Simultaneous real-time transmission of multiple Markov sources over a shared channel

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***Abstract*—Consider a communication system in which a transmitter observes n independent Markov sources and has to jointly quantize them in real-time for a single receiver. Although the model is a special case of real-time quantization of Markov sources, a direct application of the results of real-time quantization is infeasible due to computational complexity. We restrict attention to a encoding-decoding strategies having a specific structure, and identify a dynamic program to find the best strategies with that structure. This dynamic program has uncountable state space. For the special case when all source alphabets are equal to each other and to the quantization alphabet, we reduce the dynamic program to one with a countable state space. We then present a finite-state approximation of this dynamic programming. The feasibility of the approach is shown by means of examples.**

I. Introduction

A. Problem formulation

Consider the communication system in which a transmitter *causally* observes n independent first-order Markov sources $\{S_t\}_{t=0}^{\infty}$, $i \in \{1, \dots, n\}$. The sources are assumed to have a finite or countable alphabet, denoted by $\mathcal{S}^i$. The corresponding transition matrix is denoted by P^i.

To simplify the exposition, assume that the initial state $\mathbf{s}_0 = (s_0^1, \dots, s_0^n)$ of all sources is fixed and known a priori to both the transmitter and the receiver.

The transmitter *sequentially* encodes the sources to a common quantization symbol $Q_t \in \mathcal{Q}$ according to some quantization rule $\mathbf{f} = \{f_t\}_{t=1}$, i.e.,

$$Q_t = f_t(\mathbf{S}_{1:t}, Q_{1:t-1}), \quad t = 1, 2, \dots \tag{1}$$

where $\mathbf{S}_t$ denotes $(S_t^1, \dots, S_t^n)$ and $\mathbf{S}_{1:t}$ is a short-hand for $\{\mathbf{S}_\tau\}_{\tau=1}^{t}$ and a similar interpretation holds for $Q_{1:t-1}$.

The receiver *sequentially* observes the quantized symbols and generates an estimate $\hat{\mathbf{S}} = (\hat{S}_t^1, \dots, \hat{S}_t^n)$ of all sources according a decoding rule $\mathbf{g} = \{g_t\}_{t=1}^{\infty}$, i.e.,

$$\hat{\mathbf{S}}_t = g_t(Q_{1:t}), \quad t = 1, 2, \dots. \tag{2}$$

The fidelity of the reconstruction is quantified by a per-step distortion function

$$d(\mathbf{S}_t, \hat{\mathbf{S}}_t) = \sum_{i=1}^{n} d^i(S_t^i, \hat{S}_t^i). \tag{3}$$

We are interested in choosing the *encoding-decoding strategy* $(\mathbf{f}, \mathbf{g})$ to minimize the expected discounted distortion over an infinite horizon

$$J_\beta(\mathbf{f}, \mathbf{g}) = \mathbb{E}^{(\mathbf{f},\mathbf{g})}\left[\sum_{t=1}^{\infty} \beta^{t-1} d(\mathbf{S}_t, \hat{\mathbf{S}}_t) \,\middle|\, \mathbf{S}_0 = \mathbf{s}_0\right] \tag{4}$$

where $\beta \in (0, 1)$ is the discount factor.

B. Literature overview

Since each source is Markov, the "joint" source $\{\mathbf{S}_t\}_{t=1}^{\infty}$ is also Markov. Hence the model described above is a special case of the real-time quantization of a Markov source. Such a model was first considered by Witsenhausen [1] who identified the structure of optimal encoders. For the model described above, Witsenhausen's result may be stated as follows:

Theorem 1 (Witsenhausen [1]): In real-time quantization, there is no loss of optimality in restricting attention to encoding strategies of the form

$$Q_t = f_t(\mathbf{S}_t, Q_{1:t-1}). \tag{5}$$

Witsenhausen [1] also presented a *source repackaging* technique to generalize the above structural results to higher order Markov sources and finite decoding delay (also called source coding with lookahead).

