2. See Fig. 3 for details.

4. PROOF OF THE MAIN RESULTS

The main idea of the proof is similar to that of Radner (1962). However, instead of working with the observations (y_0, y_i), we work with the orthogonal random variables $(y_0, y_i - \hat{y}_i)$.

Consider agent $i \in N$ and arbitrarily fix the strategy g_{-i} of all agents other than i. A necessary condition for the strategy (g_i, g_{-i}) to be globally optimal is that g_i is the *best response* strategy to g_{-i}, i.e., for any $y_0 \in \mathbb{R}^0_y$, $y_i \in \mathbb{R}^i_y$, and $\hat{u}_i \in \mathbb{R}^i_u$ and $u_j = g_j(y_0, y_j)$, $j \in N$, we have that
$$\mathbb{E}^{g_{-i}}[c(x, u_i, u_{-i}) | y_0, y_i] \le \mathbb{E}^{g_{-i}}[c(x, \hat{u}_i, u_{-i}) | y_0, y_i].$$
A sufficient condition for the above to hold is
$$\frac{\partial}{\partial u_i} \mathbb{E}^{g_{-i}}[c(x, u_i, u_{-i}) | y_0, y_i] = 0. \tag{14}$$
Assuming that we can interchange differentiation and expectation, we get that
$$\begin{aligned}
\text{LHS of (14)} &= \mathbb{E}^{g_{-i}}\left[\frac{\partial}{\partial u_i} c(x, u_i, u_{-i}) \,\middle|\, y_0, y_i \right] \\
&= \mathbb{E}^{g_{-i}}\left[\frac{\partial}{\partial u_i} \left[\sum_{k \in N} \sum_{j \in N} u_k^\intercal R_{kj} u_j + 2 \sum_{k \in N} u_k^\intercal P_k x \right] \,\middle|\, y_0, y_i \right] \\
&= 2\mathbb{E}^{g_{-i}}\left[\sum_{j \in N} R_{ij} u_j + P_i x \,\middle|\, y_0, y_i \right]
\end{aligned}$$
Thus, a necessary condition for strategy g to be optimal is that for all $i \in N$ and all $y_k \in \mathbb{R}_{d_y^k}$, $k \in N_0$, we have that
$$\sum_{j \in N} R_{ij} \mathbb{E}[u_j | y_0, y_i] + P_i \mathbb{E}[x | y_0, y_i] = 0. \tag{15}$$
Hence, a necessary condition for the strategy described in Theorem 1 to be optimal is that for all $i \in N$ and all $y_0 \in \mathbb{R}^{d_y^0}, y_i \in \mathbb{R}^{d_y^i}$,
$$\sum_{j \in N} R_{ij} \mathbb{E}[L_j(y_j - \hat{y}_j) + H_j \hat{x}_0 | y_0, y_i] + P_i \mathbb{E}[x | y_0, y_i] = 0. \tag{16}$$
The argument so far is similar to Radner (1962). Now, to verify (16), we exploit the orthogonal projection theorem. Recall that $\hat{y}_i = \mathbb{E}[y_i | y_0]$ and $(x, y_0, y_1, \dots, y_n)$. Therefore, $(x - \hat{x}_0)$ and $(y_i - \hat{y}_i)$ are both orthogonal to y_0. Hence,
$$\begin{aligned}
\mathbb{E}[y_j - \hat{y}_i | y_0, y_i] &= \mathbb{E}[y_j - \hat{y}_j | y_0] + \mathbb{E}[y_j - \hat{y}_j | y_i - \hat{y}_i] \\
&= \hat{\Sigma}_{ji} \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i), \qquad (17) \\
\mathbb{E}[\hat{x}_0 | y_0, y_i] &= \mathbb{E}[\hat{x}_0 | y_0] + \mathbb{E}[\hat{x}_0 | y_i - \hat{y}_i] = \hat{x}_0, \qquad (18)
\end{aligned}$$
and
$$\begin{aligned}
\mathbb{E}[x | y_0, y_i] &= \mathbb{E}[x | y_0] + \mathbb{E}[x | y_i - \hat{y}_i] \\
&= \hat{x}_0 + \hat{\Theta}_i \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i). \qquad (19)
\end{aligned}$$
Substituting (17)–(19) in (16), we get that a necessary condition for the strategy described in Theorem 1 to be optimal is that for all $i \in N$ and all $y_0 \in \mathbb{R}^{d_y^0}, y_i \in \mathbb{R}^{d_y^i}$,
$$\Big[\sum_{j \in N} R_{ij} L_j \hat{\Sigma}_{ji} + P_i \hat{\Theta}_i \Big] \hat{\Sigma}_{ii}^{-1} (y_i - \hat{y}_i) + \Big[\sum_{i \in N} R_{ij} H_j + P_i \Big] \hat{x}_0 = 0. \tag{20}$$
For the above equation to hold for all realizations of $(y_i - \hat{y}_i)$ and $\hat{x}_0$, it must be the case that both terms in the square bracket are zero, i.e., for all $i \in N$,
$$\sum_{j \in N} R_{ij} L_j \hat{\Sigma}_{ji} + P_i \hat{\Theta}_i = 0, \tag{21}$$
and
$$\sum_{i \in N} R_{ij} H_j + P_i = 0. \tag{22}$$
These set of equations can be further simplified as follows. To simplify the equations for the gains $\{H_i\}_{i \in N}$, combine (22) for all $i \in N$ to get $RH + P = 0$, or equivalently, $H = -R^{-1} P$.

