Optimal Decentralized Control of Coupled Subsystems With Control Sharing

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***Abstract*—Subsystems that are coupled due to dynamics and costs arise naturally in various communication applications. In many such applications the control actions are shared between different control stations giving rise to a *control sharing* information structure. Previous studies of control-sharing have concentrated on the linear quadratic Gaussian setup and a solution approach tailored to continuous valued control actions. In this paper a three step solution approach for finite valued control actions is presented. In the first step, a person-by-person approach is used to identify redundant data or a sufficient statistic for local information at each control station. In the second step, the common-information based approach of Nayyar *et al.* (2013) is used to find a sufficient statistic for the common information shared between all control stations and to obtain a dynamic programming decomposition. In the third step, the specifics of the model are used to simplify the sufficient statistic and the dynamic program.**

***Index Terms*—Decentralized control, nonclassical information structures, stochastic optimal control.**

I. Introduction

Signaling, or the ability of one control station to communicate information about its observation to another control station, is a fundamental aspect of decentralized control. As shown in [1], the absence of signaling simplifies the structure of optimal decentralized control strategies. In this paper, we show that the reverse is also true: the presence of signaling may also simplify the structure of optimal decentralized control strategies.

To illustrate the above point, we investigate a model with explicit signaling: the *control sharing* information structure. In such an information structure, each control station observes the control actions of all other control stations after a one-step delay.

Control sharing information structures arise naturally in many communication applications such as multi-access broadcast [2], [3], paging and registration in mobile cellular systems [4], and real-time communication with feedback [5]. In these applications, each node may be treated as a controlled subsystem. These subsystems are coupled through dynamics and cost. In Section II we propose a model of coupled subsystems with control sharing that captures the different dynamical models used in the above [2]–[5].

The model considered in this paper has a nonclassical information structure. We refer the reader to [6] for an overview of the various solution approaches to decentralized control systems with nonclassical information structures. We briefly describe two approaches that are most relevant to our model.

For a linear quadratic Gaussian (LQG) model with control sharing information structure, the following solution methodology was proposed in [7], [8]. Embed the local observations in the control actions to convert the control sharing model to a one-step delay sharing model, and then solve the latter system. This embedding technique works because: i) control actions are real-valued random variables and, as such, convey infinite information (in an information theoretic sense) and ii) measurable bijections exist between Euclidean spaces. In the model of this paper, control actions take finite values. Therefore, the embedding technique of [7], [8] does not work.

Control sharing information structure is a special case of *partial history sharing* information structure, for which the following solution methodology was proposed in [9]. Split the data available at each control station into two parts: a *common information* part that is commonly known to all control stations, and a *local information* part that consists of the remaining data. Then, the decentralized stochastic control system is equivalent to a centralized stochastic control system in which a fictitious *coordinator* observes the common information and chooses functions that map the local information at each control station to its action. This solution approach extends to infinite horizons only when the equivalent centralized system is time-homogeneous. When the model of this paper is converted to a centralized system, the local information at a control station is the history of local state observations which is increasing with time and, hence, not time-homogeneous. Therefore, the common information approach of [9] is not directly applicable to the model of this paper.

The rest of this paper is organized as follows. In Section II, we present two models for coupled subsystems with control sharing: the full and the partial observation models. In Section III, we present the following three-step solution approach for the full observation model. In the first step, we use a person-by-person approach to identify redundant data or a sufficient statistic for local information at each control station. In the second step, we use the common-information approach [9] to find a sufficient statistic for the common information shared between all control stations and to obtain a dynamic programming decomposition. In the third step, we use the specifics of the model to simplify the sufficient statistic and the dynamic program. In Section IV, we extend this three-step approach to the partial observation model. In Section V we conclude by discussing some of the salient features of our solution approach.

Notation

Random variables are denoted with upper case letters (X, Y, etc.), their realization with lower case letters (x, y, etc.), and their space of realizations by script letters ($\mathcal{X}$, $\mathcal{Y}$, etc.). Subscripts denote time and superscripts denote the subsystem; e.g., X^i_t denotes the state of subsystem i at time t. The short hand notation $X^i_{1:t}$ denotes the vector $(X^i_1, X^i_2, \dots, X^i_t)$. Bold face letters denote the collection of variables at all subsystems; e.g., $\mathbf{X}_t$ denotes $(X^1_t, X^2_t, \dots X^n_t)$. The notation $\mathbf{X}^{-i}_t$ denotes the vector $(X^1_t, \dots, X^{i-1}_t, X^{i+1}_t, \dots, X^n_t)$.

$\Delta(\mathcal{X})$ denotes the probability simplex for the space $\mathcal{X}$. $\mathbb{P}(A)$ denotes the probability of an event A. $\mathbb{E}[X]$ denotes the expectation of a random variable X. $\mathbb{1}[x = y]$ denotes the indicator function of the statement $x = y$, i.e., $\mathbb{1}[x = y]$ is 1 if $x = y$ and 0 otherwise.

II. Coupled Subsystems With Control Sharing

1) System Components: Consider a discrete-time networked control system with n subsystems. The state (X^i_t, Z_t) of subsystem i, $i = 1, \dots, n$, has two components: a *local state* $X^i_t \in \mathcal{X}^i$ and a *shared state* $Z_t \in \mathcal{Z}$, which is identical for all subsystems. The initial shared state Z_1 has a distribution P_Z. Conditioned on the initial shared state Z_1, the initial local state of all subsystems are independent; initial local state X^i_1 is distributed according to $P_{X^i|Z}$, $i = 1, \dots, n$. Let $\mathbf{X}_t := (X^1_t, \dots, X^n_t)$ denote the local state of all subsystems.

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A control station is co-located with each subsystem. Let $U_t^i \in \mathcal{U}^i$ denote the control action of control station i and $\mathbf{U}_t := (U_t^1, U_t^2, \ldots, U_t^n)$ denote the collection of all control actions.