Walrand and Varaiya [2] showed that Witsenhausen's structural result extends to the case of real-time communication of *discrete* Markov sources over a DMC with noiseless feedback. Furthermore, when the receiver has no restrictions on its memory (as is the case in the above model), the structure of optimal encoders and decoders may be refined as follows:[1]

Theorem 2 (Walrand and Varaiya [2]): Let $\Delta(\prod_{i=1}^{n} \mathcal{S}^i)$ denote the space of probability distributions on $\mathbf{S}$. Define $\Pi_{t|t-1}, \Pi_{t|t} \in \Delta(\prod_{i=1}^{n} \mathcal{S}^i)$ as follows. For $\mathbf{s} \in \prod_{i=1}^{n} \mathcal{S}^i$,

$$\Pi_{t|t-1}(\mathbf{s}) = \mathbb{P}(\mathbf{S}_t = \mathbf{s} \mid Q_{1:t-1})$$
$$\Pi_{t|t}(\mathbf{s}) = \mathbb{P}(\mathbf{S}_t = \mathbf{s} \mid Q_{1:t}).$$

Then, there is no loss of optimality in restricting attention to encoding and decoding strategies of the form

$$Q_t = f_t(\mathbf{S}_t, \Pi_{t|t-1}), \qquad \hat{\mathbf{S}}_t = g_t(\Pi_{t|t}). \tag{6}$$

[1] Since fixed rate quantization is a special case of DMC with noiseless feedback, Walrand and Varaiya's result apply to that setup.

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Walrand and Varaiya also presented a dynamic programming decomposition of the problem based on Π_t. Linder and Yüksel [3] showed that Walrand-Varaiya-type structural results hold under quite general assumptions on the Markov source and the distortion function. A similar result under more restrictive assumptions was also established by Borkar *et al* [4].

These results have been generalized to other setups, including source coding with side-information [5], variable rate quantization [6], finite lookahead [7], joint source-channel coding [5], [8], joint source-channel coding with noisy feedback [9], and various multi-terminal setups [10], [11].

These structural results are useful because they identify a time-homogeneous sufficient statistic of the data available at the transmitter and the receiver. Thus, the domain of the encoders and decoders does not change with time, thereby simplifying implementation complexity. A time-homogeneous sufficient statistic also enables us to identify a dynamic programming decomposition, and thereby search for optimal encoding-decoding strategies in a systematic way. In spite of these advantages, these results have been of limited use because of the inherent computational complexity of solving the resultant dynamic programs.

In the model presented above, the source is a collection of n-independent sources. Thus, the search for optimal real-time encoding-decoding strategies is expected to be an order of magnitude more difficult than that of a single Markov source. For that reason, *we consider a simplified version of the problem by imposing assumptions on the structure of the encoding-decoding strategies.* Under these assumptions, the problem of optimal quantization of n sources decomposes into n independent problems of optimal quantization of a single source and a scheduling problem.

C. Simplifying assumptions

Instead of "joint" quantization of all sources, consider the subproblem of sequentially quantizing source $\{S^i_t\}_{t=1}^\infty$ over alphabet $\mathcal{Q}$ and per-step distortion $d^i(\cdot,\cdot)$. Let $(\mathbf{f}^i, \mathbf{g}^i)$ be a Walrand-Varaiya-type optimal encoding-decoding strategy for this subproblem. That is, for every $s^i_t \in \mathcal{S}^i$ and $\pi^i_{t|t-1} \in \Delta(\mathcal{S}^i)$, the encoding strategy $\mathbf{f}^i$ prescribes the quantization symbol

$$q^i_t = f^i_t(s^i_t, \pi^i_{t|t-1});$$

and for every $\pi^i_{t|t} \in \Delta(\mathcal{S}^i)$, the decoding strategy $\mathbf{g}^i$ prescribes the source reconstruction

$$\hat{s}^i_t = g^i(\pi^i_{t|t}).$$

Assume that such optimal *individual* encoding-decoding strategies have been determined for each source $\{S^i_t\}_{t=1}^\infty$ and per-step distortion $d^i(\cdot,\cdot)$ for $i \in \{1,\dots,n\}$. Then, for the joint quantization of the n sources, we restrict attention to *scheduling strategies* described below.

