Fundamental limits of remote state estimation

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Joint work with Jhelum Chakravorty and Jayakumar Subramanian

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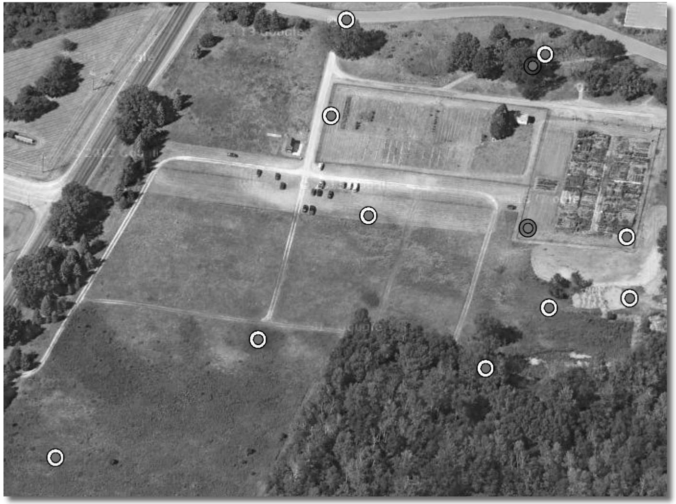
There is a need to revisit rate-distortion theory to take network access into account.

Many applications require:

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction



Sensor Networks

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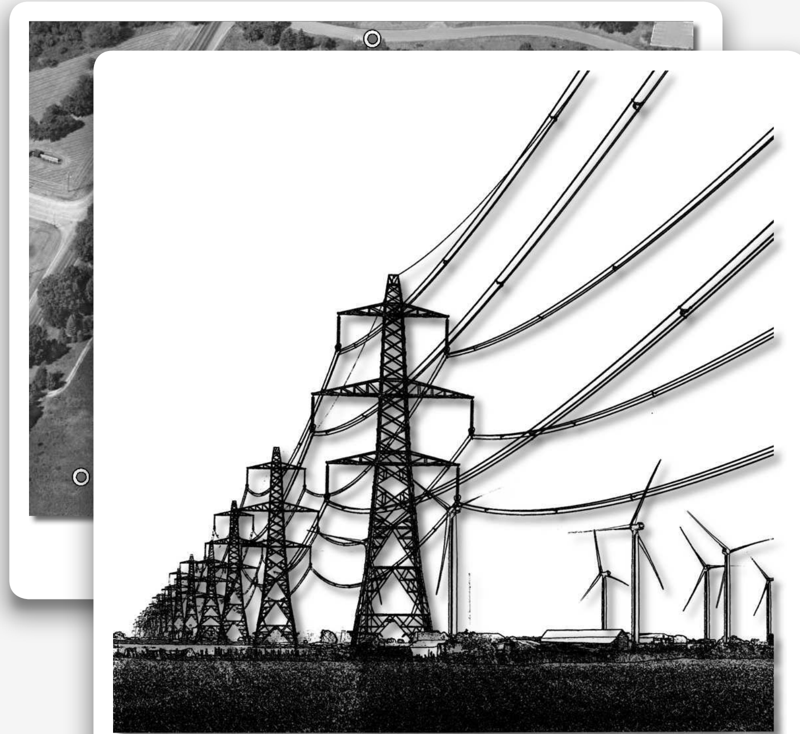

Smart Grids

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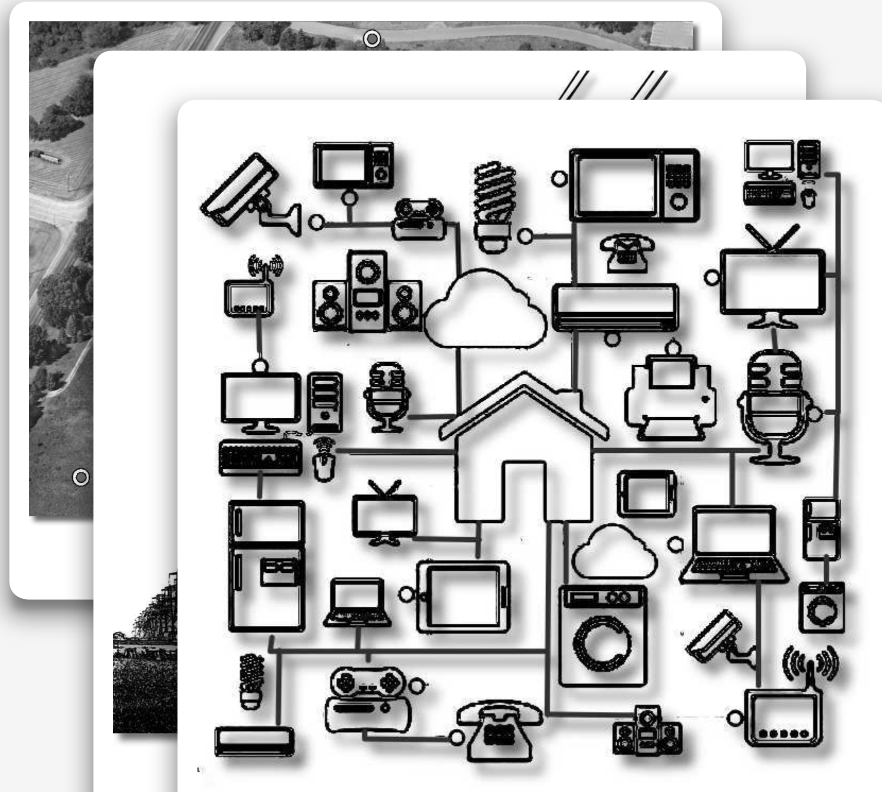

Internet of Things

Remote stat

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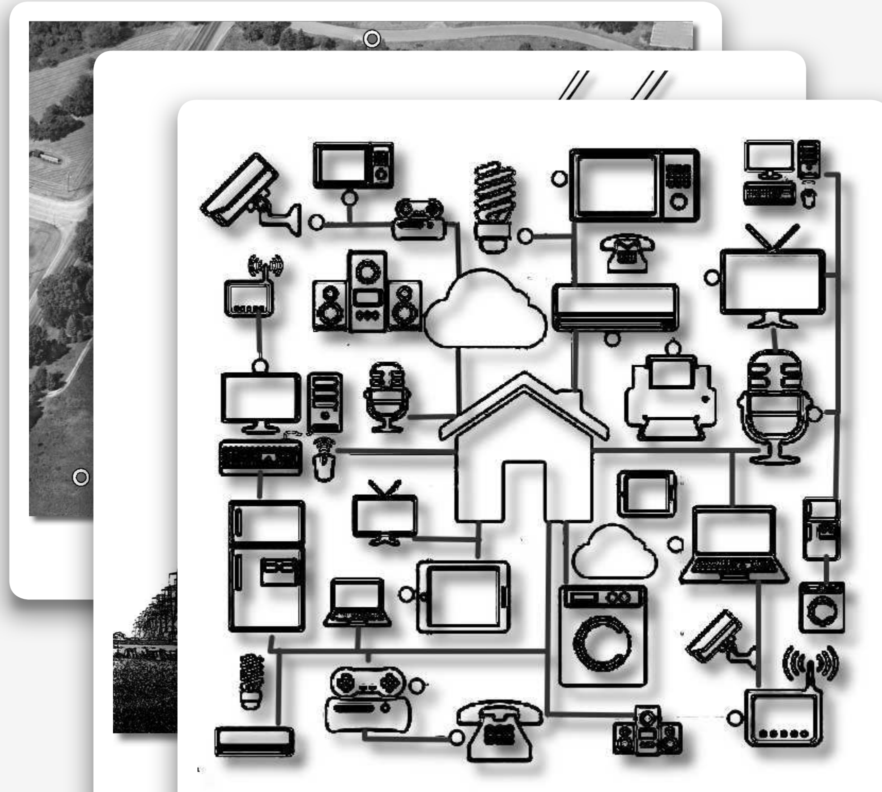

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- Sensing is cheap
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Analyze a stylized model and evaluate fundamental trade-offs

Remote stat

Communication system

Source model

$\{X_t\}_{t \geqslant 0}$, $X_t \in \mathcal{X}$, is a first-order Markov process.

For some results, we restrict to **autoregressive model**: $X_{t+1} = aX_t + W_t$, $X_t \in \mathbb{Z}/\mathbb{R}$.

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When $S_t = 1$ (Channel is ON)
channel output = channel input

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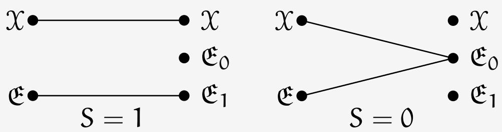
channel output = noise



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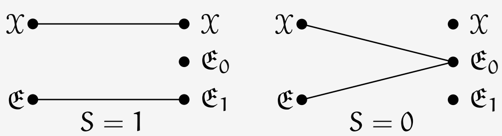
Formal definition

Input alphabet $\bar{\mathcal{X}} = \mathcal{X} \cup \{\mathfrak{E}\}$ Output alphabet $\mathcal{Y} = \mathcal{X} \cup \{\mathfrak{E}_0, \mathfrak{E}_1\}$.



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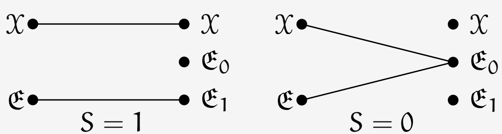
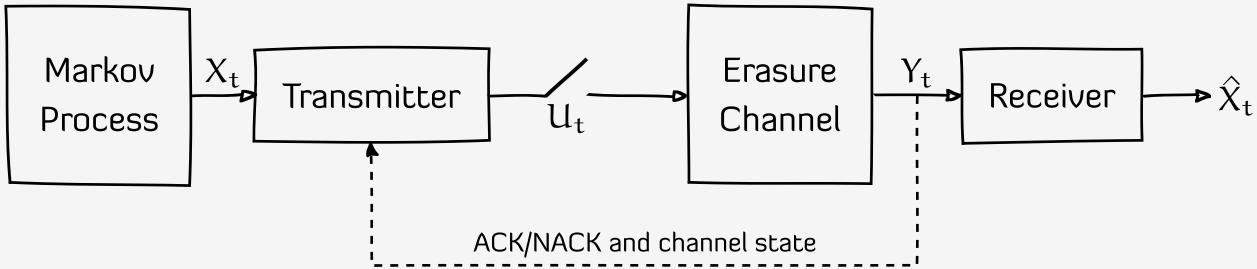
Channel input/output relationship

$$\mathbb{P}(Y_t \mid \bar{X}_{0:t}, S_{0:t}) = \mathbb{P}(Y_t \mid \bar{X}_t, S_t).$$



$$= \begin{cases} \mathfrak{E}_1, & \text{if } \bar{X}_t = \mathfrak{E} \text{ and } S_t = 1 \quad \text{(No received energy)} \\ \mathfrak{E}_0, & \text{if } S_t = 0 \quad \text{(Received energy)} \\ \bar{X}_t, & \text{if } \bar{X}_t \in \mathcal{X} \text{ and } S_t = 1 \quad \text{(Packet can be decoded)} \end{cases}$$

Communication system (cont.)



Feedback The receiver sends two bits of feedback: ACK/NACK and channel state.