To simplify the equations for the gains $\{L_i\}_{i \in N}$, vectorize both sides of (21) and use $\operatorname{vec}(ABC) = (C^\intercal \otimes A) \times \operatorname{vec}(B)$ to obtain
$$\sum_{j \in N} (\hat{\Sigma}_{ij} \otimes R_{ij}) \operatorname{vec}(L_j) + \operatorname{vec}(P_i \hat{\Theta}_i) = 0.$$
Subsitituting $\hat{\Gamma}_{ij} = \hat{\Sigma}_{ij} \otimes R_{ij}$ and $\hat{\eta}_i = \operatorname{vec}(P_i \hat{\Theta}_i)$, we get $\hat{\Gamma} L + \hat{\eta} = 0$, or equivalently, $L = -\hat{\Gamma}^{-1} \hat{\eta}$.

Thus, we have proved parts 1 and 3 of the Theorem. Part 2 follows from directly substituting the result of part 1. To prove the result of part 4, observe that
$$J^* = \mathbb{E}\big[\mathbb{E}[u^\intercal R u + 2u^\intercal P x | y_0]\big]. \tag{23}$$
Now consider

$$
\begin{aligned}
&\mathbb{E}[u^\intercal Ru + 2u^\intercal Px|y_0] \\
&\quad = \mathbb{E}[\big(L(y-\hat{y}) + H\hat{x}_0\big)^\intercal R\big(L(y-\hat{y}) + H\hat{x}_0\big)|y_0] \\
&\qquad + 2\mathbb{E}[\big(L(y-\hat{y}) + H\hat{x}_0\big)^\intercal Px|y_0] \\
&\quad = \mathbb{E}[(y-\hat{y})^\intercal L^\intercal RL(y-\hat{y})|y_0] + \hat{x}_0^\intercal H^\intercal RH\hat{x}_0 \\
&\qquad + 2\mathbb{E}[(y-\hat{y})^\intercal L^\intercal Px|y_0] + 2\mathbb{E}[\hat{x}_0^\intercal HPx|y_0] \qquad (24)
\end{aligned}
$$

We split the above sum into two parts: the first part is

$$
\begin{aligned}
&\mathbb{E}[(y-\hat{y})^\intercal L^\intercal RL(y-\hat{y})|y_0] + 2\mathbb{E}[(y-\hat{y})^\intercal L^\intercal Px|y_0] \\
&\quad = \sum_{i\in N}\sum_{j\in N} \mathbb{E}\big[(y_i-\hat{y}_i)^\intercal L_i^\intercal R_{ij} L_j(y_j-\hat{y}_j) \bigm| y_0\big] \\
&\qquad + 2\sum_{i\in N} \mathbb{E}\big[(y_i-\hat{y}_i)^\intercal L_i^\intercal P_i x \bigm| y_0\big] \\
&\quad \overset{(a)}{=} \sum_{i\in N}\sum_{i\in N} \operatorname{Tr}(L_i\hat{\Sigma}_{ij}L_j^\intercal R_{ji}) + 2\sum_{i\in N}\operatorname{Tr}(L_i\hat{\Theta}_i^\intercal P_i^\intercal) \\
&\quad = \sum_{i\in N} \operatorname{Tr}\Big(L_i\Big(\sum_{j\in N}\hat{\Sigma}_{ij}L_j^\intercal R_{ji} + 2\hat{\Theta}_i^\intercal P_i^\intercal\Big)\Big) \\
&\quad \overset{(b)}{=} \sum_{i\in N}\operatorname{Tr}(L_i\hat{\Theta}_i^\intercal P_i^\intercal) \overset{(c)}{=} \sum_{i\in N}\operatorname{vec}(L_i)^\intercal \operatorname{vec}(P_i\hat{\Theta}_i) \\
&\quad = L^\intercal\hat{\eta} = -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta}, \qquad (25)
\end{aligned}
$$