The transmitter and the receiver keep track of the posterior distributions $\Pi_{t|t-1}$ and $\Pi_{t|t}$. The update equation for these distributions will be described shortly. The transmitter picks an *index* $U_t \in \{1,\dots,n\}$ of the source to transmit and then sends

$$Q_t = (U_t, f^{U_t}_t(S^{U_t}, \Pi^{U_t}_{t|t-1})) \tag{7}$$

to the receiver,[2] where $\Pi^i_{t|t-1}$ is the marginal of $\Pi_{t|t-1}$. The index sequence $\{U_t\}_{t=1}^\infty$ is chosen according to a *scheduling strategy* $\mathbf{h} = \{h_t\}_{t=1}^\infty$ where

$$U_t = h_t(\mathbf{S}_t, \Pi_{t|t-1}), \quad t = 1, 2, \dots. \tag{8}$$

The receiver uses Q_t to update $\Pi_{t|t-1}$ to $\Pi_{t|t}$ (the update equation is described later) and generates estimates $\hat{\mathbf{S}}_t$ according to

$$\hat{S}^i_t = g^i_t(\Pi^i_{t|t}) \tag{9}$$

where $\Pi^i_{t|t}$ is the marginal of $\Pi_{t|t}$.

The above separation of quantization and scheduling is the first simplifying assumption (A1) that we make. Such a separation is not optimal but it is assumed to make the problem tractable.

Even with assumption (A1), finding the best scheduling strategy is not easy because the evolution of the posterior distribution is coupled with the scheduling strategy. In particular, suppose the posterior at the receiver is $\pi_{t|t-1}$ and quantized symbol (k, q_t) is received. Then the receiver knows that the source output S_t belongs to the set

$$\left\{ \tilde{\mathbf{s}}_t \in \prod_{i=1}^n \mathcal{S}^i : h_t(\tilde{\mathbf{s}}_t, \pi_{t|t-1}) = k \text{ and } f^k_t(s^k_t, \pi_{t|t-1}) = q_t \right\}$$

To update of the posterior $\Pi_{t|t}$, the receiver needs to know the observed quantization symbol (k, s^k_t) *and the scheduling function* h_t. Thus, the dynamic program to find the optimal scheduling strategy will be similar to the dynamic program to find the optimal quantization strategy. In particular, the information state of this dynamic program will be $\Pi_{t|t-1}$, the joint posterior on the n sources.

To simplify the optimization problem, we restrict attention to *oblivious update rules* of the posterior distribution. More precisely, the transmitter and the receiver keep track of the marginal distributions $\mathbf{\Pi}_{t|t-1} = (\Pi^1_{t|t-1}, \dots, \Pi^n_{t|t-1})$ and $\mathbf{\Pi}_{t|t} = (\Pi^1_{t|t}, \dots, \Pi^n_{t|t})$. These marginal distributions are updated as follows: for all $i \in \{1,\dots,n\}$

$$\Pi^i_{t|t} = \begin{cases} \ell^i_t(\Pi^i_{t|t}, q^i_t), & \text{if } Q_t = (i, q^i_t) \\ \Pi^i_{t|t-1}, & \text{otherwise} \end{cases} \tag{10}$$

and

$$\Pi^i_{t+1|t} = \Pi^i_{t|t} P^i \tag{11}$$

where P^i is the transition matrix of source $\{S^i_t\}_{t=1}^\infty$ and

$$\ell^i_t(\pi^i_{t|t-1}, q^i_t)(s^i) = \frac{\pi^i_{t|t-1}(s^i)\,\mathbb{1}\{f^i_t(s^i, \pi^i_{t|t-1}) = q^i_t\}}{\sum_{\tilde{s}^i \in \mathcal{S}^i} \pi^i_{t|t-1}(\tilde{s}^i)\,\mathbb{1}\{f^i_t(\tilde{s}^i, \pi^i_{t|t-1}) = q^i_t\}} \tag{12}$$