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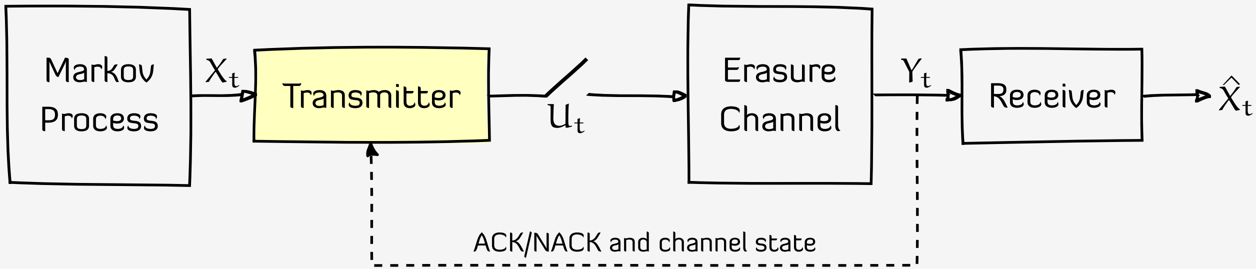

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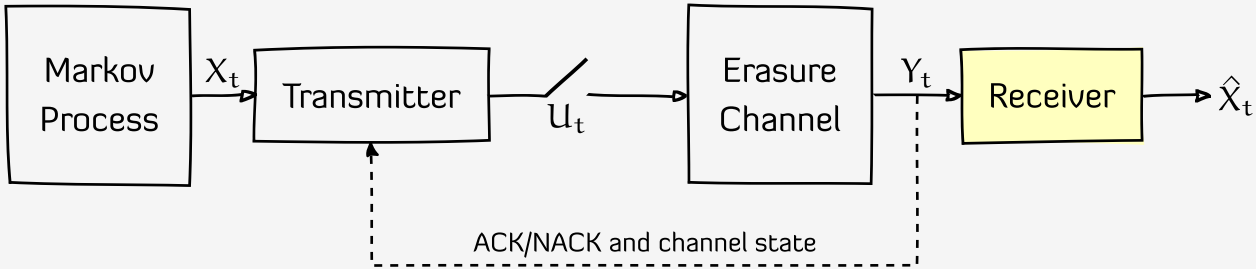
Transmitter Decides whether to transmit or not. Denoted by $U_t \in \{0, 1\}$.

If $U_t = 0$, $\bar{X}_t = \mathfrak{E}$. If $U_t = 1$, $\bar{X}_t = X_t$.

$$U_t = f_t(X_{1:t}, Y_{1:t-1}, S_{1:t-1})$$

Communication system (cont.)



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Receiver Chooses an estimate $\hat{X}_t \in \mathcal{X}$

$$\hat{X}_t = g_t(Y_{1:t}, S_{1:t})$$

An illustration

Transmitter	Receiver
$U_1 = f_1(X_1)$	

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Performance metrics

Performance metrics   Distortion D and Number of transmissions N

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1. Discounted setup, $\beta \in (0,1)$

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty}\beta^t d(X_t,\hat{X}_t)\Big]; \qquad N_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty}\beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f,g) = \limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t,\hat{X}_t)\Big]; \qquad N_1(f,g) = \limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

Optimization problems

Constrained communication

$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \big\{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \big\}$$

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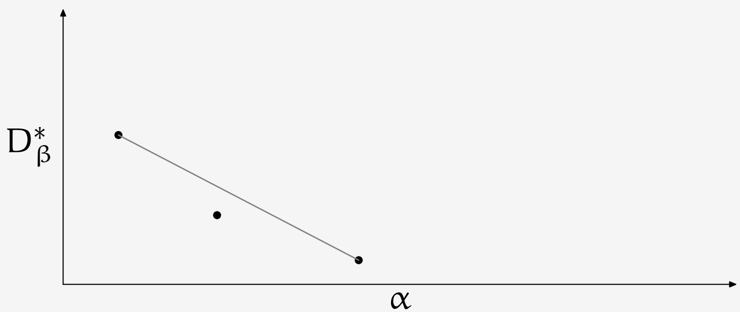

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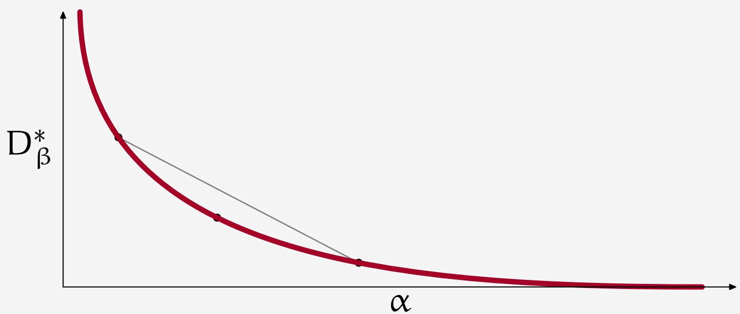

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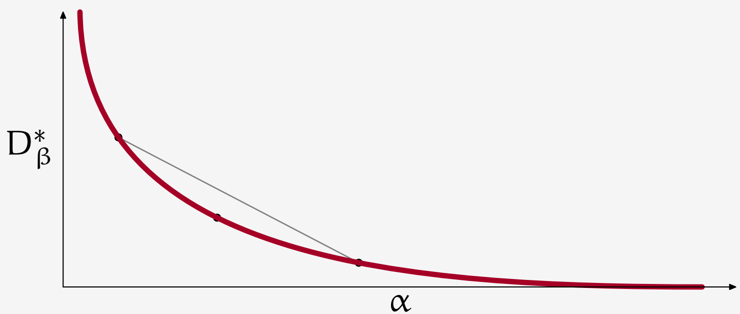

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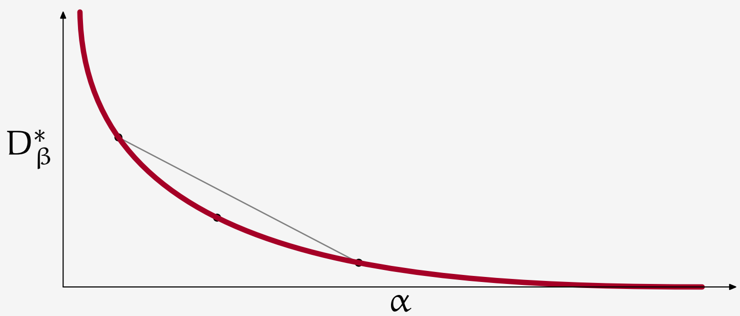

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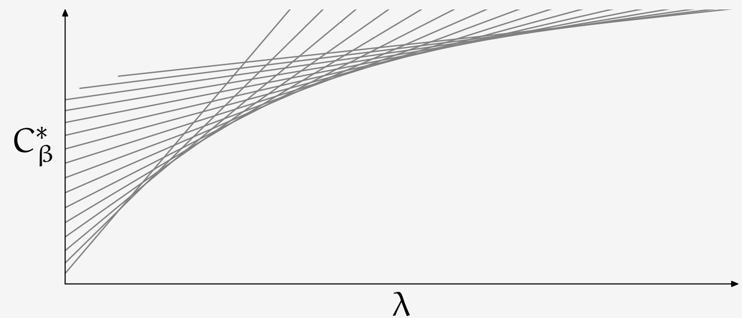

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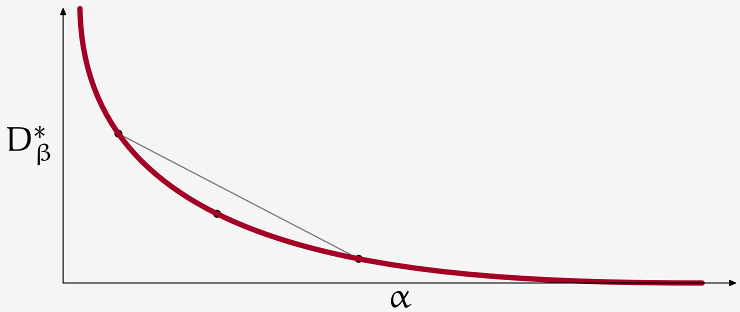

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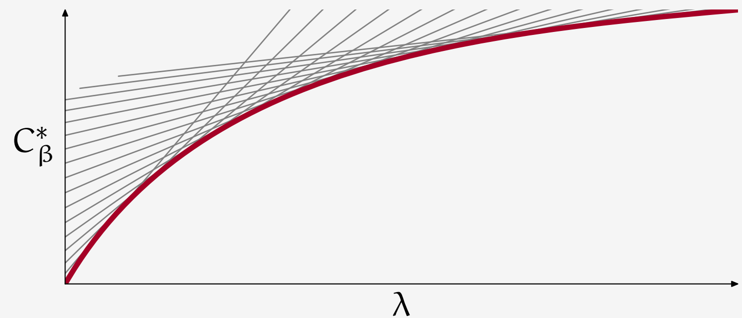

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Optimization problems

Constrained communication

Our result: Provide computable expressions for these trade-offs and identify optimal strategies that achieve them.

Costly c

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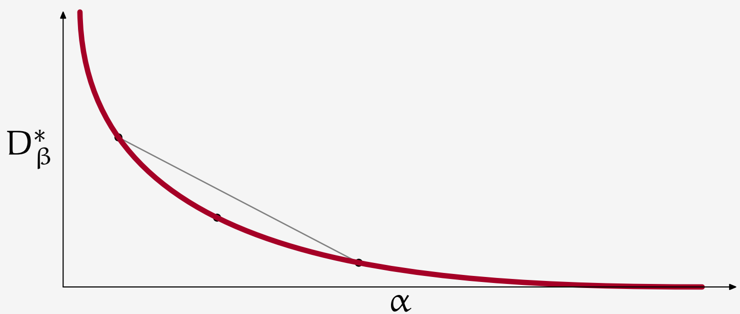

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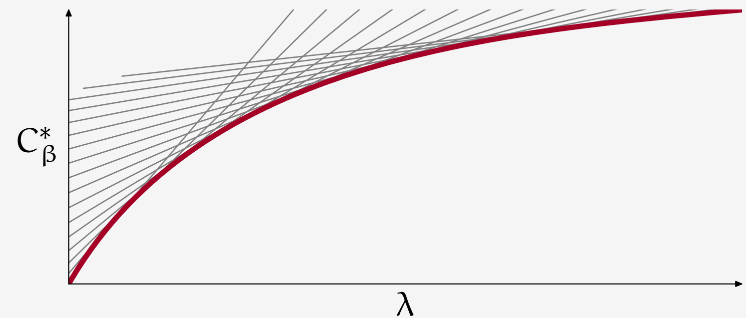

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Comparison to Information Theory

- Costly communication is analogous to communication under power constraint.
- Constrained communication is analogous to distortion-rate function.
  So, we call it **distortion-transmission** function.
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Previous work on remote-state estimation

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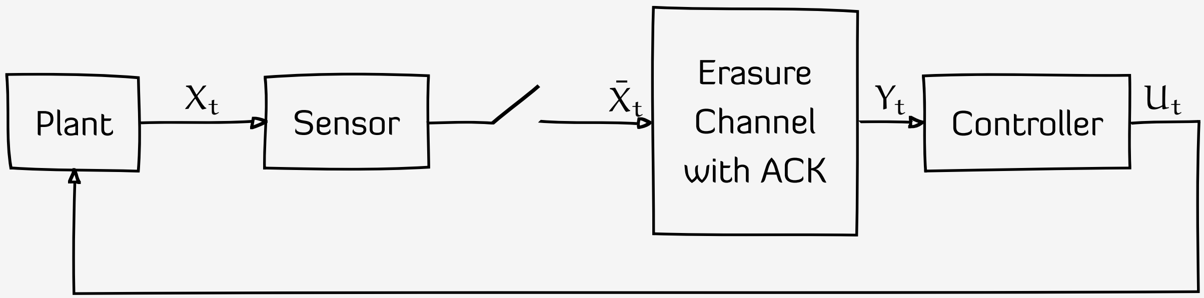
Other related work

- Event-based estimation ...
- Censoring sensors ...
- Sensor sleep scheduling ...
- Age of Information ...