2) System Dynamics: The shared and the local state of each subsystem are coupled through the control actions; the shared state evolves according to

$$Z_{t+1} = f_t^0(Z_t, \mathbf{U}_t, W_t^0) \tag{1}$$

while the local state of subsystem i, $i = 1, \ldots, n$, evolves according to

$$X_{t+1}^i = f_t^i(X_t^i, Z_t, \mathbf{U}_t, W_t^i) \tag{2}$$

where $W_t^i \in \mathcal{W}^i$, $i = 0, 1, \ldots, n$, is the plant disturbance with distribution P_{W^i}. The processes $\{W_t^i\}_{t=1}^{\infty}$, $i = 0, 1, \ldots, n$, are assumed to be independent across time, independent of each other, and also independent of the initial state $(Z_1, \mathbf{X}_1)$ of the system.

Note that the updated local state of subsystem i depends only on the previous local state of subsystem i and the previous shared state but is controlled by all control stations.

3) Observation Models and Information Structures: We consider two observation models that differ in the observation of the local state X_t^i at control station i. In the first model, called *full observation model*, control station i perfectly observes the local state X_t^i; in the second model, called *partial observation model*, control station i observes a noisy version $Y_t^i \in \mathcal{Y}^i$ of the local state X_t^i given by

$$Y_t^i = \ell_t^i(X_t^i, \tilde{W}_t^i) \tag{3}$$

where $\tilde{W}_t^i \in \tilde{\mathcal{W}}_t^i$ is the observation noise with distribution $P_{\tilde{W}^i}$. The processes $\{\tilde{W}_t^i\}_{t=1}^{\infty}$, $i = 1, \ldots, n$ are assumed to be independent across time, independent of each other, independent of $\{W_t^i\}_{t=1}^{\infty}$, $i = 0, \ldots, n$, and independent of the initial state $(\mathbf{X}_1, Z_1)$.

In both models, in addition to the local measurement of the state of its subsystem, each control station perfectly observes the shared state Z_t and the one-step delayed control actions $\mathbf{U}_{t-1}$ of all control stations. The control stations perfectly recall all the data they observe. Thus, in the full observation model, control station i chooses a control action according to

$$U_t^i = g_t^i(X_{1:t}^i, Z_{1:t}, \mathbf{U}_{1:t-1}) \tag{4}$$

while in the partial observation model, it chooses a control action according to

$$U_t^i = g_t^i(Y_{1:t}^i, Z_{1:t}, \mathbf{U}_{1:t-1}). \tag{5}$$

The function g_t^i is called the *control law* of control station i. The collection $\mathbf{g}^i := (g_1^i, g_2^i, \ldots, g_T^i)$ of control laws at control station i is called the *control strategy of control station* i. The collection $\mathbf{g} := (\mathbf{g}^1, \mathbf{g}^2, \ldots, \mathbf{g}^n)$ of control strategies of all control stations is called the *control strategy of the system*.

4) Cost and Performance: At time t, the system incurs a cost $c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t)$ that depends on the shared state, the local state of all subsystems, and the actions of all control stations. Thus, the subsystems are also coupled through cost.

The system runs for a time horizon T. The performance of a control strategy $\mathbf{g}$ is measured by the expected total cost incurred by that strategy, which is given by

$$J(\mathbf{g}) := \mathbb{E}\left[\sum_{t=1}^{T} c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t)\right] \tag{6}$$

where the expectation is with respect to a joint measure of $(\mathbf{X}_{1:T}, Z_{1:T}, \mathbf{U}_{1:T})$ induced by the choice of the control strategy $\mathbf{g}$.

We are interested in the following optimal control problem.

5) Problem 1: Given the distributions $P_Z, P_{X^i|Z}, P_{W^i}, P_{\tilde{W}^i}$ of the initial shared state, initial local state, plant disturbance of subsystem i, and observation noise of subsystem i (for the partial observation model), $i = 1, \ldots, n$, a horizon T, and the cost functions c_t, $t = 1, \ldots, T$, find a control strategy $\mathbf{g}$ that minimizes the expected total cost given by (6).

The above model and optimization problem arise in a variety of communication applications such as multi-access broadcast [2], [3], paging and registration in mobile cellular systems [4], and real-time communication with feedback [5] (see [10] for details).

III. Main Result for the Full Observation Model

In this section, we derive the structure of optimal control laws and a dynamic programming decomposition for the full observation model using the following three-step approach:

1) Use a person-by-person approach to show that the past values of the local state $X_{1:t-1}^i$ are irrelevant at control station i at time t. Thus, for any control strategy of control station i that uses $(X_{1:t}^i, Z_{1:t}, \mathbf{U}_{1:t-1})$, we can choose a control strategy that uses only $(X_t^i, Z_{1:t}, \mathbf{U}_{1:t-1})$ without any loss in performance.
2) When attention is restricted to control strategies of the form derived in Step 1, the common information C_t is $(Z_{1:t}, \mathbf{U}_{1:t-1})$ and the local information L_t^i at control station i is X_t^i. Following the common information approach of [9] show that $\Pi_t = \mathbb{P}(\mathbf{X}_t \mid C_t, Z_t)$ is a sufficient statistic for the common information C_t. Use Π_t to identify the structure of optimal control laws and dynamic programming decomposition.
3) Define $\Theta_t^i = \mathbb{P}(X_t^i \mid C_t)$ and $\mathbf{\Theta}_t = (\Theta_t^1, \ldots, \Theta_t^n)$. Use the system dynamics to show that $(Z_t, \mathbf{\Theta}_t)$ is sufficient to compute Π_t. Based on this sufficiency, replace Π_t by $(Z_t, \mathbf{\Theta}_t)$ in the structural results and the dynamic programming decomposition of Step 2.

Now, we describe each of these steps in detail. For simplicity of exposition, we assume that $\mathcal{Z}$, $\mathcal{X}^i$, $\mathcal{U}^i$, and $\mathcal{W}^i$, $i = 1, \ldots, n$, are finite. The results extend to general alphabets under suitable technical conditions (similar to those for centralized stochastic control [11]).