[2]We assume that U_t can be sent over a side-channel. If not, we can assume that the quantization of individual sources $\{S^i_t\}_{t=1}^\infty$ is done to a quantization alphabet of size $|\mathcal{Q}| - \log_2 n$ and the additional $\log_2 n$ bits are used to convey U_t.

The above specified oblivious update of the posterior is the second simplifying assumption (A2) that we make. As with our previous assumption, restricting attention to such update rules is not optimal.

Thus, given the individual (Walrand-Varaiya-type) encoding-decoding schemes $\{(\mathbf{f}^i, \mathbf{g}^i)\}_{i=1}^n$ for each source and rules (10) and (11) for updating the receiver's posterior on each source, we are interested in finding an optimal scheduling strategy $\mathbf{h}$ to minimize the expected discounted distortion

$$J_\beta(\mathbf{h}) = \mathbb{E}^{\mathbf{h}}\left[\sum_{t=1}^{\infty} \beta^{t-1} d(\mathbf{S}_t, \hat{\mathbf{S}}_t) \,\middle|\, \mathbf{S}_0 = \mathbf{s}_0\right] \tag{13}$$

where $\beta \in (0,1)$ is the discount factor.

Our motivation for the two simplifying assumptions is threefold. Firstly, as mentioned above, without these assumptions the problem is too unwieldy. Secondly, we believe that it is possible to identify models of sources for which the assumptions (A1) and (A2) do not result in a loss of optimality. Thirdly, in certain applications such as data communication in smart grids, there are privacy concerns that forces the use of such separated strategies.

The scheduling problem under (A1) and (A2) is also related to real-time broadcast over a half-duplex communication network (i.e., a network in which each node can either transmit or receive at a given time, but not both).

II. Dynamic programming decomposition and its simplification for a special case

A. Dynamic programming decomposition

Let $(\mathbf{f}^i, \mathbf{g}^i)$ be a *time-homogeneous* optimal strategy for source $\{S_t^i\}_{t=1}^\infty$, $i \in \{1, \dots, n\}$. Under assumptions (A1) and (A2), the choice of an optimal scheduling strategy is a centralized stochastic control problem which can be solved using a dynamic program. To simplify the notation of the dynamic program, define

$$D^i(\pi^i) = \sum_{s^i \in \mathcal{S}^i} d^i(s^i, g^i(\pi^i))\pi^i(s^i). \tag{14}$$

as the expected distortion at source i when the posterior $\Pi_{t|t}^i$ is π^i. Note that this expected distortion and the posterior update rule $\ell^i(\cdot)$ given by (12) do not depend on time since the encoding-decoding strategy is time-homogeneous.

Theorem 3: Let $V\colon \prod_{i=1}^n (\mathcal{S}^i \times \Delta(\mathcal{S}^i)) \to \mathbb{R}$ be the unique bounded fixed point of the following equation: for all $s^i \in \mathcal{S}^i$, $\pi^i \in \Delta(\mathcal{S}^i)$, $i \in \{1, \dots, n\}$

$$V(\mathbf{s}, \boldsymbol{\pi}) = \min_{u \in \{1,\dots,n\}} \left\{ \sum_{i=1}^n D^i(\pi_-^i) + \beta \sum_{\mathbf{s}_+} \boldsymbol{\pi}_+(\mathbf{s}_+) V(\mathbf{s}_+, \boldsymbol{\pi}_+) \right\} \tag{15}$$

where $\boldsymbol{\pi}_- = (\pi_-^1, \dots, \pi_-^n)$, $\boldsymbol{\pi}_+ = (\pi_+^1, \dots, \pi_+^n)$ and