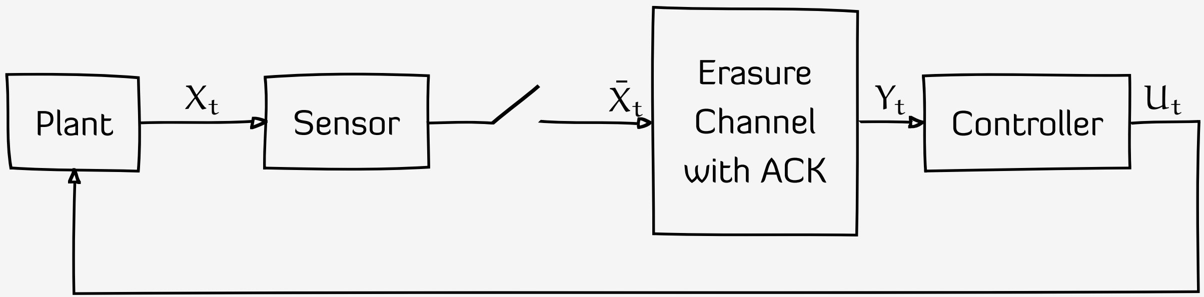
A networked control motivation

Networked control system



Model $\quad X_{t+1} = AX_t + BU_t + W_t, \quad \bar{X}_t \in \{X_t, \mathfrak{E}\}, \quad U_t = g_t(Y_{1:t}). \quad$ Min. quadratic cost

Networked control system



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Separation of estimation and control

- Consider the innovation process: $Z_t = X_t - \tilde{X}_t$, where $\tilde{X}_t = \sum_{s=0}^{t-1} A^{t-s-1} BU_s$
- There is no loss of optimality in deciding to transmit based on Z_t.
- Certainty equivalent controller is optimal: $U_t = K_t(\hat{Z}_t + \tilde{X}_t)$

- Yüksel, "Jointly Optimal LQG Quantization and Control Policies for Multi-Dimensional Systems," TAC 2014
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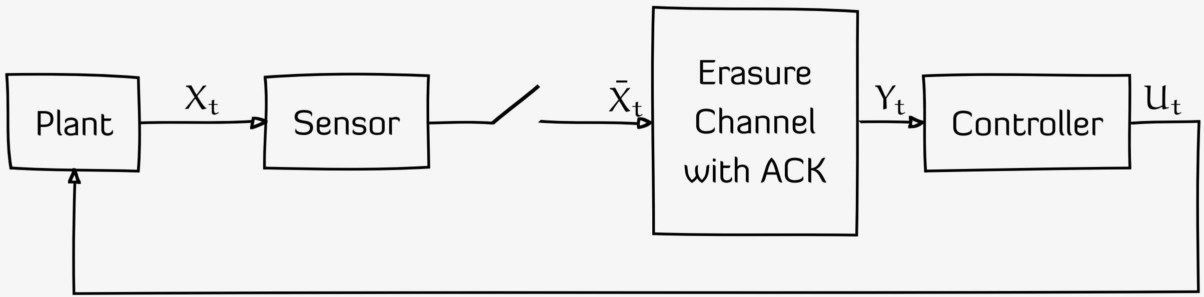

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- Certainty equivalent controller is optimal: $U_t = K_t(\hat{Z}_t + \tilde{X}_t)$
- Innovations do not depend on control $Z_{t+1} = AZ_t + W_t$

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Why bother?

How much do we gain compared to simple strategies?

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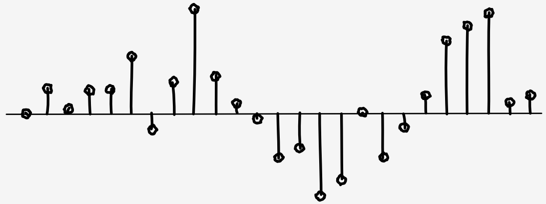

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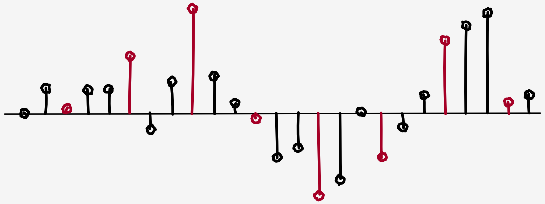
Periodic Transmission Strategy



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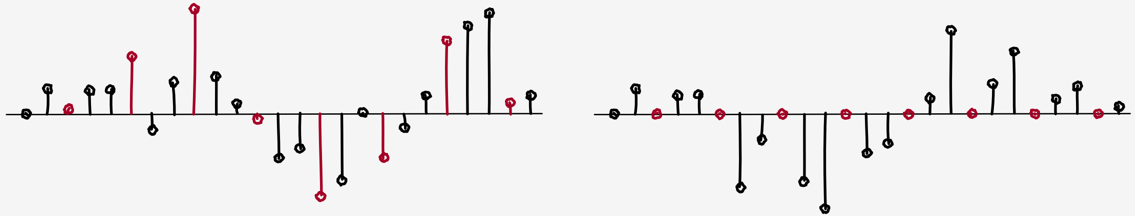

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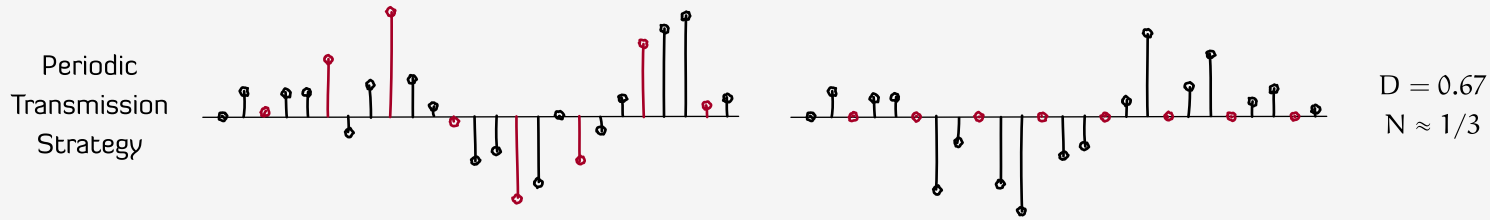

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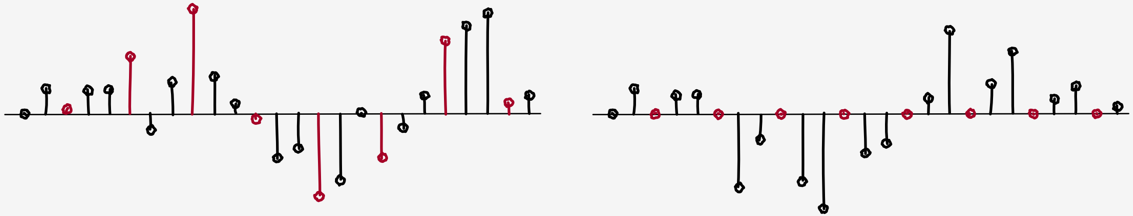
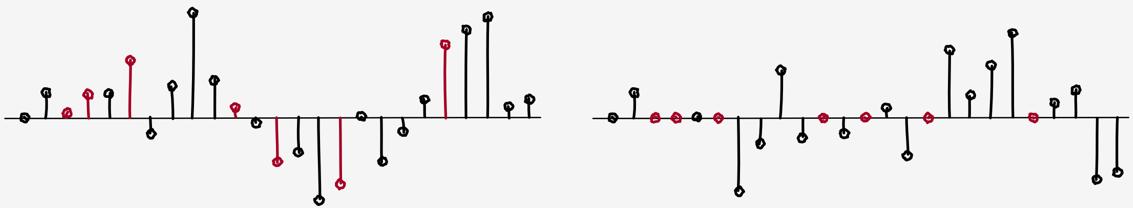