A. Step 1: Shedding of Irrelevant Information

In the full observation model, the local states of all subsystems are conditionally independent given the history of shared state and control actions. In particular,

Proposition 1: For any realization $z_t \in \mathcal{Z}$, $x_t^i \in \mathcal{X}^i$ and $u_t^i \in \mathcal{U}^i$ of X_t^i and U_t^i, $i = 1, \ldots, n$, $t = 1, \ldots, T$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{X}_{1:t} = \mathbf{x}_{1:t} \mid Z_{1:t} = z_{1:t}, \mathbf{U}_{1:t-1} = \mathbf{u}_{1:t-1}) \\ = \prod_{i=1}^{n} \mathbb{P}(X_{1:t}^i = x_{1:t}^i \mid Z_{1:t} = z_{1:t}, \mathbf{U}_{1:t-1} = \mathbf{u}_{1:t-1}).\end{aligned} \tag{7}$$

See Appendix A for proof. An immediate consequence of the above Proposition is the following:

Lemma 2: Consider the full observation model for an arbitrary but fixed choice of control strategy $\mathbf{g}$. Define $R_t^i = (X_t^i, Z_{1:t}, \mathbf{U}_{1:t-1})$. Then:

1) The process $\{R_t^i\}_{t=1}^{T}$ is a controlled Markov process with control action U_t^i, i.e., for any $x_t^i, \tilde{x}_t^i \in \mathcal{X}^i$, $z_t, \tilde{z}_t \in \mathcal{Z}$, $u_t^i, \tilde{u}_t^i \in \mathcal{U}^i$, $r_t^i = (x_t^i, z_{1:t}, \mathbf{u}_{1:t-1})$, $\tilde{r}_t^i = (\tilde{x}_t^i, \tilde{z}_{1:t}, \tilde{\mathbf{u}}_{1:t-1})$, $i = 1, \ldots, n$, and $t = 1, \ldots, T$,

$$\begin{aligned}\mathbb{P}(R_{t+1}^i = \tilde{r}_{t+1}^i \mid R_{1:t}^i = r_{1:t}^i, U_{1:t}^i = u_{1:t}^i) \\ = \mathbb{P}(R_{t+1}^i = \tilde{r}_{t+1}^i \mid R_t^i = r_t^i, U_t^i = u_t^i).\end{aligned}$$

2) The instantaneous conditional cost simplifies as follows:

$$\mathbb{E}[c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t) \mid R^i_{1:t} = r^i_{1:t}, U^i_{1:t} = u^i_{1:t}] = \mathbb{E}[c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t) \mid R^i_t = r^i_t, U^i_t = u^i_t].$$

See Appendix D for proof.

In light of Lemma 2, lets reconsider the subproblem of finding the optimal control strategy for control station i when the control strategy $\mathbf{g}^{-i}$ of all other control stations is fixed arbitrarily. In this subproblem, control station i has access to $R^i_{1:t}$, chooses U^i_t, and incurs an expected instantaneous cost $\mathbb{E}[c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t) \mid R^i_{1:t}, U^i_{1:t}]$. Lemma 2 implies that the subproblem of finding the optimal control strategy $\mathbf{g}^i$ is a Markov decision process. Thus, using Markov decision theory [11], we get that restricting attention to control laws of the form (8) at control station i is without loss of optimality. By cyclically using the same argument for all control stations we obtain the following:

Proposition 3: In the full observation model, restricting attention to control laws of the form

$$U^i_t = \bar{g}^i_t(X^i_t, Z_{1:t}, \mathbf{U}_{1:t-1}) \tag{8}$$

at all control stations $i, i = 1, \ldots, n$, is without loss of optimality.

Thus, the past values of local state $X^i_{1:t-1}$ are irrelevant at control station i at time $t, i = 1, \ldots, n$. However, even after shedding $X^i_{1:t-1}$, the data at each control station is still increasing with time. In the next step, we show how to "compress" this data into a sufficient statistic.

B. Step 2: Sufficient Statistic for Common Data

Consider Problem 1 for the full observation model and restrict control strategies of the form (8). Proposition 3 shows that this restriction is without loss of optimality. We use the results of [9] for this restricted setup.

Split the data at each control station into two parts: the common data $C_t = (Z_{1:t}, \mathbf{U}_{1:t-1})$ that is observed by all control stations and the local (or private) data $L^i_t = X^i_t$ that is observed by only control station i. Note that the common information is increasing with time (i.e., $C_t \subset C_{t+1}$), while the local information L^i_t has a fixed size. Thus, the system has *partial history sharing* information structure with finite local memory. Nayyar *et al.* [9] derived structural properties of optimal controllers and a dynamic programming decomposition for such an information structure.

To present the result, we first define the following:

Definition 1: Given any control strategy $\bar{\mathbf{g}}$ of the form (8), let Π_t, $t = 1, \ldots, T$, denote the posterior probability of $(\mathbf{X}_t, Z_t)$ given the common information C_t; i.e., for any $z \in \mathcal{Z}$ and $x^i \in \mathcal{X}^i$, the component $(\mathbf{x}, z)$ of Π_t is given by

$$\Pi_t(\mathbf{x}, z) := \mathbb{P}^{\bar{\mathbf{g}}}(\mathbf{X}_t = \mathbf{x}, Z_t = z \mid C_t).$$

The update of Π_t follows the standard nonlinear filtering equation. It is shown in [9] that Π_t is a sufficient statistic for C_t; in particular, we have the following structural result.

Proposition 4: ([9, Theorem 2] applied to model of Proposition 3) In the full observation model, restricting attention to control laws of the form

$$U^i_t = \hat{g}^i_t(X^i_t, \Pi_t) \tag{9}$$

at all control stations $i, i = 1, \ldots, n$, is without loss of optimality.