$$\boldsymbol{\pi}_+(\mathbf{s}_+) = \prod_{i=1}^n \pi_+^i(s_+^i)$$

with

$$\pi_-^i = \begin{cases} \ell^i(\pi^i, f^i(s^i, \pi^i)), & \text{if } i = u; \\ \pi^i, & \text{otherwise} \end{cases}$$

and

$$\pi_+^i = \pi_-^i P^i.$$

Moreover, let $h^*(\mathbf{s}, \boldsymbol{\pi})$ denote (any of the) arg min of the right hand side of (15). Then, the time-homogeneous scheduling strategy $\mathbf{h}^* = (h^*, h^*, \dots)$ is optimal for Problem (13).

Proof: To prove the result, we need to show that the process $\{(\mathbf{S}_t, \mathbf{\Pi}_{t|t-1})\}_{t=1}^\infty$ is a controlled Markov process controlled by $\{U_t\}_{t=1}^\infty$. Then the result follows from standard results for controlled Markov processes [12]. The details of establishing the controlled Markov property are omitted due to space limitations. ■

B. A special case

To get some insight into the nature of the solution, consider the following special case:

Assumption (A3): The alphabet sizes of all the sources are equal to the quantization alphabet, i.e., $|\mathcal{S}^i| = |\mathcal{Q}|$ (or if a side-channel is not available to send U_t, then $|\mathcal{S}^i| = |\mathcal{Q}| - \log_2 n$).

For this special case, the optimal encoding-decoding strategy for a single source is straight forward. The optimal encoding strategy is to send the source uncoded, i.e.,

$$f_t^i(S_t^i, \Pi_{t|t-1}^i) = S_t^i;$$

the optimal decoding strategy is the solution to a filtering problem, i.e.,

$$g^i(\Pi_{t|t}^i) = \arg\min_{\hat{s} \in \mathcal{S}} \sum_{s \in \mathcal{S}} d^i(s, \hat{s}) \Pi_{t|t}^i(s).$$

Note that both the encoding and decoding strategies are time-invariant.

When source i is transmitted, the update function of the posterior distribution $\Pi_{t|t-1}^i$ simplifies as follows:

$$\ell^i(\pi_{t|t-1}^i, q_t) = \delta_{q_t}^i$$

where $\delta_{q_t}^i$ denotes the Dirac distribution on $\mathcal{S}^i$ with the unit mass at q_t.

Under assumption (A3), the dynamic program of Theorem 3 simplifies as follows. When the transmitter decides to transmit source u, then:

1) $\pi_-^u = \delta_{s^u}^u$, therefore $D^u(\pi_-^u) = 0$ and $\pi_+^u = \delta_{s^u}^u P^u$. Since the size of all the sources is the same, we drop the superscript u in $\delta_{s^u}^u$ and simply denote it as δ_{s^u}.
2) for $i \neq u$, $\pi_-^i = \pi^i$, therefore $D^i(\pi_-^i) = D^i(\pi^i)$ and $\pi_+^i = \pi^i P^i$.

Thus, the dynamic program of (15) simplifies to

$$V(\mathbf{s}, \boldsymbol{\pi}) = \min_{u \in \{1,\dots,n\}} \left\{ \sum_{i \neq u} D^i(\pi^i) + \beta \sum_{\mathbf{s}_+} \boldsymbol{\pi}_+(\mathbf{s}_+) V(\mathbf{s}_+, \boldsymbol{\pi}_+) \right\} \tag{16}$$

where $\boldsymbol{\pi}_+(\mathbf{s}_+)$ is defined as before and

$$\pi_+^i = \begin{cases} \delta_{s^i} P^i, & \text{if } u = i \\ \pi^i P^i, & \text{otherwise.} \end{cases} \tag{17}$$

Even after all these simplifications, the above dynamic program is difficult to solve because part of the state space, $\boldsymbol{\pi}$, is a vector of probability distributions. Such dynamic programs may either be solved by working with piecewise-linear and concave envelop of the value function (see [13] and its generalizations), or by discretizing the state-space [14], or by using point-based methods [15], [16].