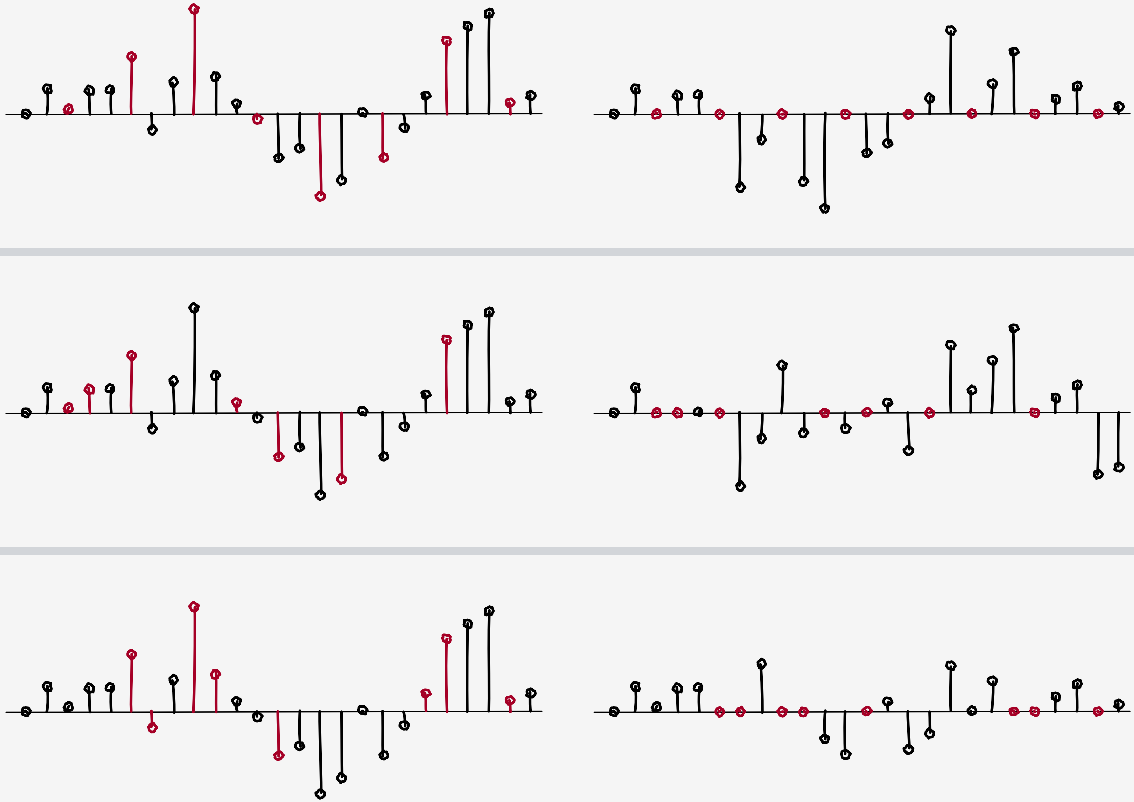
$D = 0.67$
$N \approx 1/3$

Randomized Transmission Strategy



$D = 2.00$
$N \approx 1/3$

$X_{t+1} = X_t + W_t$, $W_t \sim \mathcal{N}(0,1)$. Perfect channel

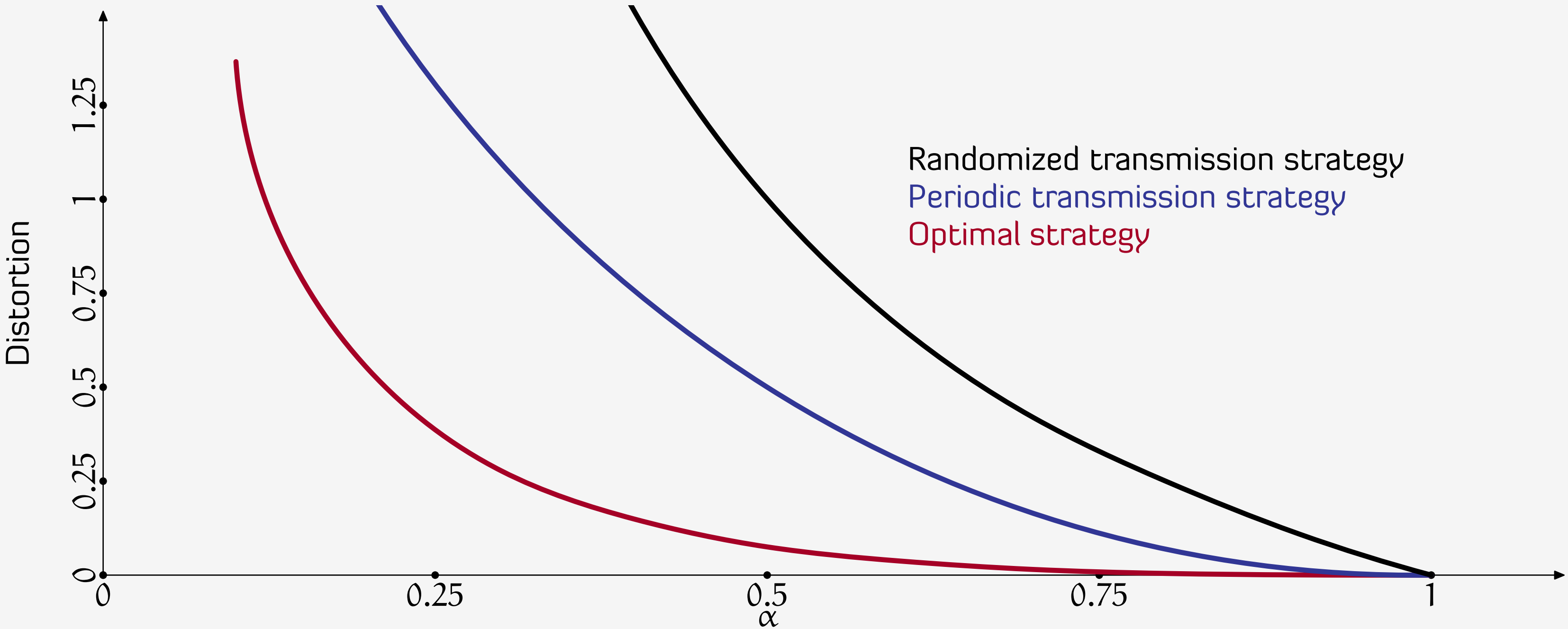


Periodic Transmission Strategy: $D = 0.67$, $N \approx 1/3$

Randomized Transmission Strategy: $D = 2.00$, $N \approx 1/3$

Optimal Transmission Strategy: $D = 0.24$, $N \approx 1/3$

Distortion-transmission trade-off: Perfect channel



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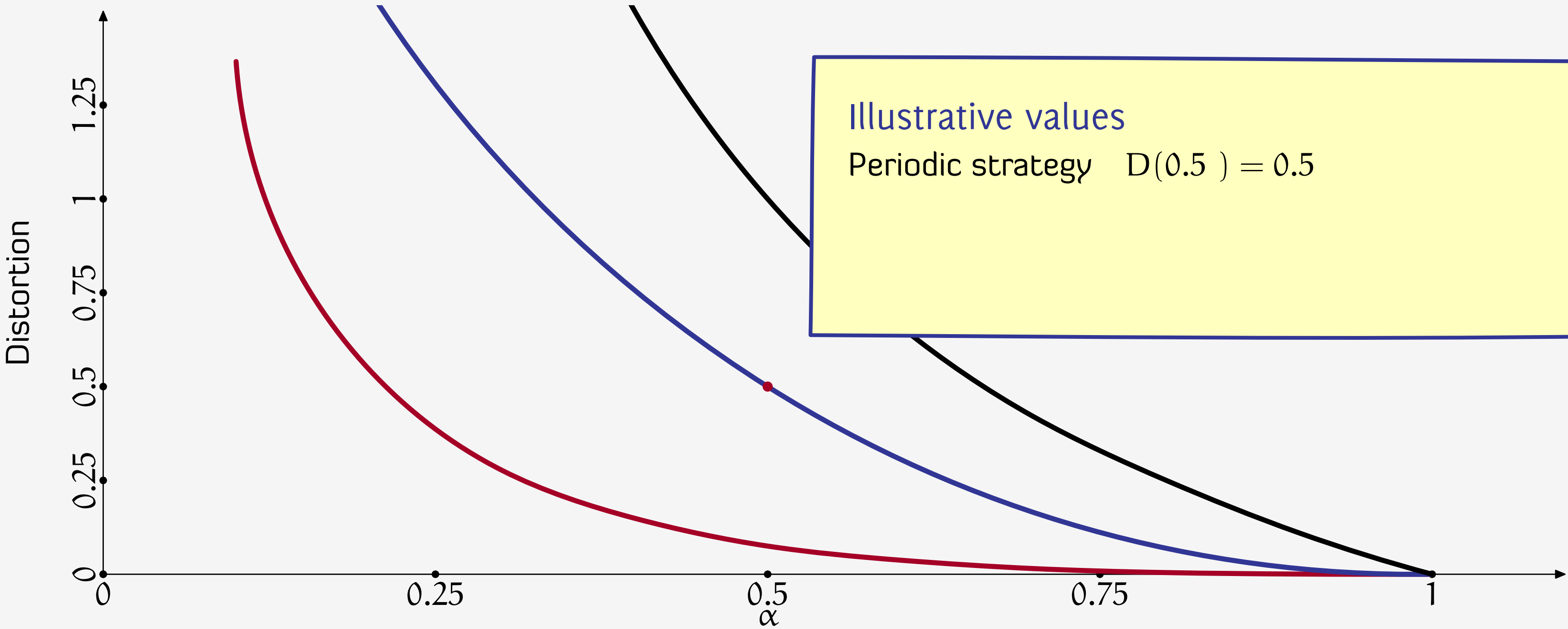
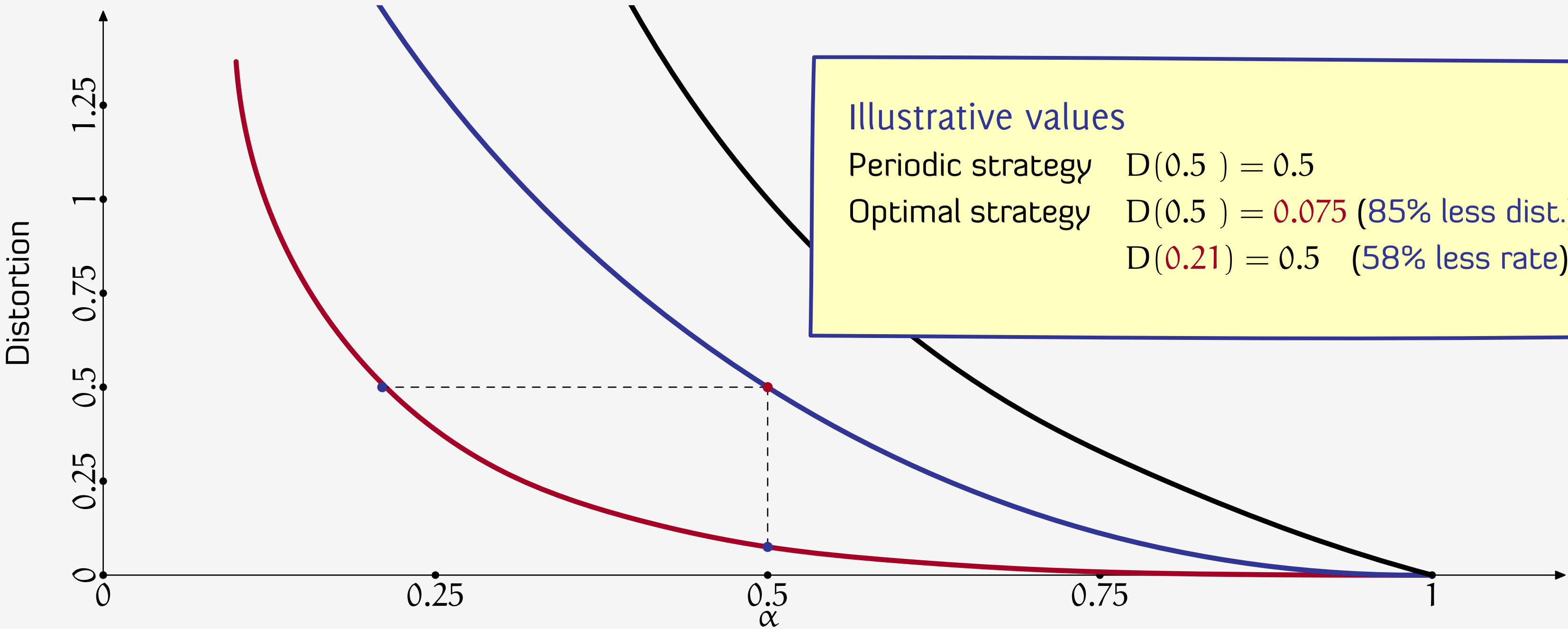


Illustrative values

Periodic strategy $D(0.5\ ) = 0.5$

Distortion-transmission trade-off: Perfect channel



Illustrative values

Periodic strategy $D(0.5) = 0.5$

Optimal strategy $D(0.5) = 0.075$ (85% less dist.)

$D(0.21) = 0.5$ (58% less rate)

What's the conceptual difficulty?

Static (one-shot) problem

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Total expected cost

$$c(\hat{x}, \mathcal{S}) \coloneqq \lambda\, \mathbb{P}(X \notin \mathcal{S}) + \varepsilon \sum_{x \notin \mathcal{S}} \mathbb{P}(X = x) d(x - \hat{x}) + \sum_{x \in \mathcal{S}} \mathbb{P}(X = x) d(x - \hat{x})$$

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Choose $(\hat{x}, \mathcal{S})$ to minimize $c(\hat{x}, \mathcal{S})$.

Set-valued (or combinatorial) optimization problem.

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Sequential optimization problem where the optimization problem at each step is a set-valued optimization problem that depends on a history of previously chosen sets!.

Exhaustive search complexity: $\left(|\mathcal{X}|2^{|\mathcal{X}|}\right)^{(2^{|\mathcal{X}|})^T}$

Main results

Optimal strategies and their performance

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

Distortion $d(x, \hat{x}) = d(x - \hat{x})$ where $d(\cdot)$ is symmetric and quasi-convex.

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Optimal estimation strategy

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

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Optimal transmission strategy

$$U_t = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geqslant k(S_{t-1}) \\ 0, & \text{otherwise} \end{cases}$$

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- Optimal strategies are simple and intuitive

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Salient features

- Optimal strategies are simple and intuitive
- The transmitter does not try to send information through timing events (or length of silence intervals).
- The estimation strategy does not depend on the value of the threshold
- When the estimator does not receive a packet, it behaves as if the packet was dropped by the channel, **even when the channel is perfect!**

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Performance of threshold based strategies

- $K_\beta^{(k)}$: Expected discounted number of transmissions until first successful reception.
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- $K^{(k)}_\beta$: Expected discounted number of transmissions until first successful reception.
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Then, $D^{(k)}_\beta = \dfrac{L^{(k)}_\beta}{M^{(k)}_\beta}$ and $N^{(k)}_\beta = \dfrac{K^{(k)}_\beta}{M^{(k)}_\beta}$. **(Renewal Relationships)**

Optimal trade-offs for discrete sources

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$$C_\beta^*(\lambda) \coloneqq \inf_{(f,g)} \big\{ D_\beta(f,g) + \lambda N_\beta(f,g) \big\}$$

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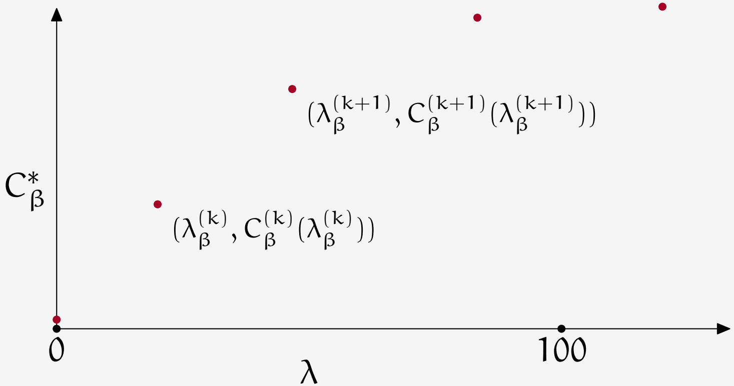

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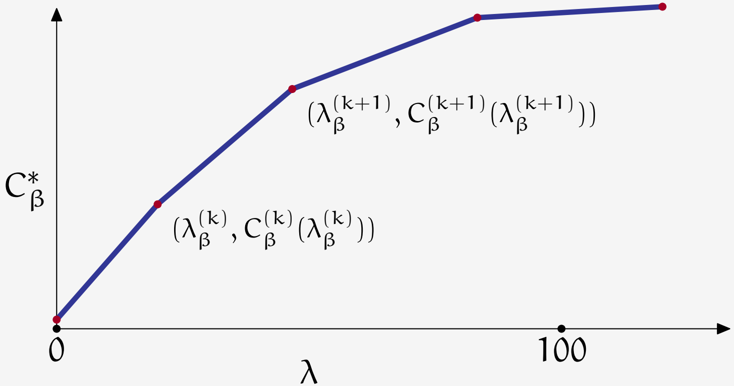

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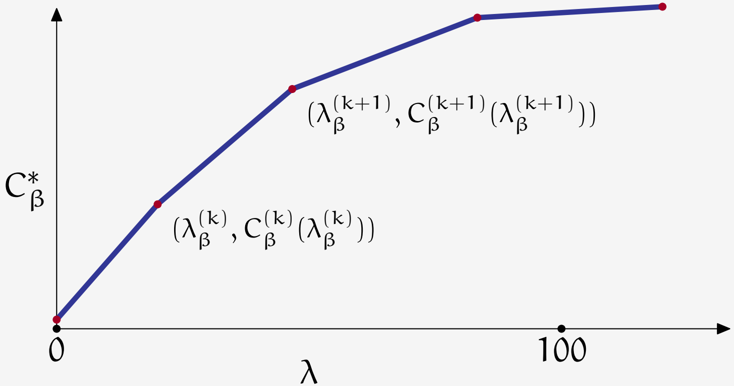