To obtain a dynamic programming decomposition to find optimal control strategies of the form (9), the following *partially evaluated control laws* were defined in [9]: For any control strategy of the form (9), and any realization π_t of Π_t, let

$$\hat{d}^i_t(\cdot) = \hat{g}^i_t(\cdot, \pi_t)$$

denote a mapping from $\mathcal{X}^i_t$ to $\mathcal{U}^i_t$. When Π_t is a random variable, the above mapping is a random mapping denoted by $\hat{D}^i_t$. Let $\hat{\mathbf{d}}_t = (\hat{d}^1_t, \ldots, \hat{d}^n_t)$ and $\hat{\mathbf{D}}_t = (\hat{D}^1_t, \ldots, \hat{D}^n_t)$. Then optimal control strategies of the form (9) are obtained as follows.

Proposition 5: ([9, Theorem 3] applied to model of Proposition 3) For any $\pi_t \in \Delta(\mathcal{Z} \times \mathcal{X}^1 \times \cdots \times \mathcal{X}^n)$, define

$$V_T(\pi_T) = \min_{\hat{\mathbf{d}}_T} \mathbb{E}[c_T(\mathbf{X}_T, Z_T, \mathbf{U}_T) \mid \Pi_T = \pi_T, \hat{\mathbf{D}}_T = \hat{\mathbf{d}}_T]$$

and for $t = T-1, T-2, \ldots, 1$,

$$V_t(\pi_t) = \min_{\hat{\mathbf{d}}_t} \mathbb{E}[c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t) + V_{t+1}(\Pi_{t+1}) \mid \Pi_t = \pi_t, \hat{\mathbf{D}}_t = \hat{\mathbf{d}}_t].$$

Let $\hat{\Psi}_t(\pi_t)$ denote the set of $\arg\min$ of the right hand side of $V_t(\pi_t)$, and $\hat{\Psi}^i_t$ denote the i-th component of $\hat{\Psi}_t$. Then, a control strategy

$$\hat{g}^i_t(x^i_t, \pi_t) \in \hat{\Psi}^i_t(\pi_t)(x^i_t)$$

is optimal for Problem 1 with the full observation model.

C. Step 3: Simplification of the Sufficient Statistic

In this step, we use Proposition 1 to simplify the sufficient statistic Π_t used in Step 2, and thereby simplify Propositions 4 and 5. For that matter, we define the following.

Definition 2: Given any control strategy $\hat{\mathbf{g}}$ of the form (9), let Θ^i_t, $t = 1, \ldots, T$, denote the posterior probability of X^i_t given the common information $(Z_{1:t}, \mathbf{U}_{1:t-1})$, i.e., for any $x^i \in \mathcal{X}^i$, the component x^i of Θ^i_t is given by

$$\Theta^i_t(x^i) := \mathbb{P}^{\hat{\mathbf{g}}}(X^i_t = x^i \mid Z_{1:t}, \mathbf{U}_{1:t-1}).$$

We now show that $(Z_t, \boldsymbol{\Theta}_t)$ is a sufficient to compute Π_t. More precisely:

Lemma 6: For any $z \in \mathcal{Z}$, $x^i \in \mathcal{X}^i$, $i = 1, \ldots, n$, the values $(z, \boldsymbol{\Theta}_t(\mathbf{x}))$ are sufficient to compute $\Pi_t(\mathbf{x}, z)$.

Proof: The proof follows directly from the definition of Π_t, Θ^i_t and Proposition 1. Let $C_t = (Z_{1:t}, \mathbf{U}_{1:t-1})$ and consider the component $(\mathbf{x}, z)$ of Π_t:

$$\Pi_t(\mathbf{x}, z) \overset{(a)}{=} \mathbb{1}[Z_t = z] \cdot \mathbb{P}(\mathbf{X}_t = \mathbf{x} \mid Z_{1:t}, \mathbf{U}_{1:t-1}) \overset{(b)}{=} \mathbb{1}[Z_t = z] \cdot \prod_{i=1}^{n} \Theta^i_t(x^i)$$

where (a) follows form the law of total probability and (b) follows from Proposition 1. ■

Therefore, we can substitute $(Z_t, \boldsymbol{\Theta}_t)$ for Π_t in Proposition 4 to get the following:

Theorem 1 (Structure of Optimal Controllers): In the full observation model, restricting attention to control laws of the form

$$U^i_t = \tilde{g}^i_t(X^i_t, Z_t, \boldsymbol{\Theta}_t) \tag{10}$$

at all control stations $i, i = 1, \ldots, n$, is without loss of optimality.

To obtain a similar simplification for the dynamic program of Proposition 5, we need to show that $(Z_t, \boldsymbol{\Theta}_t)$ updates in a state-like manner (i.e., it is an information state). That is established by the following Lemma.

Lemma 7: There exists a deterministic function F_t such that

$$\boldsymbol{\Theta}_{t+1} = F_t(\boldsymbol{\Theta}_t, Z_{t+1}, Z_t, \mathbf{U}_t, \hat{\mathbf{D}}_t). \tag{11}$$

Furthermore,

$$\mathbb{P}(Z_{t+1} \mid Z_{1:t}, \mathbf{\Theta}_{1:t}, \hat{\mathbf{D}}_{1:t}) = \mathbb{P}(Z_{t+1} \mid Z_t, \mathbf{\Theta}_t, \hat{\mathbf{D}}_t). \qquad (12)$$

See Appendix B for proof.

To simplify the dynamic program of Proposition 5, proceed as follows. For any control strategy of the form (10), and any realization $\boldsymbol{\theta}_t$ of $\mathbf{\Theta}_t$, let

$$\tilde{d}_t^i(\cdot) = \tilde{g}_t^i(\cdot, z_t, \boldsymbol{\theta}_t)$$

denote a mapping from $\mathcal{X}_t^i$ to $\mathcal{U}_t^i$. When $\mathbf{\Theta}_t$ is a random variable, the above mapping is a random mapping denoted by $\tilde{D}_t^i$. Let $\tilde{\mathbf{d}}_t = (\tilde{d}_t^1, \dots, \tilde{d}_t^n)$ and $\tilde{\mathbf{D}}_t = (\tilde{D}_t^1, \dots, \tilde{D}_t^n)$. Then optimal control strategies of the form (9) are obtained as follows.

