Fixed Delay Joint Source Channel Coding for Finite Memory Systems

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Fixed Delay & Fixed Complexity

Motivation

- Classical Information Theory does not take delay and complexity into account.
- Why consider delay and complexity?
- **Delay:**
  - QoS (end–to–end delay) in communication networks
  - Control over communication channels.
  - Decentralized detection in sensor networks.
- **Complexity:** (size of lookup table)
  - cost
  - power consumption

Finite Delay Communication

- Separation Theorem: distortion d is feasible is

$$\text{Rate Distortion of Source} < \text{Channel Capacity}$$

$$R(d) \qquad < \qquad C$$

- For finite delay system Separation Theorem does not hold.
- What is equivalent of rate distortion and channel capacity?
- Find a metric to check whether distortion level d is feasible or not.
- Metric will depend on the source and the channel.

Problem Formulation

Objective

Evaluate optimal performance $R^{-1}(C)$ for the simplest non–trivial system

– Markov Source
– memoryless noisy channel
– additive distortion

Constraints

– Use stationary encoding and decoding schemes.
– Fixed memory available at the encoder and the decoder.

Model

- **Markov Source:**
  - Source Output $\{X_1, X_2, \dots\}, \quad X_n \in \mathcal{X}$.
  - Transition probability matrix P
- **Finite State Encoder:**
  - Input X_n, State S_n, Output Z_n

$$Z_n = f(X_n, S_{n-1}), \quad Z_n \in \mathcal{Z}$$
$$S_n = h(X_n, S_{n-1}), \quad S_n \in \mathcal{S}$$

- **Memoryless Channel:**

$$\Pr\left(Y_n \,\middle|\, Z^n, Y^{n-1}\right) = \Pr(Y_n \mid Z_n) = Q(Y_n, Z_n)$$

Model

- **Finite State Decoder:**
  - Input Y_n, State M_n, Output $\hat{X}_n$

$$\hat{X}_n = g(Y_n, M_{n-1}), \quad \hat{X}_n \in \mathcal{X}$$
$$M_n = h(Y_n, M_{n-1}), \quad M_n \in \mathcal{M}$$

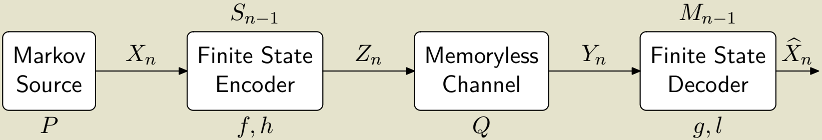
- **Distortion Metric:**

$$\rho : \mathcal{X} \times \mathcal{X} \to [0, K], \quad K < \infty$$

- D **step delay**

$$\rho(X_{n-D}, \hat{X}_n)$$

Problem Formulation



Problem (P1)

Given source $(\mathcal{X}, P)$, channel $(\mathcal{Z}, \mathcal{Y}, Q)$, memory $(\mathcal{S}, \mathcal{M})$ and distortion (ρ, D), determine encoder (f, h) and decoder (g, l) so as to minimize

$$\mathcal{J}(f, h, g, l) \triangleq \limsup_{N \to \infty} \frac{1}{\widetilde{N}} E\left\{ \sum_{n=D+1}^{N} \rho\left(X_{n-D}, \widehat{X}_n\right) \,\middle|\, f, h, g, l \right\}$$

where $\widetilde{N} = N - D + 1$

Literature Overview

- Transmitting Markovian source through finite–state machines as encoders and decoders.

- Problem considered by Gaarder and Slepian in mid 70's.

- N. T. Gaarder and D. Slepain
  On optimal finite–state digital communication systems,
  ISIT, Grignano, Italy, 1979
  TIT, vol. 28, no. 2, pp. 167–186, 1982.

Our Approach

- Start with a simpler (to analyze) problem
  - Finite horizon
  - zero delay
  - time–varying design
- dynamic team problem—solved using Stochastic Optimization Techniques
- finite delay problem
- infinite horizon problem
- Find conditions under which time invariant (stationary) designs are optimal.
- Low complexity algorithms to obtain optimal performance and optimal design.