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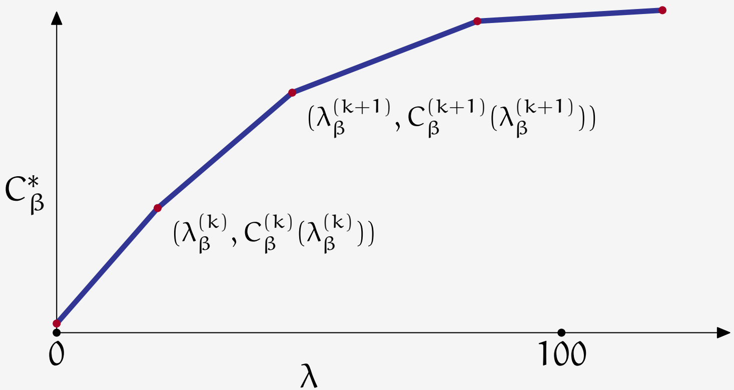

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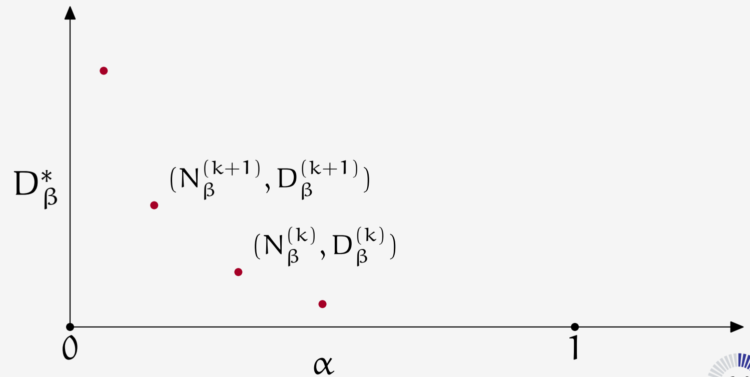

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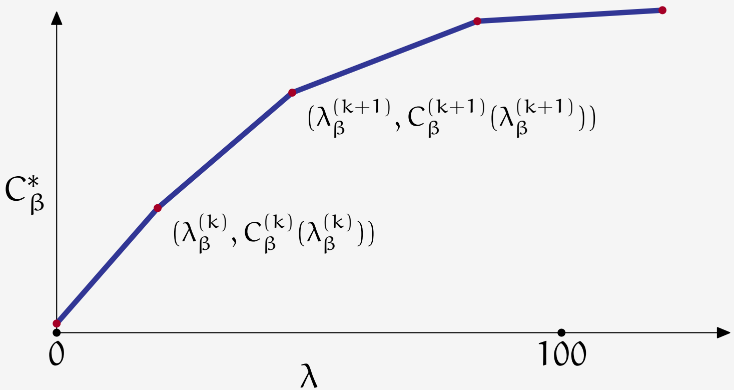

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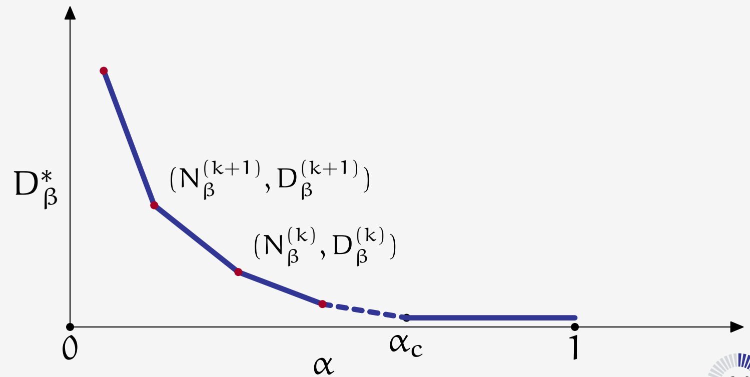

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Again assume i.i.d. packet drops.

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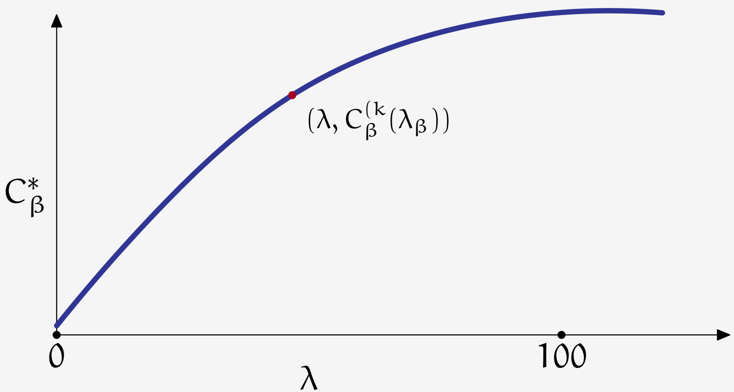

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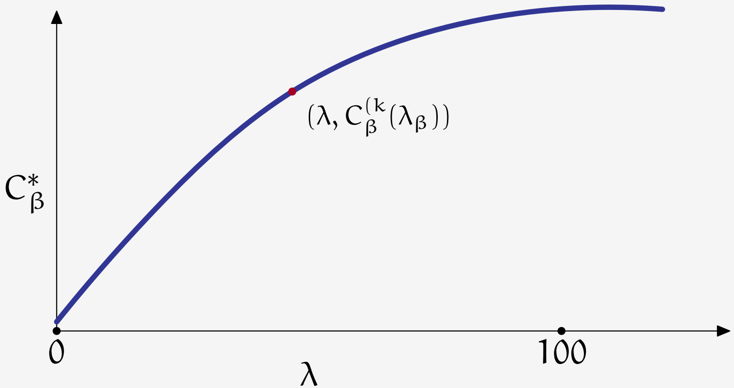

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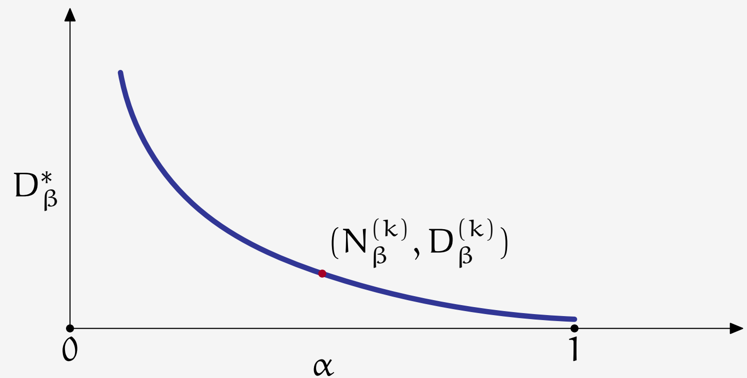

Proof outline

How to prove the optimality of a coding scheme?

Information theory approach

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.

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- Dynamic program: Identify sufficient statistics dynamic program
- Structural results: Determine qualitative properties of optimal solutions

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Stochastic control approach

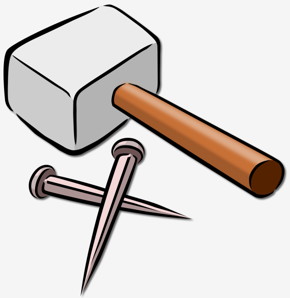
- Dynamic program: Identify sufficient statistics dynamic program
- Structural results: Determine qualitative properties of optimal solutions

> Structural results are hard! Especially for **multi-agent systems**.

- Related results (real-time comm.): [Witsenhausen 1979, Walrand-Varaiya 1983, Teneketzis 2006, Mahajan-Teneketzis 2009, Kaspi-Merhav 2012, Asnani-Weissman 2013, Yüksel 2013 . . .]

So how do we start?

Decentralized stochastic control

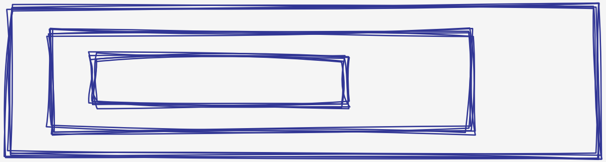


Dealing with non-classical information structure

- **Structure of optimal strategies**
  Instead of $f(\text{history of obs})$ use $f(\text{info state})$.

- **Compute optimal strategy using DP**
  $V(\text{info state}) = \min_{\text{action}} \left[\mathcal{B}_{\text{action}} V\right](\text{info state})$

Dealing with non-classical information structure



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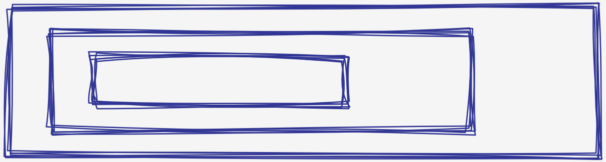

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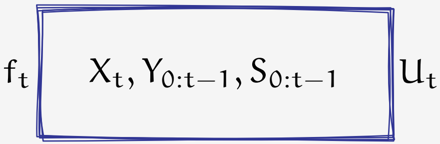

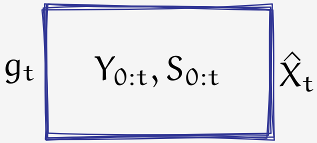

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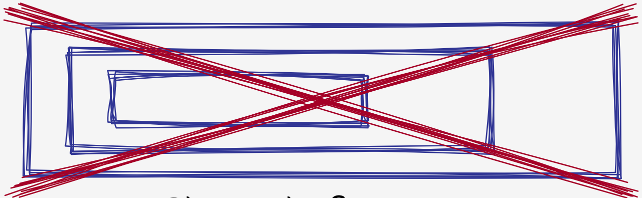

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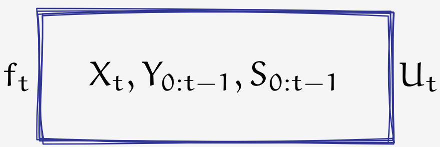

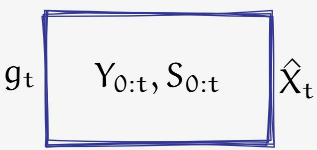

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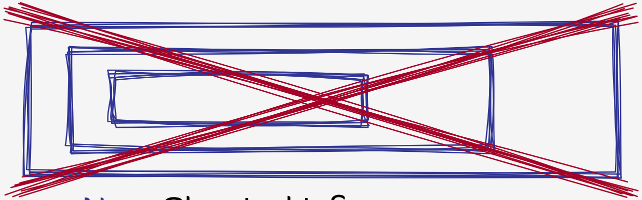

Non-Classical info. struct.

f_t $\boxed{X_t, Y_{0:t-1}, S_{0:t-1}}$ U_t

g_t $\boxed{Y_{0:t}, S_{0:t}}$ $\hat{X}_t$

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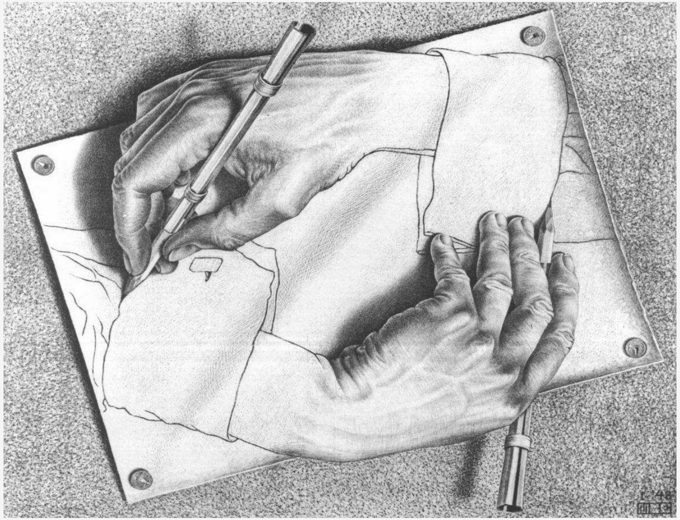