Theorem 2 (Dynamic Programming Decomposition): For any $z_t \in \mathcal{Z}$ and $\theta_t^i \in \Delta(\mathcal{X}^i)$, $i = 1, \dots, n$, define

$$V_T(z_T, \boldsymbol{\theta}_T) = \min_{\tilde{\mathbf{d}}_T} \mathbb{E}[c_T(\mathbf{X}_T, Z_T, \mathbf{U}_T) \mid Z_T = z_T, \mathbf{\Theta}_T = \boldsymbol{\theta}_T, \tilde{\mathbf{D}}_T = \tilde{\mathbf{d}}_T]$$

and for $t = T-1, T-2, \dots, 1$,

$$V_t(z_t, \boldsymbol{\theta}_t) = \min_{\tilde{\mathbf{d}}_t} \mathbb{E}[c_t(\mathbf{X}_t, Z_t, \mathbf{U}_t) + V_{t+1}(\Pi_{t+1}) \mid Z_t = z_t, \mathbf{\Theta}_t = \boldsymbol{\theta}_t, \tilde{\mathbf{D}}_t = \tilde{\mathbf{d}}_t].$$

Let $\tilde{\Psi}_t(z_t, \boldsymbol{\theta}_t)$ denote the set of $\arg\min$ of the right hand side of $V_t(z_t, \boldsymbol{\theta}_t)$, and $\tilde{\Psi}_t^i$ denote the i-th component of $\tilde{\Psi}_t$. Then, a control strategy

$$\tilde{g}_t^i(x_t^i, z_t, \boldsymbol{\theta}_t) \in \tilde{\Psi}_t^i(z_t, \boldsymbol{\theta}_t)(x_t^i)$$

is optimal for Problem 1 with the full observation model.

IV. Main Result for the Partial Observation Model

In this section, we derive the structure of optimal control laws and a dynamic programming decomposition for the partial observation model. As in the full observation model, we cannot directly use the results of [9] because the local observations $Y_{1:t}^i$ at each control station are increasing with time. To circumvent this difficulty, we follow a three step approach, similar to the one taken for the full observation model.

A. Step 1: Sufficient Statistic for Local Observations

In this step, we find a sufficient statistic for the local observations $Y_{1:t}^i$ at control station i. For that matter, we define the following:

Definition 3: Given any control strategy $\mathbf{g}$ of the form (5), let Ξ_t^i, $i = 1, \dots, n, t = 1, \dots, T$ denote the posterior probability of the local state X_t^i of substation i given all the information $(Y_{1:t}^i, Z_{1:t}, \mathbf{U}_{1:t-1})$ at control station i, i.e., for any $x^i \in \mathcal{X}^i$, the component x^i of Ξ_t^i is given by

$$\Xi_t^i(x^i) := \mathbb{P}^{\mathbf{g}}(X_t^i = x^i \mid Y_{1:t}^i, Z_{1:t}, \mathbf{U}_{1:t-1}) \stackrel{(a)}{=} \mathbb{P}^{\mathbf{g}}(X_t^i = x^i \mid Y_{1:t}^i, Z_{1:t-1}, \mathbf{U}_{1:t-1})$$

where (a) follows because $\{W_t^0\}_{t=1}^T$ is independent of $\{W_t^i\}_{t=1}^T$.

The update of Ξ_t^i follows a nonlinear filtering equation as shown below.

Lemma 8: For every i, $i = 1, \dots, T$, there exist a deterministic function $\tilde{F}_t^i$ such that

$$\Xi_{t+1}^i = \tilde{F}_t^i(\Xi_t^i, Y_{t+1}^i, Z_t, \mathbf{U}_t). \qquad (13)$$

The proof follows from the law of total probability and Bayes rule and is similar to the proof of Appendix B.

We want to establish that Ξ_t^i is a sufficient statistic for the local observations $Y_{1:t}^i$ at control station i. For that matter, we need the following two conditional independence properties.

Proposition 9: Proposition 1 is also true for the partial observation model for an arbitrary but fixed choice of control strategy $\mathbf{g}$ of the form (5).

The proof is along the same lines as the proof of Proposition 1. See [10] for details.

Proposition 10: In the partial observation model, the posterior probability Ξ_t^i of the local states of all subsystems are conditionally independent given the history of shared state and control actions. Specifically, for any Borel subsets E_t^i of $\Delta(\mathcal{X}^i)$, $\mathbf{E}_t = (E_t^1, \dots, E_t^n)$, $u_t^i \in \mathcal{U}^i$, $z_t \in \mathcal{Z}$, $i = 1, \dots, n$ and $t = 1, \dots, T$, we have

$$\mathbb{P}(\mathbf{\Xi}_{1:t} \in \mathbf{E}_{1:t} \mid Z_{1:t} = z_{1:t}, \mathbf{U}_{1:t-1} = \mathbf{u}_{1:t-1}) = \prod_{i=1}^{n} \mathbb{P}(\Xi_{1:t}^i \in E_{1:t}^i \mid Z_{1:t} = z_{1:t}, \mathbf{U}_{1:t-1} = \mathbf{u}_{1:t-1}). \qquad (14)$$

See Appendix C for proof.

An immediate consequence of Proposition 10 and Lemma 8 is the following.

Lemma 11: Lemma 2 is also true for the partial observation model with R_t^i defined as $(\Xi_t^i, Z_{1:t}, \mathbf{U}_{1:t-1})$.

The proof is omitted due to space limitation. See [10] for details.

By repeating an argument similar to the argument after Lemma 2, we get the following:

Proposition 12: In the partial observation model, restricting attention to control laws of the form

$$U_t^i = \bar{g}_t^i(\Xi_t^i, Z_{1:t}, \mathbf{U}_{1:t-1}) \qquad (15)$$

at all control stations i, $i = 1, \dots, n$, is without loss of optimality.