We take an alternative approach and present a simpler method to solve the above dynamic program. We show that under any policy, the reachable set of information states $(\mathbf{s}, \boldsymbol{\pi})$ is countable and an optimal solution may be found based on a finite state approximation of the countable state.

III. Finite state approximation of the dynamic program

A. Reachability analysis and countable state-space representation

For notational convenience, in this section we restrict attention to the case of two sources (i.e., $n = 2$). The results extend naturally to multiple sources as well. For two sources, the dynamic program of (16) may be written as

$$V(s^1, s^2, \pi^1, \pi^2) = \min\left\{W^1(s^1, \pi^2), W^2(s^2, \pi^1)\right\} \quad (18)$$

where W^u corresponds to continuation cost for choosing action u and is given by

$$W^1(s^1, \pi^2) = D^2(\pi^2) + \beta \sum_{(s^1_+, s^2_+)} [\delta_{s^1} P^1]_{s^1 s^1_+} [\pi^2 P^2]_{s^2 s^2_+} V(s^1_+, s^2_+, P^1 \delta^1_{s^1}, P^2 \pi^2)$$

and W^2 defined in a symmetric manner.

Proposition 1: Under any scheduling strategy, the reachable set of $\{(\Pi^1_t, \Pi^2_t)\}_{t=1}^{\infty}$ is given by $\mathcal{R}^1 \times \mathcal{R}^2$ where

$$\mathcal{R}^i = \{\delta_z (P^i)^k \in \Delta(\mathcal{S}^i) : z \in \mathcal{S}^i \text{ and } k \in \mathbb{Z}_{>0}\}$$

Note that $\mathcal{R}^i$ is countable and isomorphic to $\mathcal{S}^i \times \mathbb{Z}_{>0}$ and any $\pi^i = \delta_z (P^i)^k \in \mathcal{R}^i$ may be denoted by $(z, k) \in \mathcal{S}^i \times \mathbb{Z}_{>0}$.

Proof: We prove the result using induction. In particular,

1) The initial state $\pi^i_1 = \delta_{s^i_0} P^i$ belongs to $\mathcal{R}^i$.
2) For any realization π^i_t of Π^i_t and any choice u_{t+1} of U_{t+1}, π^i_{t+1} is given by (17). Thus, if $\pi^i_t \in \mathcal{R}^i$, then so does π^i_{t+1}. ∎

A consequence of the above result is the following.

Proposition 2: An optimal scheduling strategy is given as follows. Let $\hat{V} \colon (\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^1, \mathbb{Z}_{>0}, \mathcal{S}^2, \mathbb{Z}_{>0}) \to \mathbb{R}$ be the unique bounded fixed point of the following equation. For any $s^i, z^i \in \mathcal{S}^i$ and $k^i \in \mathbb{Z}_m$

$$\hat{V}(s^1, s^2, z^1, k^1, z^2, k^2) = \min\{\hat{W}^1(s^1, z^2, k^2), \hat{W}^2(s^2, z^1, k^1)\} \quad (19)$$

where

$$\hat{W}^1(s^1, z^2, k^2) = D^2(\delta_{z^2}(P^2)^{k^2}) + \beta \sum_{(s^1_+, s^2_+)} [\delta_{s^1} P^1]_{s^1_+} [\delta_{z^2}(P^2)^{k^2+1}]_{s^2_+} V(s^1_+, s^2_+, s^1, 1, z^2, k^2+1)$$

and $\hat{W}^2$ is defined in a symmetric manner. Let $\hat{h}^*(s^1, s^2, z^1, k^1, z^2, k^2)$ denote (any of the) arg min of the right hand side of (19). For any $s^1 \in \mathcal{S}^i$ and $\pi^i = \delta_{z^i}(P^i)^{k^i} \in \mathcal{R}^i$, define

$$h^*(s^1, s^2, \pi^1, \pi^2) = \hat{h}^*(s^1, s^2, z^1, k^1, z^2, k^2). \quad (20)$$

Then, the stationary strategy $\mathbf{h} = (h^*, h^*, \dots)$ is optimal for Problem (13) under assumption (A3).