Finite Horizon Problem

Finite Horizon Case — Model

- **Encoder and Tx Memory Update**

$$Z_n = f_n(X_n, S_{n-1}) \qquad f \triangleq (f_1, \ldots, f_N)$$

$$S_n = h_n(X_n, S_{n-1}) \qquad h \triangleq (h_1, \ldots, h_N)$$

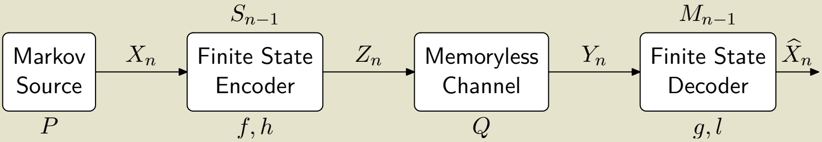
- **Decoder and Rx Memory Update**

$$\widehat{X}_n = g_n(Y_n, M_{n-1}) \qquad g \triangleq (g_1, \ldots, g_N)$$

$$M_n = l_n(Y_n, M_{n-1}) \qquad l \triangleq (l_1, \ldots, l_N)$$

- **Delay** $D = 0$

Finite Horizon Problem Formulation



Problem (P2)

Given source $(\mathcal{X}, P)$, channel $(\mathcal{Z}, \mathcal{Y}, Q)$, memory $(\mathcal{S}, \mathcal{M})$, distortion $(\rho, D = 0)$ and horizon N, determine encoder (f, h) and decoder (g, l) so as to minimize

$$\mathcal{J}_N(f, h, g, l) \triangleq E\left\{ \sum_{n=1}^{N} \rho(X_n, \widehat{X}_n) \,\middle|\, f, h, g, l \right\}$$

where $f \triangleq (f_1, \ldots, f_N)$, and so on for h, g, l.

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Solution Concept in Seq. Stoch. Opt

- **One Step Optimization**

$$\min_{\substack{f_1, f_2, \ldots, f_N \\ h_1, h_2, \ldots, h_N \\ g_1, g_2, \ldots, g_N \\ l_1, l_2, \ldots, l_N}} E\left\{ \sum_{n=1}^{N} \rho(X_n, \widehat{X}_n) \,\middle|\, f^N, h^N, g^N, l^N \right\}$$

- $4N$ **Step Optimization—Sequential Decomposition**

$$\min_{f_1}\left\{ \min_{g_1}\left\{ \min_{l_1}\left\{ \min_{h_1}\left\{ \cdots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. \cdots \min_{f_N}\left\{ \min_{g_N}\left\{ \min_{l_N}\left\{ \min_{h_N}\left\{ \Box \right\}\right\} \cdots \right\}\right\}\right\}\right\}$$

Dynamic Team Problems

- **Team Decision Theory:** distributed agents with common objective
  - Marshak and Radner
  - Witsenhausen
- Decentralized of information—encoder and decoder have different view of the world.
- Non–classical information pattern
- Non–convex functional optimization problem
- Most important step is identifying information state

Information State

If φ_{n-1} is the information state at n^- (and $\gamma = (f, h, g, l)$)

- **State** in the sense of

$$\longrightarrow \varphi_{n-1} \xrightarrow{T_{n-1}(\gamma_n)} \varphi_n \xrightarrow{T_n(\gamma_{n+1})} \varphi_{n+1} \longrightarrow$$

- Absorbs the effect of past decision rules on future performance.

$$E\left\{ \sum_{i=n}^{N} \rho(X_i, \widehat{X}_i) \,\middle|\, \gamma_1^N \right\} = E\left\{ \sum_{i=n}^{N} \rho(X_i, \widehat{X}_i) \,\middle|\, \pi_{n-1}^0, \gamma_n^N \right\}$$

Find an information state for Problem (P2)

Find an information state for Problem (P2)

Guess & Verify

Information State for (P2)

- **Definition**

$$\pi_n^1 \triangleq \Pr(X_n, Y_n, S_{n-1}, M_{n-1})$$
$$\pi_n^2 \triangleq \Pr(X_n, \quad S_{n-1}, M_n)$$
$$\pi_n^0 \triangleq \Pr(X_n, \quad S_n, \quad M_n)$$

Information State for (P2)

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$$\pi_n^0 \triangleq \Pr(X_n, \quad S_n, \quad M_n)$$

Lemma

For all $n = 1, \ldots, N$,

- there exist linear transforms T^0, T^1, T^2 such that

$$\longrightarrow \pi_{n-1}^0 \xrightarrow{T_{n-1}^0(f_n)} \pi_n^1 \xrightarrow{T_n^1(l_n)} \pi_n^2 \xrightarrow{T_n^2(h_n)} \pi_n^0 \longrightarrow$$

Information State for (P2)

- **Definition**

$$\pi_n^1 \triangleq \Pr(X_n, Y_n, S_{n-1}, M_{n-1})$$
$$\pi_n^2 \triangleq \Pr(X_n, \quad S_{n-1}, M_n)$$
$$\pi_n^0 \triangleq \Pr(X_n, \quad S_n, \quad M_n)$$

Lemma (cont...)