The common information approach

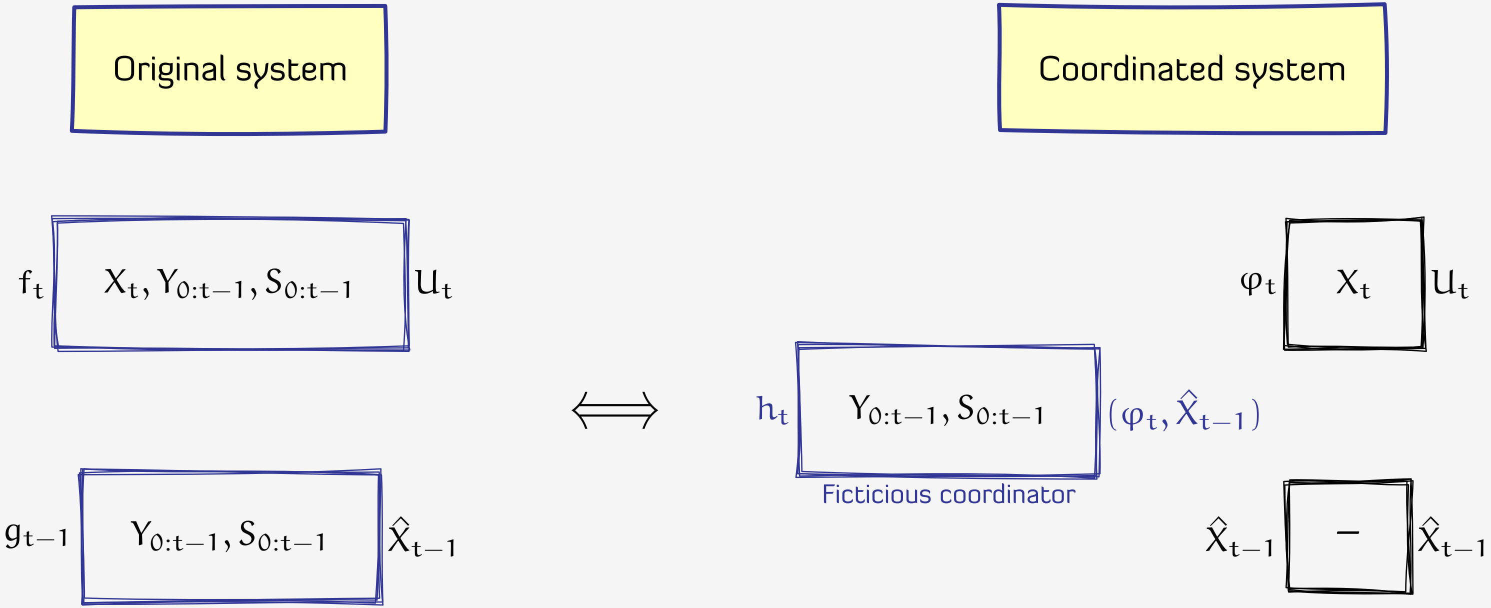
Original system

f_t $X_t, Y_{0:t-1}, S_{0:t-1}$ U_t

g_{t-1} $Y_{0:t-1}, S_{0:t-1}$ $\hat{X}_{t-1}$

▷ Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

The common information approach



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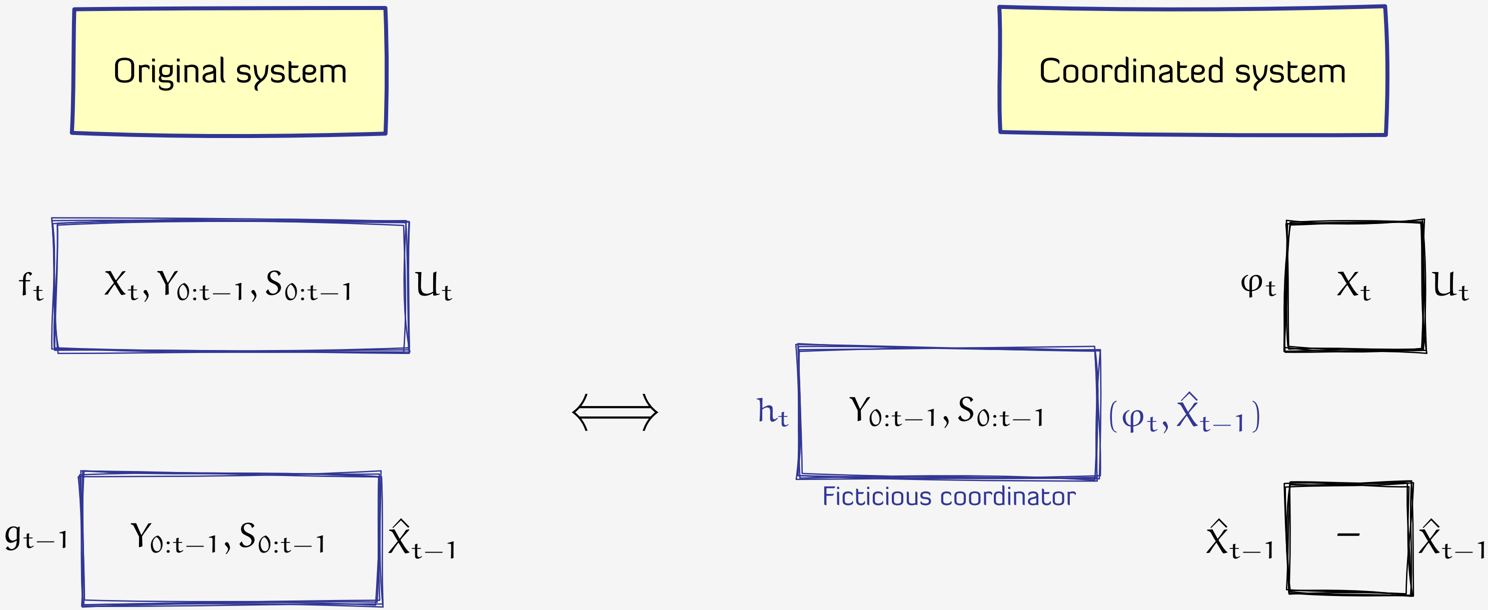


- The coordinated system is equivalent to the original system.

$$f_t(x, y_{0:t-1}, s_{0:t-1}) = h_t^1(y_{0:t-1}, s_{0:t-1})(x).$$

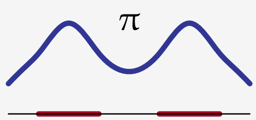
- The coordinated system is centralized. Belief state $\mathbb{P}(X_t \mid Y_{0:t-1}, S_{0:t-1})$.

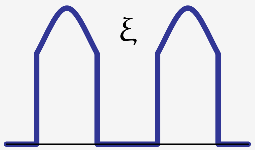
- Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Information states or sufficient statistics

Notation For any $\pi \in \Delta(\mathcal{X})$ and $\varphi \colon \mathcal{X} \to \{0, 1\}$

- $B_i(\varphi) = \{x \in \mathcal{X} : \varphi(x) = i\}, \; i \in \{0, 1\}$
- $\xi = \pi|_\varphi$ means $\xi(x) = \dfrac{\mathbb{1}_0\{\varphi(x)\}\pi(x)}{\pi(B_0(\varphi))}$





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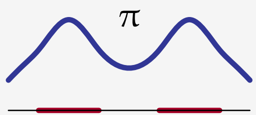

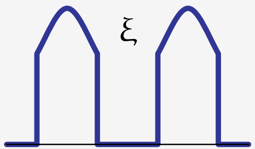

Pre-transmission belief $\pi^1_t(x) = \mathbb{P}(X_t = x | S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}).$

Post-transmission belief $\pi^2_t(x) = \mathbb{P}(X_t = x | S_{0:t} = s_{0:t}, Y_{0:t} = y_{0:t}).$

Information states or sufficient statistics

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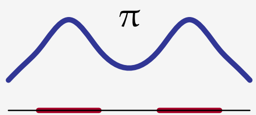
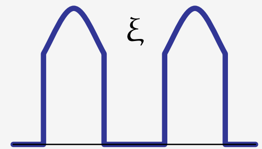




Pre-transmission belief

$$\pi_t^1(x) = \mathbb{P}(X_t = x | S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}).$$

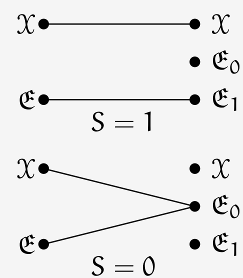
Post-transmission belief

$$\pi_t^2(x) = \mathbb{P}(X_t = x | S_{0:t} = s_{0:t}, Y_{0:t} = y_{0:t}).$$

Belief update

$$\pi_{t+1}^1 = \pi_t^2 P$$

$$\pi_t^2 = F^2(\pi_t^1, \varphi_t, y_t) = \begin{cases} \delta_{y_t}, & \text{if } y_t \in \mathcal{X} \\ \pi_t^1|_{\varphi_t}, & \text{if } y_t = \mathfrak{E}_1 \\ \pi_t^1, & \text{if } y_t = \mathfrak{E}_0 \end{cases}$$



Dynamic program

$$V^1_{T+1}(s, \pi^1) = 0$$

and for $t \in \{T, \dots, 0\}$

$$V^1_t(s, \pi^1) = \min_{\varphi : \mathcal{X} \to \{0,1\}} \Big\{ \lambda \pi^1(B_1(\varphi)) + \pi^1(B_0(\varphi)) W^0_t(\pi^1, \varphi) + \sum_{x \in B_1(\varphi)} \pi^1(x) W^1_t(\pi^1, \varphi, x) \Big\}$$

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Salient features

- Minimization over functions φ
- Similar to DP for POMDPs. Can be solved using similar numerical techniques.

Can we use the DP to say something more about the optimal strategy?

Simplifying modeling assumptions

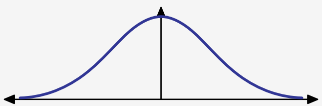
Markov process

$X_{t+1} = aX_t + W_t$

- Discrete state process: $X_t,\ a,\ W_t \in \mathbb{Z}$
- Continuous state process: $X_t,\ a,\ W_t \in \mathbb{R}$

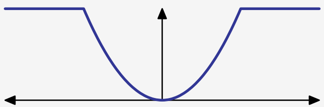
Noise Distribution

Unimodal and symmetric

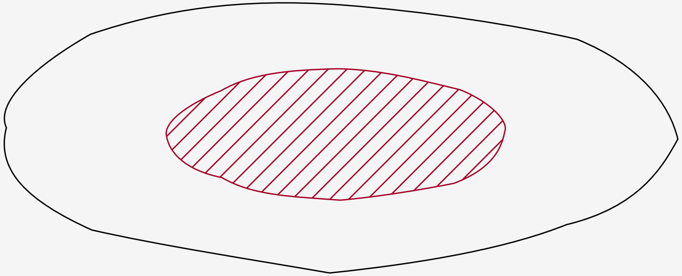


Distortion function

Symmetric and quasi-convex



Step 1 Threshold strategies are optimal



Search space of strategies (f, g)

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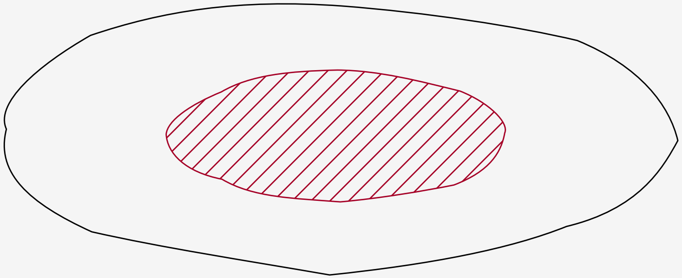
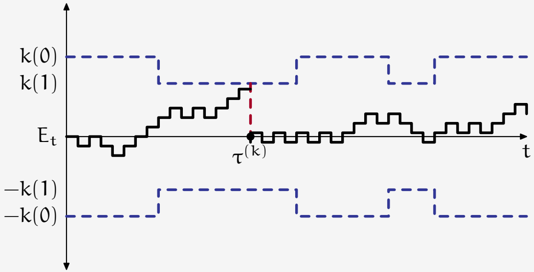


Search space of strategies (f, g)

Step 2 Performance of threshold strategies



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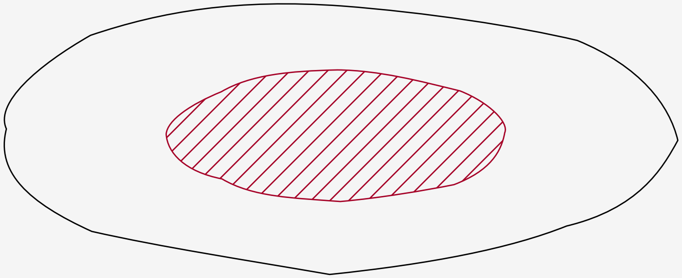