B. Finite state approximation

The dynamic program described above is a countable state dynamic program and it can be solved by finite state approximation methods described in [17], [18]. One such approximation method is as follows.

Let $\mathbb{Z}_m$ denote the set $\{1, \dots, m\}$. Define an approximation sequence $\{\hat{V}_m\}_{m=1}^{\infty}$ of $\hat{V}$ where $\hat{V}_m \colon (\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^1, \mathbb{Z}_m, \mathcal{S}^2, \mathbb{Z}_m) \to \mathbb{R}$ is the unique bounded fixed point of the equation

$$\hat{V}_m(s^1, s^2, z^1, k^1, z^2, k^2) = \min\{\hat{W}^1_m(s^1, z^2, k^2), \hat{W}^2_m(s^2, z^1, k^1)\}$$

and $\hat{W}^i_m$ has a definition similar to $\hat{W}^i$ in which $k^i + 1$ is replaced by $\min\{k^i + 1, m\}$. Let $\hat{h}^*_m$ denote the corresponding optimal strategy and h^*_m be defined similar to (20).

Proposition 3: $\lim_{m \to \infty} \hat{V}_m \to \hat{V}$. Furthermore, any limit point of the sequence of scheduling functions $\{h^*_m\}_{m=1}^{\infty}$ is optimal for Problem (13) under assumption (A3).

Proof: The sequence of finite-state models described above is a *augmentation type approximation sequence* (see [18, Definition 2.5.3]). Therefore, the existence of a limit point of $\{\hat{h}^*_m\}_{m=1}^{\infty}$ follows from [18, Proposition B.5].

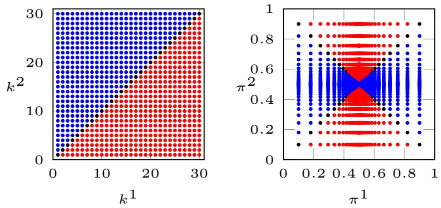
The underlying state spaces $\mathcal{S}^i$ are finite; hence the expected distortion $D^i(\cdot)$ is finitely bounded. Therefore, the DC(β) conditions hold [18, Proposition 4.7.1]. It follows from [18, Theorem 4.6.3] that any of the limit points of $\{\hat{h}^*_m\}_{m=1}^{\infty}$ is optimal for (19). The result follows from Proposition 2. ∎

As stated in the beginning of this section, these results extend naturally to multiple sources as well.

IV. Numerical examples

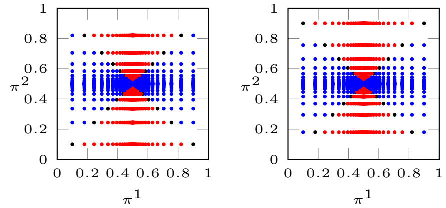
We investigate the setup of simultaneously transmitting two binary sources with the Hamming distortion and discount factor $\beta = 0.9$. We consider three cases, and for each case simulations suggest that the strategy has converged when $m = 30$. We describe the features of the strategy $\hat{h}^*_{30}$ and h^*_{30}.

