QUESTION ABOUT MEASURABILITY OF OPTIMAL CONTROL STRATEGIES IN STATIC TEAMS

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Let

- $(\mathcal{X}_0, \mathfrak{F}_0)$, $(\mathcal{X}_1, \mathfrak{F}_1)$, $(\mathcal{X}_2, \mathfrak{F}_2)$, $(\mathcal{U}_1, \mathfrak{G}_1)$, $(\mathcal{U}_2, \mathfrak{G}_2)$, $(\mathcal{W}, \mathfrak{H})$ be Polish spaces.
- (X_0, X_1, X_2, W) be random variables defined on a common probability space $(\Omega, \mathfrak{F}, P)$, where $X_i \in \mathcal{X}_i$ and is $\mathfrak{F}/\mathfrak{F}_i$ measurable, for $i \in \{0, 1, 2\}$, and $W \in \mathcal{W}$ is $\mathfrak{F}/\mathfrak{H}$ measurable.
- For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be the class of measurable functions from $(\mathcal{X}_0 \times \mathcal{X}_i, \mathfrak{F}_0 \otimes \mathfrak{F}_i)$ to $(\mathcal{U}_i, \mathfrak{G}_i)$
- For $i \in \{1, 2\}$, let $\mathcal{D}_i$ be the class of measurable functions from $(\mathcal{X}_i, \mathfrak{F}_i)$ to $(\mathcal{U}_i, \mathfrak{G}_i)$.
- Let ℓ be a measurable function from $(\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{W}, \mathfrak{G}_1 \otimes \mathfrak{G}_2 \otimes \mathfrak{H})$ to $(\mathbb{R}, \mathcal{B})$.

Consider the following optimization problem:

$$\inf_{\substack{f_1 \in \mathcal{C}_1 \\ f_2 \in \mathcal{C}_2}} \mathbb{E}\big[\ell\big(f_1(X_0, X_1), f_2(X_0, X_2), W\big)\big] \tag{P1}$$

This problem can be simplified under the following assumption:

(A) The spaces $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{W}$, are finite sets.

Under (A), we can ignore measurability constraints. Define:

- Let $\mathcal{A}$ be the space of functions from $\mathcal{X}_0$ to $\mathcal{D}_1 \times \mathcal{D}_2$.

Consider the following family of optimization problems:

$$\forall x_0 \in \mathcal{X}_0, \quad \min_{\substack{g_1 \in \mathcal{D}_1 \\ g_2 \in \mathcal{D}_2}} \mathbb{E}\big[\ell\big(g_1(X_1), g_2(X_2), W\big) \,\big|\, X_0 = x_0\big] \tag{P2}$$

Let $h(x_0)$ denote the arg min (assuming there is a fixed rule for breaking ties). Note that the function $h \in \mathcal{A}$.

Definition 1. We say $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ and $h \in \mathcal{A}$ are *consistent* if for all $(x_0, x_1, x_2) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2$

$$f_1(x_0, x_1) = g_1(x_1) \text{ and } f_2(x_0, x_2) = g_2(x_2), \quad \text{where } (g_1, g_2) = h(x_0). \tag{1}$$

An immediate implication of this defintiion is that if (f_1, f_2) and h are consistent, then

$$\mathbb{E}\big[\ell\big(f_1(X_0, X_1), f_2(X_0, X_2), W\big)\big] = \mathbb{E}\big[\ell\big(h^1(X_0)(X_1), h^2(X_0)(X_2), W\big)\big]$$

where $h^i(x)$ denotes the i-th component.

Proposition 1. *Under (A), the solution of Problem (P1) can be obtained by solving Problem (P2) and vice versa.*

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(1) *For all* $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$*, there exists an* $h \in \mathcal{A}$ *that is consistent with* (f_1, f_2)*, and hence achieves the same expected cost.*
(2) *For all* $h \in \mathcal{A}$*, there exists* $(f_1, f_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ *that is consistent with* h*, and hence achieves the same expected cost.*

Therefore, Problems (P1) and (P2) are equivalent.

The proof is immediate. We can use the definition of consistency to construct the respective functions.

Note that Problem (P2) is simpler to solve than Problem (P1). In particular, if $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2, \mathcal{U}_1, \mathcal{U}_2$ are binary valued, a brute force solution of (P1) requires 2^8 computations (there are 4^2 possibilities for both f_1 and f_2) while a brute force solution of (P2) requires 2^5 computations (for each value of x_0, there are 2^2 possibilities for g_1 and g_2).

Question

Does there exist a sigma algebra on $\mathcal{A}$ such that we can make a claim similar to Proposition 1, when Assumption (A) is not true. We may need to weaken the definition of consistency such that the equality in (1) hold almost everywhere.