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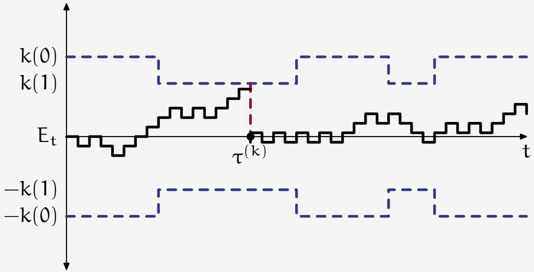
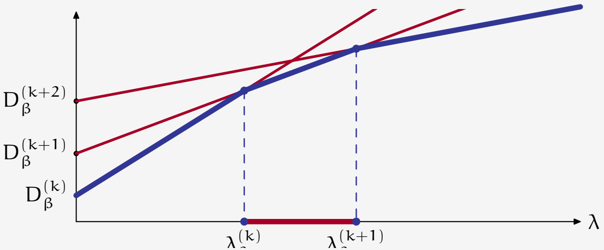


Step 3 Optimal costly communication



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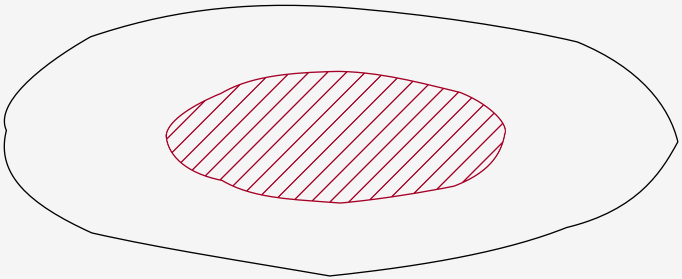

Search space of strategies (f, g)

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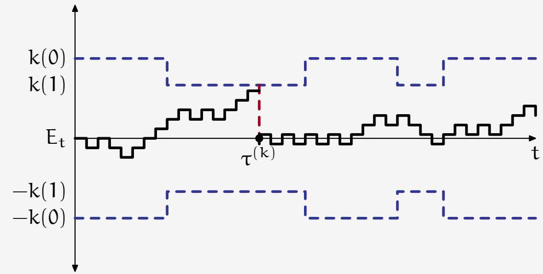

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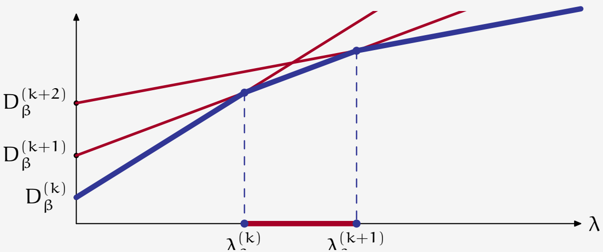
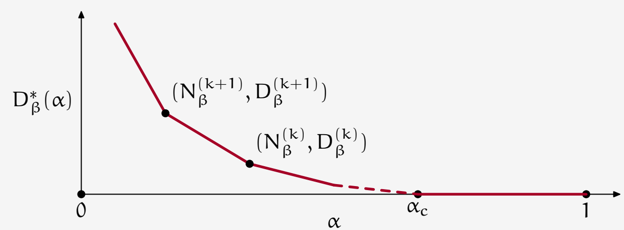


Step 4 Optimal constrained communication



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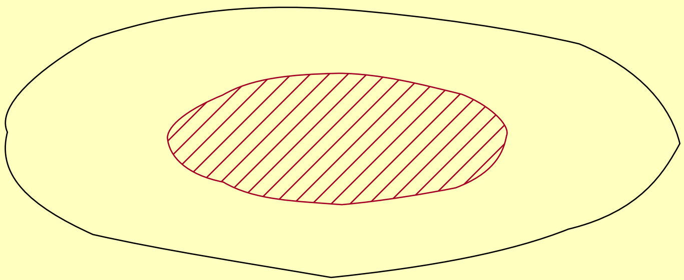

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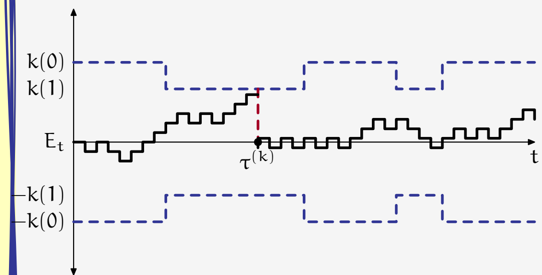

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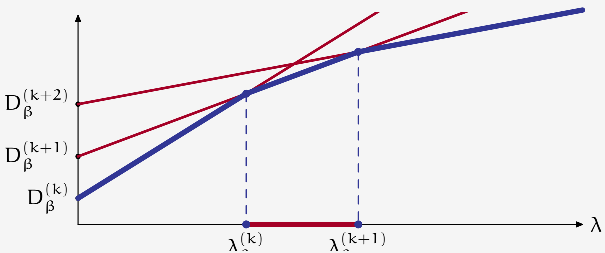

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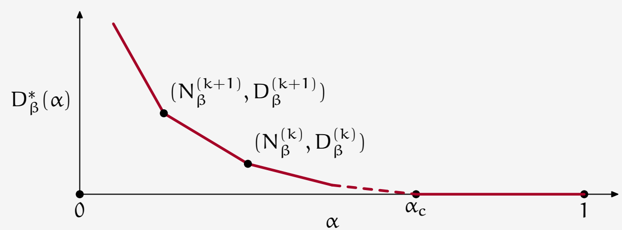

Step 1 A change of variables

Define $$Z_0 = 0 \text{ and } Z_t = \begin{cases} aZ_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$ (Observable at both Tx and Rx)

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$$E_t = X_t - aZ_{t-1}, \quad E_t^+ = X_t - Z_t, \quad \hat{E}_t = \hat{X}_t - Z_t$$

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Thus, these are related as

$$E_t^+ = \begin{cases} E_t, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ 0, & \text{if } Y_t \in \mathcal{X} \end{cases} \qquad \text{and} \qquad E_{t+1} = aE_t^+ + W_t$$

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Note $X_t - \hat{X}_t = E_t^+ - \hat{E}_t$ and hence $d(X_t - \hat{X}_t) = d(E_t^+ - \hat{E}_t)$.

Implication of change of variables

Pre-transmission belief $\pi_t^1(e) = \mathbb{P}(E_t = e | S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1}).$

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Pre-transmission belief $\pi^1_t(e) = \mathbb{P}(E_t = e | S_{0:t-1} = s_{0:t-1}, Y_{0:t-1} = y_{0:t-1})$.

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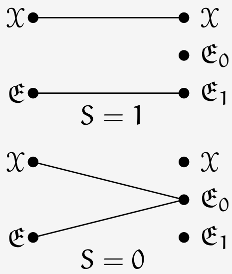
Belief update

$$\pi^1_{t+1} = \pi^2_t P$$

$$\pi^2_t = F^2(\pi^1_t, \varphi_t) = \begin{cases} \delta_0, & \text{if } y_t \in \mathcal{X} \\ \pi^1_t|_{\varphi_t}, & \text{if } y_t = \mathfrak{E}_1 \\ \pi^1_t, & \text{if } y_t = \mathfrak{E}_0 \end{cases}$$



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$$V^1_t(s, \pi^1) = \min_{\varphi : \mathcal{X} \to \{0,1\}} \Big\{ \lambda \pi^1(B_1(\varphi)) + \pi^1(B_0(\varphi)) W^0_t(\pi^1, \varphi) + \sum_{x \in B_1(\varphi)} \pi^1(x) W^1_t(\pi^1, \varphi, x) \Big\}$$

$$V^2_t(s, \pi^2) = \min_{\hat{x} \in \mathcal{X}} \sum_{x \in \mathcal{X}} \pi^2(x) d(x, \hat{x}) + V^1_{t+1}(s, \pi^2 P)$$

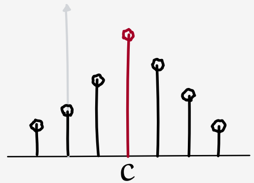
where $W^0_t(\pi^1, \varphi) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \pi^1|_\varphi)$

$$W^1_t(\pi^2, \varphi, x) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \delta_x) \quad Q_{s1} V^2_t(1, \delta_0)$$

Step 1 Preliminaries: Majorization

[Hajek Mitzel Yang 2008, Lipsa Martins 2011, Nayyar et. al. 2013]

Almost uniform and unimodal (ASU) distribution about c



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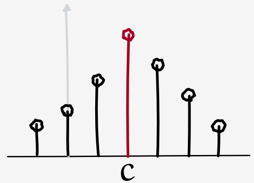

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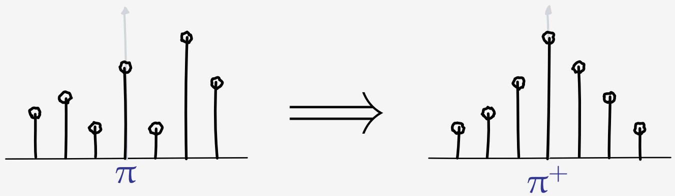
ASU Rearrangement



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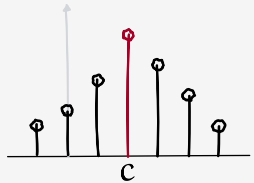

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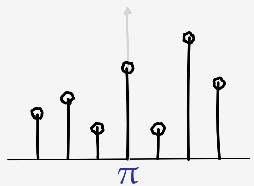
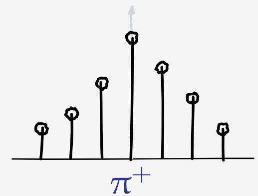
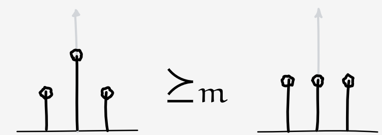


$\Longrightarrow$





Majorization

$$\xi \succeq_m \pi \text{ iff } \sum_{i=-n}^{n} \xi_i^+ \geqslant \sum_{i=-n}^{n} \pi_i^+ \quad \text{and} \quad \sum_{i=-n}^{n+1} \xi_i^+ \geqslant \sum_{i=-n}^{n+1} \pi_i^+$$

Invariant to permutations.

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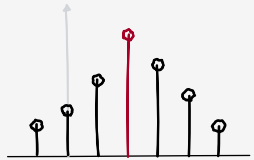

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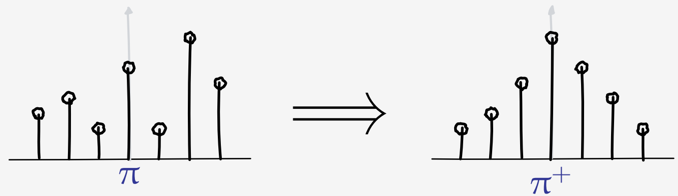

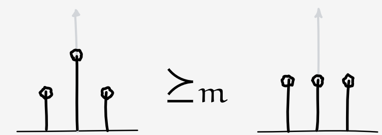

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$\xi \succeq_m \pi$ iff $\sum_{i=-n}^{n} \xi_i^+ \geqslant \sum_{i=-n}^{n} \pi_i^+$ and $\sum_{i=-n}^{n+1} \xi_i^+ \geqslant \sum_{i=-n}^{n+1} \pi_i^+$

Invariant to permutations.