where (a) uses the following: for any vectors a and b and matrices A and B of appropriate dimensions, $\mathbb{E}[a^\intercal A^\intercal Bb] = \mathbb{E}[\operatorname{Tr}(a^\intercal A^\intercal Bb)] = \mathbb{E}[\operatorname{Tr}(Aab^\intercal B^\intercal)] = \operatorname{Tr}(A\mathbb{E}[ab^\intercal]B^\intercal)$; (b) uses (21); and (c) uses the following: for any matrices A and B of appropriate dimensions $\operatorname{Tr}(AB^\intercal) = \operatorname{vec}(A)^\intercal\operatorname{vec}(B)$.

The second part of (24) is

$$
\begin{aligned}
&\hat{x}_0^\intercal H^\intercal RH\hat{x}_0 + 2\mathbb{E}[\hat{x}_0^\intercal H^\intercal Px|y_0] \\
&\quad = \hat{x}_0^\intercal H^\intercal RH\hat{x}_0 + 2\hat{x}_0^\intercal H^\intercal P\hat{x}_0 = \hat{x}_0^\intercal H^\intercal(RH+2P)\hat{x}_0 \\
&\quad = \hat{x}_0^\intercal H^\intercal P\hat{x}_0 = -\hat{x}_0^\intercal P^\intercal R^{-1}P\hat{x}_0. \qquad (26)
\end{aligned}
$$

where the last two equalities use $RH + P = 0$.

Subsituting (25) and (26) in (24), we get

$$
\mathbb{E}[u^\intercal Ru + 2u^\intercal Px|y_0] = -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta} - \hat{x}_0^\intercal P^\intercal R^{-1}P\hat{x}_0. \qquad (27)
$$

Substituting (27) in (23), we get

$$
\begin{aligned}
J^* &= -\mathbb{E}[\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta}, +\hat{x}_0^\intercal P^\intercal R^{-1}P\hat{x}_0] \\
&= -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta} - \bar{x}^\intercal P^\intercal R^{-1}P\bar{x} \\
&\quad - \mathbb{E}[(y_0-\bar{y}_0)^\intercal\Sigma_{00}^{-1}\Theta_0^\intercal P^\intercal R^{-1}P\Theta_0\Sigma_{00}^{-1}(y_0-\bar{y}_0)] \\
&= -\hat{\eta}^\intercal\hat{\Gamma}^{-1}\hat{\eta} - \bar{x}^\intercal P^\intercal R^{-1}P\bar{x} - \operatorname{Tr}(\Theta_0\Sigma_{00}^{-1}\Theta_0^\intercal P^\intercal R^{-1}P).
\end{aligned}
$$

This completes the proof of part 4.

5. CONCLUSION

We investigate static teams with common information and present two structures of optimal strategies. The complexity of the proposed solution methodology is significantly less that naively using the existing results for static teams.

The first structure of optimal strategies can be interpreted as follows. For the given realization of the common information, all agents compute the conditional means and covariances given the common information and compute the gains corresponding to this conditional system. The equations describing the gains depends only on the conditional covariances. Since all the random variables are Gaussian, the conditional covariances do not depend on the realization of the common information and, therefore, neither do the optimal gains.

By a simple algebraic manipulation of the structure of the optimal controller, it can also be viewed as a hierarchical controller where each agent receives a "global correction signal" that it applies to its local control action. Such an implementation is more efficient if the common information is a high-dimensional signal (e.g., video).

The solution methodology developed in this paper could be useful for dynamic team problems as well. We plan to explore that direction in the future.

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