B. Steps 2 and 3: Sufficient Statistic for Common Data and Its Simplification

Compare Proposition 3 of the full observation model with Proposition 12 of the partial observation model. The posterior probability Ξ_t^i in the latter model plays the role of local state X_t^i in the former model. This suggests that we may follow Steps 2 and 3 of the full observation model in the partial observation model by replacing X_t^i by Ξ_t^i. Following this suggestion, define:

Definition 4: Let $\mathring{\Pi}_t$ denote the posterior probability on $(Z_t, \mathbf{\Xi}_t)$ given the common information C_t, i.e., for any $z \in \mathcal{Z}$ and any Borel subsets E^i of $\Delta(\mathcal{X}^i)$ and $\mathbf{E} = (E^1, \dots, E^n)$,

$$\mathring{\Pi}_t(\mathbf{E}, z) = \mathbb{P}(\mathbf{\Xi}_t \in \mathbf{E}, Z_t = z \mid C_t). \qquad (16)$$

Definition 5: Let $\mathring{\Theta}_t^i$, $t = 1, \dots, T$, denote the posterior probability of Ξ_t^i given the common information $(Z_{1:t}, \mathbf{U}_{1:t-1})$, i.e., for any Borel subset E^i of $\Delta(\mathcal{X}^i)$,

$$\mathring{\Theta}_t^i(E^i) := \mathbb{P}(\Xi_t^i \in E^i \mid Z_{1:t}, \mathbf{U}_{1:t-1}).$$

Now, by following the exact same argument as in Steps 2 and 3 for the full observation model, we get that Propositions 4 and 5 and Theorems 1 and 2 are also true for the partial observation model if we replace Π_t and Θ_t^i by $\mathring{\Pi}_t$ and $\mathring{\Theta}_t^i$, respectively.

V. Discussion and Conclusion

A. Infinite Horizon Setup

Although we restricted attention to a finite horizon setup, our results also extend to the infinite horizon setup as follows. Step 1 of our approach remains valid for the infinite horizon setup as well. In Step 2, the local information takes value in a time-invariant space. Therefore, the result of Proposition 5 generalizes to infinite horizon setup along the lines of [9, Theorem 5]. The simplification of Step 3, which relies on Lemma 6, proceeds as in the finite horizon setup.

B. Salient Features of the Result

It is generally believed that signaling makes decentralized control problems harder. The results of this paper show that when signaling induces common information between control stations, it may simplify a decentralized control problem. The reason for this simplification is two-fold.

Firstly, common information allows us to use the solution framework of [9]. Secondly, common information may induce appropriate conditional independence which, in turn, may simplify the structure of optimal control laws (Step 1) and the information state (Step 3). For example, in our model, $X^i_{1:t}$ is not conditionally independent of $X^j_{1:t}$ given $Z_{1:t}$, but is conditionally independent when also conditioned on the signaled information $\mathbf{U}_{1:t-1}$.

Whether or not the signaled common information leads to an appropriate conditional independence hinges on the system dynamics. For example, the above conditional independence between $X^i_{1:t}$ and $X^j_{1:t}$ given $(Z_{1:t}, \mathbf{U}_{1:t-1})$ would break if the system dynamics were

$$X^i_{t+1} = f^i_t(\mathbf{X}_t, Z_t, \mathbf{U}_t, W^i_t).$$

The above conditional independence is critical for Step 1 of our approach. If it were not true, the local information in Step 2 would not be time-invariant, and our result would not extend to infinite horizon setup (see Section V-A).

The above conditional independence is also critical for Step 3 of our approach. It allows us to use $(Z_t, \mathbf{\Theta}_t)$ instead of Π_t as the information state in the dynamic program. $\Pi_t \in \Delta(\mathcal{Z}_t \times \mathcal{X}^1 \times \cdots \times \mathcal{X}^n)$, so its size is doubly exponential in the number of subsystems. On the other hand $\mathbf{\Theta}_t \in \Delta(\mathcal{X}^1) \times \cdots \times \Delta(\mathcal{X}^n)$, so its size is exponential in the number of subsystems. Consequently, Step 3 reduces the size of the information state by an exponential factor.

In our model, the induced common information is equal to the signaled information. In general, this need not be the case. A natural next step is to investigate the relationship between signaling and common information when the signaling is implicit through the system dynamics.

Appendix A
Proof of Proposition 1

For simplicity of notation, we use $\mathbb{P}(\mathbf{x}_{1:t}, z_{1:t}, \mathbf{u}_{1:t-1})$ to denote $\mathbb{P}(\mathbf{X}_{1:t} = \mathbf{x}_{1:t}, Z_{1:t} = z_{1:t}, \mathbf{U}_{1:t-1} = \mathbf{u}_{1:t-1})$ and a similar notation for conditional probability. Define:

- $\alpha^i_t := \mathbb{P}(u^i_t \mid z_{1:t}, x^i_{1:t}, \mathbf{u}_{1:t-1})$, $\beta^i_t := \mathbb{P}(x^i_t \mid z_{t-1}, x^i_{t-1}, \mathbf{u}_{t-1})$, $\gamma^i_t := \mathbb{P}(z_t \mid z_{t-1}, \mathbf{u}_{t-1})$; and
- $A^i_t := \prod_{s=1}^t \alpha^i_s$, $B^i_t := \prod_{s=1}^t \beta^i_s$, $\Gamma_t := \prod_{s=1}^t \gamma_s$.

From the law of total probability it follows that $\mathbb{P}(\mathbf{x}_{1:t}, z_{1:t}, \mathbf{u}_{1:t-1}) = \left(\prod_{i=1}^n A^i_{t-1} B^i_t\right)\Gamma_t$. Summing over all realizations of $\mathbf{x}_{1:t}$ and observing that A^i_{t-1} and B^i_t depend only on $(x^i_{1:t}, z_{1:t}, \mathbf{u}_{1:t-1})$, we get

$$\begin{aligned}\mathbb{P}(z_{1:t}, \mathbf{u}_{1:t-1}) &= \sum_{x^1_{1:t}} \sum_{x^2_{1:t}} \cdots \sum_{x^n_{1:t}} \left(\prod_{i=1}^n A^i_{t-1} B^i_t\right) \Gamma_t \\ &= \left(\prod_{i=1}^n \left(\sum_{x^i_{1:t}} A^i_{t-1} B^i_t\right)\right) \Gamma_t.\end{aligned}$$

Thus, using Bayes rule we get

$$\mathbb{P}(\mathbf{x}_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) = \prod_{i=1}^n \frac{A^i_{t-1} B^i_t}{\sum\limits_{x^i_{1:t}} A^i_{t-1} B^i_t}. \tag{17}$$

Summing both sides over $x^i_{1:t}$, $i \neq j$, we get

$$\mathbb{P}(x^j_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) = \frac{A^j_{t-1} B^j_t}{\sum\limits_{x^j_{1:t}} A^j_{t-1} B^j_t}. \tag{18}$$

The result follows from combining (17) and (18).