The strategy $\hat{h}^*_m$ is a mapping from $(\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^1, \mathbb{Z}_m, \mathcal{S}^2, \mathbb{Z}_m)$ to $\{1, 2\}$. We fix the value of (s_1, s_2, z_1, z_2) and show $\hat{h}^*_m(s^1, s^2, z^1, k^1, z^2, k^2)$ as a function of (k^1, k^2) on a two-dimensional scatter plot where the color of the dot indicates the optimal action: red means $u = 1$, blue means $u = 2$, and black means that both actions are optimal. We use a similar technique to show the strategy $h^*_m(s^1, s^2, \delta_{z^1}(P^1)^{k^1}, \delta_{z^2}(P^2)^{k^2})$ as a function of $\delta_{z^1}(P^1)^{k^1}, \delta_{z^2}(P^2)^{k^2}$. The cases that we consider are:



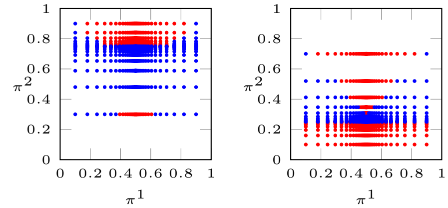
(a) Strategy $\hat{h}^*_{30}$ (b) Strategy h^*_{30}

Fig. 1. The optimal strategy for Case 1. The strategy $\hat{h}^*30$ and h^*_{30} have the shapes shown in (a) and (b).



(a) h^*_{30} for $z^2 = 1$. (b) h^*_{30} for $z^2 = 2$.

Fig. 2. The optimal strategy for Case 2. The strategy $\hat{h}_{30}$ has the shape shown in Fig. 1(a). The shape of h^*_{30} is shown in (a) and (b).



(a) h^*_{30} for $s^2 = 1$. (b) h^*_{30} for $s^2 = 2$.

Fig. 3. The optimal strategy for Case 3. The strategy $\hat{h}^*_{30}$ is not shown while the shape of h^*_{30} is shown in (a) and (b).

Case 1: Identical symmetric sources with $P^1 = P^2 = \begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix}$. The optimal strategy is shown in Fig. 1. Under the optimal strategy, the reachable values of the states $(s^1, s^2, z^1, k^1, z^2, k^2)$ are of the form: $(k^1, k^2) \in \{(1, 2), (2, 1)\}$ and other variables take all possible values.

Case 2: Complementary symmetric sources with $P^1 = \begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix}$ and $P^2 = \begin{bmatrix}0.1 & 0.9\\ 0.9 & 0.1\end{bmatrix}$. The optimal strategy is shown in Fig. 2. Under the optimal strategy, the reachable values of the states $(s^1, s^2, z^1, k^1, z^2, k^2)$ are the same as in Case 1. The reachable values in term of (π^1, π^2) differ because the transition matrices are different.

Case 3: One symmetric and one asymmetric source with $P^1 = \begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.9\end{bmatrix}$ and $P^2 = \begin{bmatrix}0.9 & 0.1\\ 0.7 & 0.3\end{bmatrix}$. The optimal strategy is shown in Fig. 3. Under the optimal strategy, the reachable values of $(s^1, s^2, z^1, k^1, z^2, k^2)$ are of the following form: $(k^1, k^2) \in \{(1, 2), (2, 1)\}$ or $(z^2, k^1, k^2) = (1, 3, 1)$ or $(z^2, k^1) = (1, 1)$, $k^2 \in \mathbb{Z}_m$ where the unspecified variables take all possible values.

In all three cases, *for the states that are reachable under the optimal strategy*, the optimal strategy may be represented as a finite state machine. We do not know if the optimal strategy always has such a structure.

V. Conclusion

We consider the problem of simultaneously transmitting multiple Markov sources over a common channel. We derive a dynamic programming decomposition under assumptions (A1) and (A2). We believe that for certain types of symmetric sources where the decoding problem decouples from the encoding strategy [19], [20], these assumptions are without any loss of optimality. For other cases, it is important to characterize the sub-optimality introduced by (A1) and (A2).

For the special case when all sources alphabets are equal (assumption (A3)), we show that the above dynamic program is equivalent to a countable state MDP. We then provide a sequence of finite-state approximations of the dynamic program that converges to the solution of the countable state MDP. Assumption (A3) limits the applicability of the model; as such it is worthwhile to investigate other setups where the dynamic program has tractable solutions.

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