For all $n = 1, \ldots, N$,

- the expected instantaneous cost can be written as

$$E\left\{ \rho(X_n, \widehat{X}_n) \,\middle|\, f^n, h^n, g^n, l^n \right\} = \widetilde{\rho}(\pi_n^1, g_n)$$

Solution of (P2)

Dynamic Program

- For $n = 1, \ldots, N$

$$V^0_{n-1}(\pi^0_{n-1}) = \min_{f_n} \left\{ V^1_n\left(T^0(f_n)\pi^0_{n-1}\right) \right\},$$

$$V^1_n(\pi^1_n) = \overline{V}_n(\pi^1_n) + \min_{l_n} \left\{ V^2_n\left(T^1(l_n)\pi^1_n\right) \right\},$$

$$\overline{V}_n(\pi^1_n) = \min_{g_n} \left\{ \widetilde{\rho}(\pi^1_n, g_n) \right\},$$

$$V^2_n(\pi^2_n) = \min_{h_n} \left\{ V^0_n\left(T^2(h_n)\pi^2_n\right) \right\},$$

and

$$V^0_N(\pi^0_N) \triangleq 0.$$

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Solution of (P2)

- The $\arg\min$ at each step determines the corresponding optimal design rule.

- The optimal performance is given by

$$\mathcal{J}_N^* = V_0^0(\pi_0^0)$$

- **Computations:** Numerical methods from Markov decision theory can be used.

Next steps . . .

Finite Delay Problem

- Delay $D \neq 0$
- Sliding window transformation of the source

$$\overline{X}_n = (X_{n-D}, \ldots, X_n)$$
$$\overline{\rho}(\overline{X}_n, \widehat{X}_n) = \rho(X_{n-D}, \widehat{X}_n)$$

- Reduces to problem (P2).

Infinite Horizon Problem

- First consider delay $D = 0$
- Two related ways to making the horizon $N \to \infty$.
- **Expected Discounted Cost Problem**

$$\mathcal{J}^{\beta}(f, h, g, l) \triangleq E\left\{ \sum_{n=1}^{\infty} \beta^{n-1} \rho(X_n, \widehat{X}_n) \;\middle|\; f, h, g, l \right\}$$

- **Average Cost Per Unit Time Problem**

$$\overline{\mathcal{J}}(f, h, g, l) = \limsup_{N \to \infty} \frac{1}{N} E\left\{ \sum_{n=1}^{N} \rho(X_n, \widehat{X}_n) \;\middle|\; f, h, g, l \right\}$$

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Expected Discounted Cost Problem

- $\mathcal{J}^{\beta}(f,h,g,l) \triangleq E\left\{ \sum_{n=1}^{\infty} \beta^{n-1} \rho(X_n, \hat{X}_n) \,\middle|\, f,h,g,l \right\}$

- Find a fixed point $(V^0, V^1, \overline{V}, V^2)$ of

$$V^0(\pi^0) = \min_f \left\{ V^1\left(T^0(f)\pi^0_{n-1}\right) \right\},$$

$$V^1(\pi^1) = \beta \overline{V}(\pi^1) + \min_l \left\{ V^2\left(T^1(l)\pi^1\right) \right\},$$

$$\overline{V}(\pi^1) = \min_g \left\{ \tilde{\rho}(\pi^1, g) \right\},$$

$$V^2(\pi^2) = \min_h \left\{ V^0\left(T^2(h)\pi^2\right) \right\},$$

- Fixed point exists and is unique provided the distortion ρ is uniformly bounded.

Average Cost per Unit Time

- Average Cost

$$\bar{\mathcal{J}}(f, h, g, l) = \limsup_{N\to\infty} \frac{1}{N} E\left\{ \sum_{n=1}^{N} \rho(X_n, \hat{X}_n) \,\middle|\, f, h, g, l \right\}$$

- Define

$$\gamma_n \triangleq (f, h, g, l)$$

$$\widehat{T}(\gamma) \triangleq T^0(f) \circ T^1(l) \circ T^2(h)$$

$$\hat{\rho}(\pi^0_{n-1}, \gamma_n) \triangleq \tilde{\rho}\big(T^0(f_n)\pi^0_{n-1}, g_n\big)$$

