# Optimal Decentralized Control of Two Agent Linear Systems with Partial Output Feedback


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## Linear-Quadratic Centralized System

**Certainty equivalence:**

$$u(t) = -L(t)\,\mathbb{E}[x(t) | y(1:t)]$$ even for non-Gaussian noise

**Separation of estimation and control:**

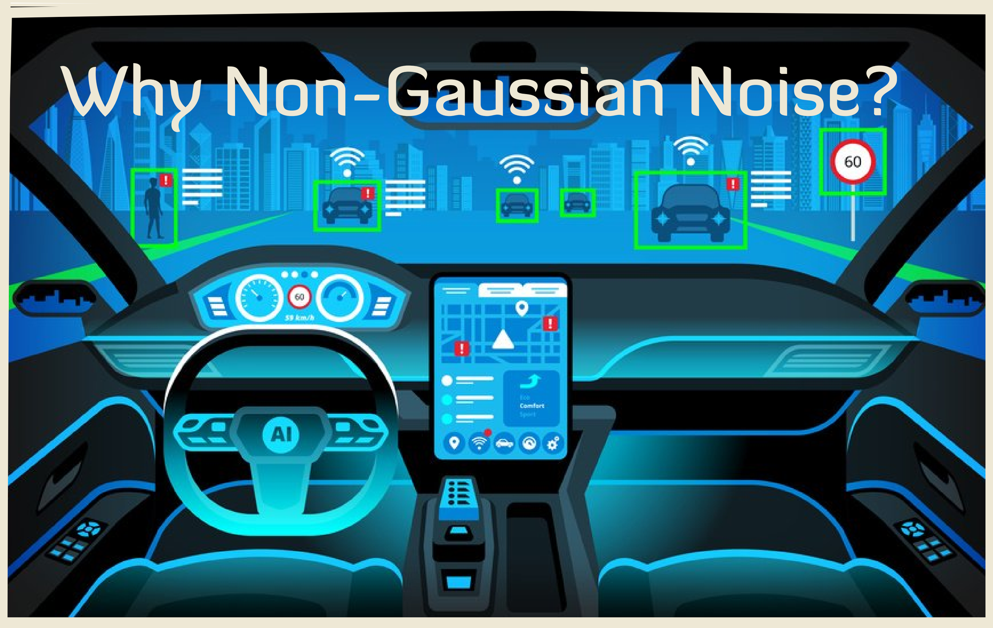
The gains are: $L(t) = [R + B^T S(t+1) B]^{-1} B^T S(t+1) A$

where $S(1:T) = \text{Riccati}(A, B, Q, R)$.

The estimates are: $\mathbb{E}[x(t) | y(1:t)] = x^c(t) + \mathbb{E}[x^s(t) | y^s(1:t)]$

For Gaussian noise, the estimate is a linear function of the data. Hence, optimal control law is linear.

## Why Non-Gaussian Noise?



## Linear-Quadratic Decentralized System

Even for Gaussian noise, linear control laws are not optimal.

**Partially nested LQG teams:** linear control laws are optimal. How to find sufficient statistics?

**Restrict to linear control laws:** How to find sufficient statistics?

What about separation principle?

What about certainty equivalence?

## Information Structure

$$I^1(t) = \{x_1(1:t), u_1(1:t-1)\}$$

$$I^2(t) = \{x_1(1:t), u_1(1:t-1), y_2(1:t), u_2(1:t-1)\}$$

## Common-Information based decomposition

Common Information:

$$I^c(t) = I^1(t) \cap I^2(t) = I^1(t)$$

Local Information:

$$I^{\ell,1}(t) = \emptyset$$

$$I^{\ell,2}(t) = \{y_2(1:t), u_2(1:t-1)\}$$

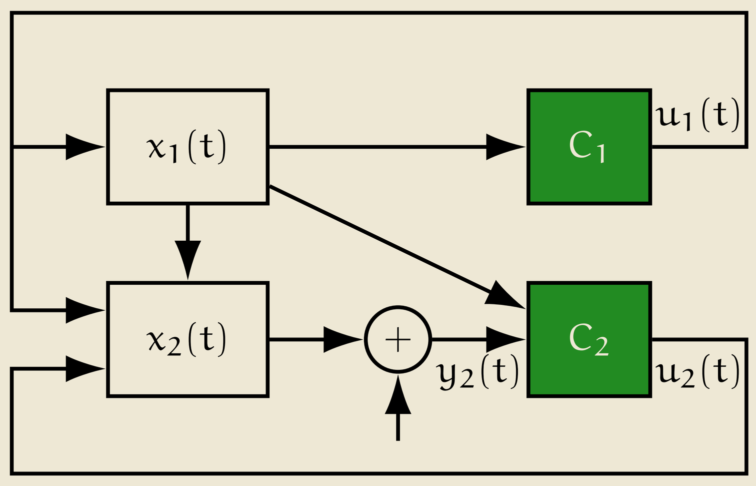
## Control splitting

Common Ctrl: $u^c(t) = \mathbb{E}[u(t) | I^c(t)]$

Local Ctrl: $u^\ell(t) = u(t) - u^c(t)$

## Static Reduction

$$I^{1,s}(t) = \{x_1^s(1:t)\}, \quad I^{2,s}(t) = \{y_2^s(1:t)\}$$




$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ v_2(t) \end{bmatrix}$$

Noise need not be Gaussian

$$J = \mathbb{E}\left[\sum_{t=1}^{T-1} \left[\|x(t)\|_Q^2 + \|u(t)\|_R^2\right] + \|x(T)\|_Q^2\right]$$

**Two-agent system with partial output feedback**

## Certainty Equivalent Controllers are Optimal

$$u^c(t) = -L^c(t)\,\mathbb{E}[x(t) | I^c(t)],$$

$$u_2^\ell(t) = -L^\ell(t)\,(\mathbb{E}[x_2(t) | I^2(t)] - \mathbb{E}[x_2(t) | I^c(t)])$$

## Separation of Estimation and Control

The gains are:

$$L^c(t) = [R + B^T S^c(t+1) B]^{-1} B^T S^c(t+1) A,$$

$$L^\ell(t) = [R_{22} + B_{22}^T S^\ell(t+1) B_{22}]^{-1} B_{22}^T S^\ell(t+1) A_{22}$$

where $S^c(1:T) = \text{Riccati}(A, B, Q, R)$

$S^\ell(1:T) = \text{Riccati}(A_{22}, B_{22}, Q_{22}, R_{22})$

The estimators are:

$$\mathbb{E}[x(t) | I^c(t)] = x^c(t) + \mathbb{E}[x^s(t) | I^{1,s}(t)]$$

$$\mathbb{E}[x(t) | I^2(t)] = x^\ell(t) + x^c(t) + \mathbb{E}[x^s(t) | I^{2,s}(t)]$$

## Key steps of the proof

### State Splitting

$$x(t) = x^c(t) + x^\ell(t) + x^s(t)$$

| | | |
|---|---|---|
| Commonly controlled part | $x^c(1) = 0,$ | $x^c(t+1) = Ax^c(t) + Bu^c(t)$ |
| Locally controlled part | $x^\ell(1) = 0,$ | $x^\ell(t+1) = Ax^\ell(t) + Bu^\ell(t)$ |
| Stochastic part | $x^s(1) = x(1),$ | $x^s(t+1) = Ax^s(t) + w(t)$ |

### Cost splitting

$$z^c(t) = x^c(t) + x^s(t)$$

$$z_2^\ell(t) = x_2^\ell(t) + x_2^s(t)$$

$$\mathbb{E}\left[\|u(t)\|_R^2\right] = \mathbb{E}\left[\|u^c(t)\|_R^2\right] + \mathbb{E}\left[\|u_2^\ell(t)\|_{R_{22}}^2\right]$$

$$\mathbb{E}\left[\|x(t)\|_Q^2\right] = \mathbb{E}\left[\|z^c(t)\|_Q^2\right] + \mathbb{E}\left[\|z_2^\ell(t)\|_{Q_{22}}^2\right] - \mathbb{E}\left[\|x^s(t)\|_Q^2\right]$$

### Completion of squares

$$J = \mathbb{E}\left[\|x(1)\|_{S^c(1)}^2 + \|x_2(t)\|_{S^\ell(1)}^2 + \sum_{t=1}^{T}\left[\|w(t)\|_{S^c(t+1)}^2 + \|w_2(t)\|_{S^\ell(t+1)}^2 - \|x_2^s(t)\|_{Q_{22}}^2\right]\right]$$

$$+ \mathbb{E}\left[\sum_{t=1}^{T} (A_{21} x_1^s(t))^T S^\ell(t+1)(A_{21} x_1^s(t) + 2A_{22} x_2^s(t))\right]$$

$$+ \mathbb{E}\left[\sum_{t=1}^{T} [\|u^c(t) + L^c(t) z^c(t)\|_{\Delta^c}^2 + \|u_2^\ell(t) + L^\ell(t) z_2^\ell(t)\|_{\Delta^\ell(t)}^2\right]$$

$$\Delta^c(t) = R + B^T S^c(t+1) B, \quad \Delta^\ell(t) = R_{22} + B_{22}^T S^\ell(t+1) B_{22}.$$

## Salient Features

The optimal control strategy is a linear function of the estimates even though the optimal estimates may not be a linear function of the data!

Proof technique combines ideas of linear systems (state splitting and completion of squares), estimation theory (orthogonality of estimate and the estimation error), and stochastic systems (static reduction).