Appendix B
Proof of Lemma 7

Consider the system for a particular realization $(\mathbf{x}_{1:T}, z_{1:T}, \mathbf{u}_{1:T}, \hat{\mathbf{d}}_{1:T})$ of $(\mathbf{X}_{1:T}, Z_{1:T}, \mathbf{U}_{1:T}, \hat{\mathbf{D}}_{1:T})$. For ease of notation, we use $\mathbb{P}(x^i_{t+1} \mid z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t})$ to denote $\mathbb{P}(X^i_{t+1} = x^i_{t+1} \mid Z_{1:t+1} = z_{1:t+1}, \mathbf{U}_{1:t} = \mathbf{u}_{1:t}, \hat{\mathbf{D}}_{1:t} = \hat{\mathbf{d}}_{1:t})$. Define

$$\begin{aligned}&A(x^i_{t+1}, \mathbf{x}_t, z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t}) \\ &\quad := \mathbb{P}(x^i_{t+1}, \mathbf{x}_t, z_{t+1}, \mathbf{u}_t \mid z_{1:t}, \mathbf{u}_{1:t-1}, \mathbf{d}_{1:t});\end{aligned}$$

and

$$\begin{aligned}&B(x^i_{t+1}, \mathbf{x}_t, z_{t+1}, z_t, \mathbf{u}_t, \boldsymbol{\theta}_t) \\ &\quad := \mathbb{P}(x^i_{t+1} \mid x^i_t, z_t, \mathbf{u}_t) \cdot \mathbb{P}(z_{t+1} \mid \mathbf{x}_t, z_t, \mathbf{u}_t) \cdot \prod_{i=1}^n \theta^i_t(x^i_t).\end{aligned}$$

The system dynamics and Proposition 1 implies that

$$\begin{aligned}&A(x^i_{t+1}, \mathbf{x}_t, z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t}) \\ &\quad = B(x^i_{t+1}, \mathbf{x}_t, z_{t+1}, z_t, \mathbf{u}_t, \boldsymbol{\theta}_t) \mathbb{1}[\mathbf{u}_t = \hat{\mathbf{d}}_t(\mathbf{x}_t)].\end{aligned} \tag{19}$$

Consider component-i of the realization $\boldsymbol{\theta}_{t+1}$ of $\mathbf{\Theta}_{t+1}$

$$\begin{aligned}\theta^i_{t+1}(x^i_{t+1}) &= \mathbb{P}(x^i_{t+1} \mid z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t}) \\ &= \sum_{\{\mathbf{x}_t : \hat{\mathbf{d}}_t(\mathbf{x}_t) = \mathbf{u}_t\}} \frac{A(x^i_{t+1}, \mathbf{x}_t, z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t})}{\sum\limits_{\tilde{x}^i_{t+1}} A(\tilde{x}^i_{t+1}, \mathbf{x}_t, z_{1:t+1}, \mathbf{u}_{1:t}, \hat{\mathbf{d}}_{1:t})} \\ &\overset{(a)}{=} \sum_{\{\mathbf{x}_t : \hat{\mathbf{d}}_t(\mathbf{x}_t) = \mathbf{u}_t\}} \frac{B(x^i_{t+1}, \mathbf{x}_t, z_{t+1}, z_t, \mathbf{u}_t, \boldsymbol{\theta}_t)}{\sum\limits_{\tilde{x}^i_{t+1}} B(\tilde{x}^i_{t+1}, \mathbf{x}_t, z_{t+1}, z_t, \mathbf{u}_t, \boldsymbol{\theta}_t)} \\ &=: F^i_t(\boldsymbol{\theta}_t, z_{t+1}, z_t, \mathbf{u}_t, \hat{\mathbf{d}}_t)(x^i_{t+1})\end{aligned} \tag{20}$$

where (a) follows from (19). Combining (20) for all i, $i = 1, \ldots, n$, proves (11).

Now to prove (12), consider

$$\begin{aligned}&\mathbb{P}(z_{t+1} \mid z_{1:t}, \mathbf{\Theta}_{1:t}, \hat{\mathbf{d}}_{1:t}) \\ &\quad = \sum_{\mathbf{x}_t, \mathbf{u}_t} \mathbb{P}(z_{t+1} \mid z_t, \mathbf{u}_t) \cdot \prod_{i=1}^n \left[\mathbb{1}[\hat{d}^i_t(x^i_t) = u^i_t] \Theta^i_t(x^i_t)\right] \\ &\quad = \mathbb{P}(z_{t+1} \mid z_t, \mathbf{\Theta}_t, \hat{\mathbf{d}}_t).\end{aligned} \tag{21}$$

Appendix C
Proof of Proposition 10

Consider

$$\mathbb{P}(\mathbf{\Xi}_{1:t} \in \mathbf{E}_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) = \int_{\mathbf{E}_{1:t}} d\mathbb{P}(\boldsymbol{\xi}_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}).$$

From Proposition 9 and the law of total probability, we get

$$\begin{aligned} &d\mathbb{P}(\boldsymbol{\xi}_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) \\ &= \sum_{\mathbf{x}_{1:t}, \mathbf{y}_{1:t}} \left(\prod_{i=1}^{n} d\mathbb{P}(\xi^i_t \mid y^i_{1:t}, z_{1:t}, \mathbf{u}_{1:t-1}) \cdot \mathbb{P}(y^i_{1:t} \mid x^i_{1:t}) \cdot \mathbb{P}(x^i_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) \right) \\ &= \prod_{i=1}^{n} \left(\sum_{x^i_{1:t}, y^i_{1:t}} d\mathbb{P}(\xi^i_t \mid y^i_{1:t}, z_{1:t}, \mathbf{u}_{1:t-1}) \cdot \mathbb{P}(y^i_{1:t} \mid x^i_{1:t}) \cdot \mathbb{P}(x^i_{1:t} \mid z_{1:t}, \mathbf{u}_{1:t-1}) \right) \end{aligned}$$

which completes the proof of the Proposition.