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Average Cost per Unit Time

- **Assumption (A1):** for some $\epsilon > 0$ there exist bounded measurable functions $v(\cdot)$ and $r(\cdot)$ and design γ_0 such that for all π^0

$$v(\pi^0) = \min_{\gamma} \left\{ v\left(\hat{T}(\gamma)\pi^0\right) \right\} = v\left(\hat{T}(\gamma_0)\pi^0\right)$$

$$\min_{\gamma} \left\{ \hat{\rho}(\pi^0, \gamma) + r\left(\hat{T}(\gamma)\pi^0\right) \right\} \leqslant v(\pi^0) + r(\pi^0)$$

$$\leqslant \hat{\rho}(\pi^0, \gamma_0) + r\left(\hat{T}(\gamma_0)\pi^0\right) + \epsilon$$

Average Cost per Unit Time

- If (A1) holds then, $\gamma_0^\infty \triangleq (\gamma_0, \gamma_0, \ldots)$ is ϵ–optimal, that is, for any other design γ'

$$\overline{J}(\gamma_0^\infty) = v(\pi_0^0) \leqslant \underline{J}(\gamma') + \epsilon$$

where

$$\underline{J}(\gamma') \triangleq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \hat{\rho}(\pi_{n-1}^0, \gamma_n').$$

- Conditions sufficient to ensure (A1) are known.

- Not easy to translate them into conditions on the problem

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Some Comments

- For the expected discounted cost problem, time–invariant designs are optimal.
- For the average cost per unit time problem, time–invariant designs are optimal under certain conditions.
- The two problems are related via a Tauberian theorem

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} \leqslant \liminf_{\beta\to 1^-} (1-\beta) \sum_{i=1}^{\infty} \beta^{i-1} a_i$$

$$\leqslant \limsup_{\beta\to 1^-} (1-\beta) \sum_{i=1}^{\infty} \beta^{i-1} a_i \leqslant \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n}$$

Solution Framework . . .

General Methodology

- Given source $(\mathcal{X}, P)$, channel $(\mathcal{Z}, \mathcal{Y}, Q)$, memory $(\mathcal{S}, \mathcal{M})$ and distortion (ρ, D).
- Convert to zero delay problem
- Find ϵ–optimal design and performance for the discounted cost problem for β close to 1.
- This can be done using a polynomial complexity algorithms.
- The resultant design is ϵ–optimal for the average cost per unit time problem (if an ϵ–optimal design for the average cost per unit time problem exists)

Some Interesting Cases

- **Fixed Delay Source Coding Problem**
- The technique presented here can be extended to non–stochastic min–max problems.
- Used to study fixed delay encoding/decoding of individual sequences.
- Interesting to compare the results with “standard” fixed delay source coding techniques.

Some Interesting Cases

- **Fixed Delay Channel Coding Problem**
- Fixed delay decoding of convolutional codes.
- Most researchers focus on computationally efficient algorithms to determine the MAP bit decoding rule.
- The problem of efficiently storing the observations has not been considered.
- If receiver memory $|\mathcal{M}| = k\,|\mathcal{Y}|$, should one store the previous k channel observations?
- Can all the past observations be compressed in $k\,|\mathcal{Y}|$ to get better performance.
- How can such "compression" functions be found.
- This problem fits naturally in the framework presented here.

Conclusion.

Summary

- Consider fixed delay, fixed complexity communication system
- Markov source and noisy memoryless channel
- Objective: Minimize total (or discounted or average) distortion
- Provide a systematic methodology for determining optimal encoding–decoding strategies and optimal performance
- There exist low complexity algorithms to find such solutions
- Interesting special cases of the framework

Thank You

Gaarder and Slepian's Approach

- Fix a design (f, h, g, l).
- $\{X_{n-D}^{n}, S_n, Z_n, Y_n, M_n, \widehat{X}_n\}$ forms a Markov chain.
- Find its steady–state distribution.
- Find the steady–state distortion

$$\lim_{n\to\infty} E\left\{ \rho(X_{n-D}, \widehat{X}_n) \right\}.$$

- **Cezáro Mean:** For any sequence of real numbers (a_n),

$$\text{If } \lim_{n\to\infty} a_n = a \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i = a$$

- Repeat for all designs (f, h, g, l).

Gaarder and Slepian's Approach

- **Difficulty:** Evaluating asymptotic (steady–state) performance is difficult.

$$\min_{f,h,g,l} \lim_{n\to\infty} E\left\{ \rho(X_{n-D}, \widehat{X}) \right\}$$

- "A sore point here is the very complicated way in which the stationary distribution of a Markov chain depends on the elements of its transition matrix "
- The matrix elements change discontinuously with a change in design (f, h, g, l).
- The resultant Markov chain can have several recurrence classes, be periodic, have several transient states etc., depending on the nature of the design (f, h, g, l).