Appendix D
Proof of Lemma 2

For ease of notation, we use $\mathbb{P}(\tilde{r}^i_{t+1} \mid r^i_{1:t}, u^i_{1:t})$ to denote $\mathbb{P}(R^i_{t+1} = \tilde{r}^i_{t+1} \mid R^i_{1:t} = r^i_{1:t}, U^i_{1:t} = u^i_{1:t})$ and a similar notation for other probability statements.

Consider

$$\begin{aligned} \mathbb{P}(\tilde{r}^i_{t+1} \mid r^i_{1:t}, u^i_{1:t}) &= \mathbb{P}(\tilde{x}^i_{t+1} \mid x^i_t, \tilde{z}_t, \tilde{\boldsymbol{u}}_t) \cdot \mathbb{P}(\tilde{z}_{t+1} \mid \tilde{z}_t, \tilde{\boldsymbol{u}}_t) \\ &\cdot \mathbb{1}[\tilde{\boldsymbol{u}}_{1:t-1} = \boldsymbol{u}_{1:t-1}] \cdot \mathbb{1}[\tilde{u}^i_t = u^i_t] \cdot \mathbb{1}[\tilde{z}_{1:t} = z_{1:t}] \\ &\cdot \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid x^i_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}, u^i_t) \end{aligned} \quad (22)$$

Simplify the last term of (22) as follows:

$$\begin{aligned} \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid x^i_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}, u^i_t) &\overset{(a)}{=} \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid x^i_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &= \sum_{\boldsymbol{x}^{-i}_{1:t}} \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid \boldsymbol{x}^{-i}_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}) \cdot \mathbb{P}(\boldsymbol{x}^{-i}_{1:t} \mid x^i_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &\overset{(b)}{=} \sum_{\boldsymbol{x}^{-i}_{1:t}} \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid \boldsymbol{x}^{-i}_{1:t}, z_{1:t}, \boldsymbol{u}_{1:t-1}) \cdot \mathbb{P}(\boldsymbol{x}^{-i}_{1:t} \mid z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &= \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid z_{1:t}, \boldsymbol{u}_{1:t-1}) \end{aligned} \quad (23)$$

where (a) is true because u^i_t is determined by $x^i_{1:t}$ $z_{1:t}$ and $\boldsymbol{u}_{1:t-1}$ and (b) follows from Proposition 1. Substituting (23) in (22), we get

$$\begin{aligned} \mathbb{P}(\tilde{r}^i_{t+1} \mid r^i_{1:t}, u^i_{1:t}) &= \mathbb{P}(\tilde{x}^i_{t+1} \mid x^i_t, \tilde{z}_t, \tilde{\boldsymbol{u}}_t) \cdot \mathbb{P}(\tilde{z}_{t+1} \mid \tilde{z}_t, \tilde{\boldsymbol{u}}_t) \\ &\cdot \mathbb{1}[\tilde{\boldsymbol{u}}_{1:t-1} = \boldsymbol{u}_{1:t-1}] \cdot \mathbb{1}[\tilde{z}_{1:t} = z_{1:t}] \cdot \mathbb{1}[\tilde{u}^i_t = u^i_t] \\ &\cdot \mathbb{P}(\tilde{\boldsymbol{u}}^{-i}_t \mid z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &= \mathbb{P}(\tilde{x}^i_{t+1}, \tilde{z}_{1:t+1}, \tilde{\boldsymbol{u}}_{1:t} \mid x^i_t, u^i_t, z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &= \mathbb{P}(\tilde{r}^i_{t+1} \mid r^i_t, u^i_t). \end{aligned} \quad (24)$$

This completes the proof of part 1) of the Lemma.

To prove part 2), it is sufficient to show that $\mathbb{P}(\tilde{z}_t, \tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{u}}_t \mid r^i_{1:t}, u^i_{1:t}) = \mathbb{P}(\tilde{z}_t, \tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{u}}_t \mid r^i_t, u^i_t)$. Consider

$$\begin{aligned} \mathbb{P}(\tilde{z}_t, \tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{u}}_t \mid r^i_{1:t}, u^i_{1:t}) &= \mathbb{1}[(\tilde{z}_t, \tilde{x}^i_t, \tilde{u}^i_t) = (z_t, x^i_t, u^i_t)] \\ &\cdot \mathbb{P}(\tilde{\boldsymbol{x}}^{-i}_t, \tilde{\boldsymbol{u}}^{-i}_t \mid x^i_{1:t}, u^i_t, z_t, \boldsymbol{u}_{1:t-1}) \\ &\overset{(c)}{=} \mathbb{1}[(\tilde{z}_t, \tilde{x}^i_t, \tilde{u}^i_t) = (z_t, x^i_t, u^i_t)] \cdot \mathbb{P}(\tilde{\boldsymbol{x}}^{-i}_t, \tilde{\boldsymbol{u}}^{-i}_t \mid z_{1:t}, \boldsymbol{u}_{1:t-1}) \\ &= \mathbb{P}(\tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{u}}_t \mid r^i_t, u^i_t) \end{aligned} \quad (25)$$

where (c) follows from an argument similar to (23)[1]. This completes the proof of part 2) of the Lemma.

[1]Recall that $\boldsymbol{x}^{-i}_t$ denotes the vector $(x^1_t, \ldots, x^{i-1}_t, x^{i+1}_t, \ldots, x^n_t)$.

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