ASU Majorization

$\xi \succeq_a \pi$ iff ξ is ASU and $\xi \succeq_m \pi$

Step 1 Implication of Majorization

Recall DP

$$V_t^1(s, \pi^1) = \min_{\varphi : \mathcal{X} \to \{0,1\}} \{\lambda \pi^1(B_1(\varphi)) + \pi^1(B_0(\varphi)) W_t^0(\pi^1, \varphi) + \pi^1(B_1(\varphi)) W_t^1(\pi^1, \varphi)\}$$

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Proposition

V_t^1 and V_t^2 satisfy the following property:

- For any $s \in \{0, 1\}$ and $\pi \succeq_a \xi$, then $V_t^i(s, \pi) \geqslant V_t^i(s, \xi)$
  (Similar to Schur convexity, so we call it ASU Schur convexity)

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Theorem

There is no loss optimality in restricting attention to threshold based transmission strategies and using estimation strategies of form

$$\hat{E}_t = \begin{cases} 0, & \text{if } Y_t\in\mathcal{X} \\ aE_{t-1}, & \text{if } Y_t\in\{\mathfrak{E}_0,\mathfrak{E}_1\} \end{cases}$$

Structure of optimal strategies

Theorem For the infinite horizon costly communication problem, we have the following:

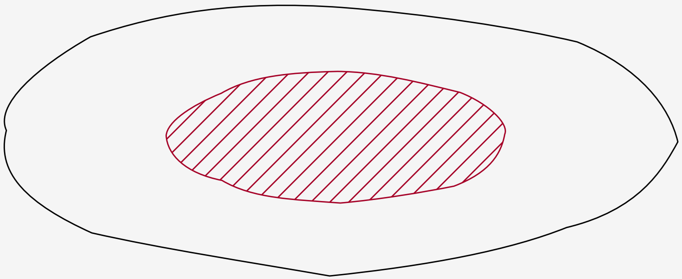
- **Structure of optimal estimation strategies**: The optimal estimation strategy is $\hat{X}_0 = 0$ and for $t > 0$

$$\hat{X}_t = \begin{cases} Y_t, & \text{if } Y_t \in \mathcal{X} \\ a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \end{cases}$$

- **Structure of optimal transmission strategy**: There exist time-invariant thresholds $k(0), k(1) \in \mathcal{X}$ such that the strategy

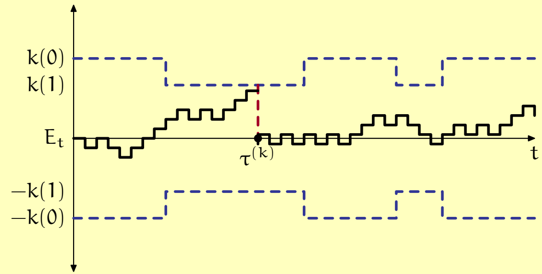
$$U_t = \begin{cases} 1, & \text{if } |X_t - aX_{t-1}| \geqslant k(S_{t-1}) \\ 0, & \text{otherwise} \end{cases}$$

Step 1 Threshold strategies are optimal

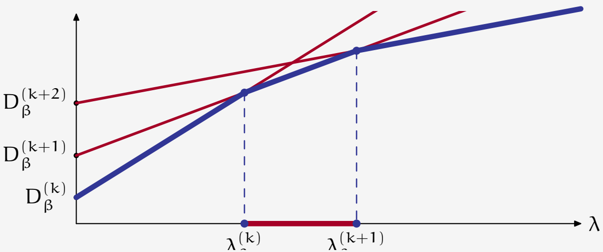


Search space of strategies (f, g)

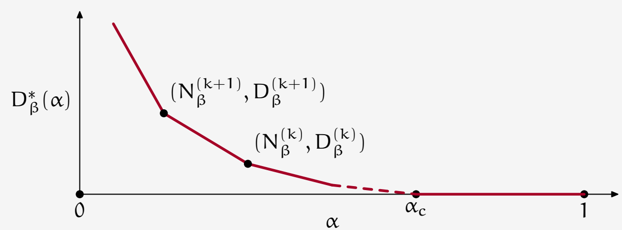
Step 2 Performance of threshold strategies



Step 3 Optimal costly communication



Step 4 Optimal constrained communication



Step 2 Performance of threshold-based strategies

Consider a threshold-based strategy

$$f^{(k)}(e, s) = \begin{cases} 1 & \text{if } |e| \geqslant k(s) \\ 0 & \text{otherwise} \end{cases}$$

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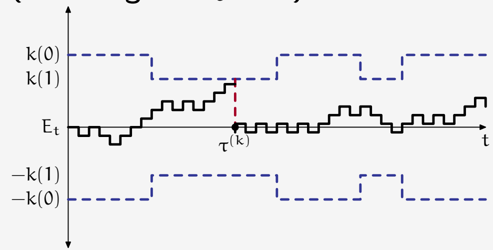
Let $\tau^{(k)}$ denote the stopping time of first reception (starting at $E_0 = 0$).



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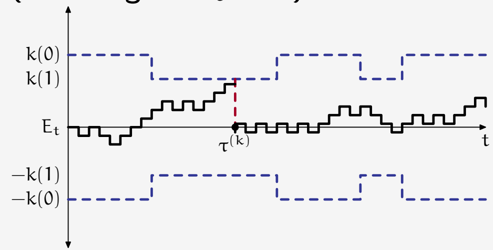

Define

$$L_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big].$$ (Distortion until first reception)

$$M_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Big| E_0 = e\Big].$$ (Time until the first reception)

$$K_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}} \beta^t U_t \Big| E_0 = e\Big].$$ (Transmissions until the first reception)

n

Proposition $\{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal relationships, we have:

$$D_\beta^{(k)} \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)}$$

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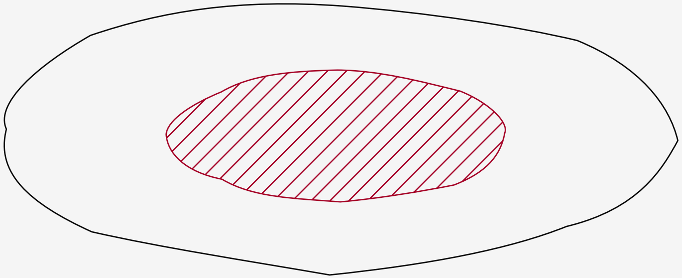
Computing $L_\beta^{(k)}$, $M_\beta^{(k)}$, $K_\beta^{(k)}$ is sufficient to compute the performance of $f^{(k)}$ (i.e., to compute $D_\beta^{(k)}$ and $N_\beta^{(k)}$).

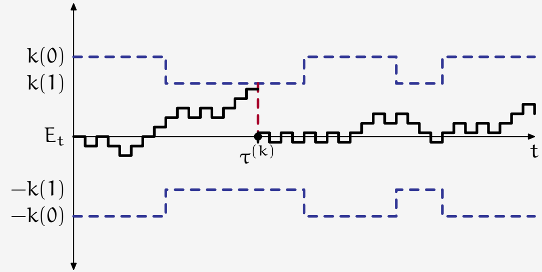
These can be computed using standard Markov chain formulas.

Step 1 Threshold strategies are optimal

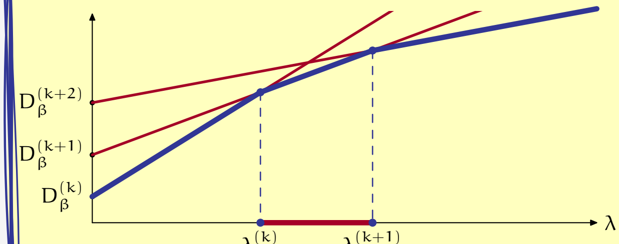


Search space of strategies (f, g)

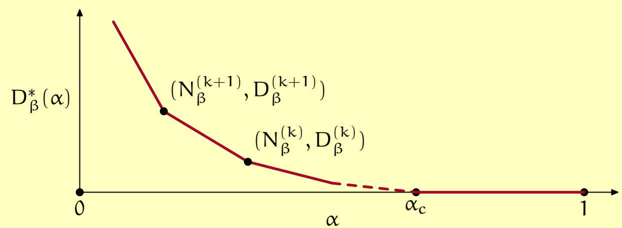
Step 2 Performance of threshold strategies



Step 3 Optimal costly communication



Step 4 Optimal constrained communication

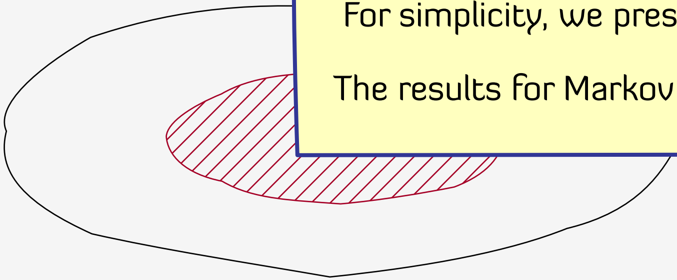


Step 1 Threshold strategies are optimal

Step 2 Performance of threshold strategies

For simplicity, we present the results for i.i.d. packet drops.

The results for Markov packet drops are similar (but harder to describe).



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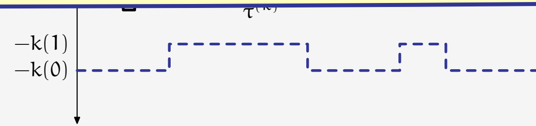

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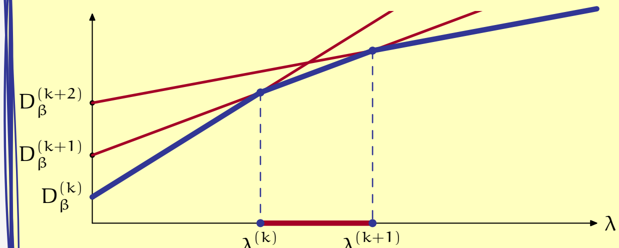

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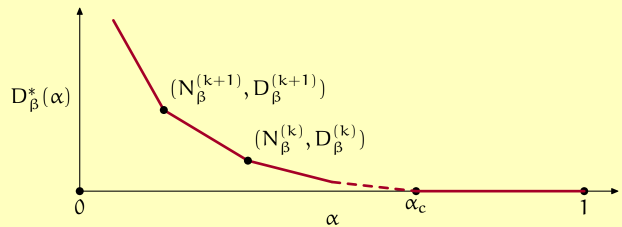

Step 3 Solution to costly comm. for discrete sources

Proposition

- $C^{(k)}_\beta(\lambda) \coloneqq D^{(k)}_\beta + \lambda N^{(k)}_\beta$ is submodular in (k, λ).
- Hence, $k^*_\beta(\lambda) \coloneqq \arg\min_{k \geq 0} C^{(k)}_\beta(\lambda)$ is increasing in λ

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Proposition

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- Hence, $k_\beta^*(\lambda) \coloneqq \arg\min_{k \geqslant 0} C_\beta^{(k)}(\lambda)$ is increasing in λ

Define $\Lambda_\beta^{(k)} \coloneqq \{\lambda \in \mathbb{R}_{\geqslant 0} : k_\beta^*(\lambda) = k\}$
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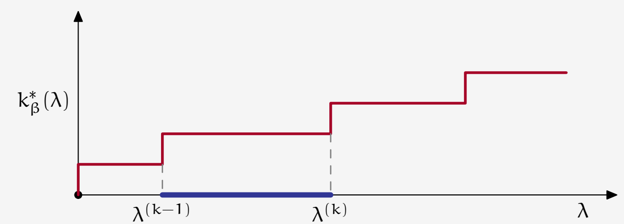

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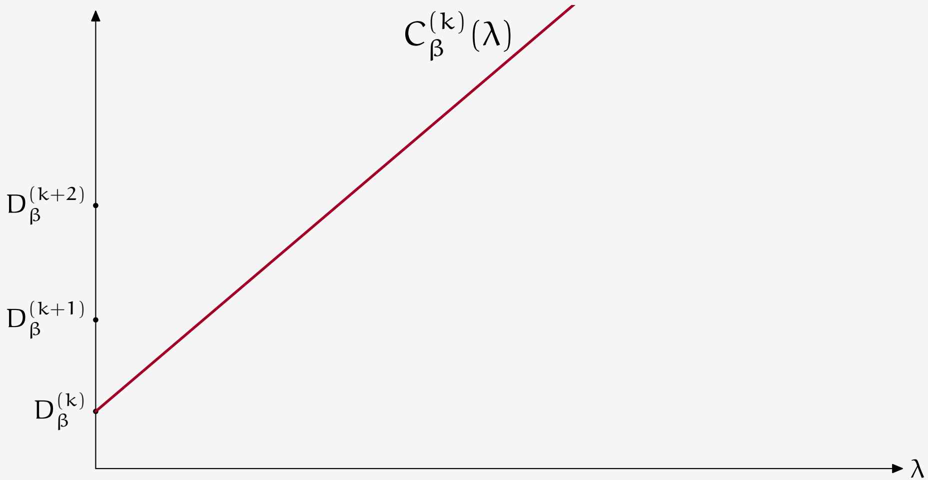
Solution to costly comm. for discrete sources



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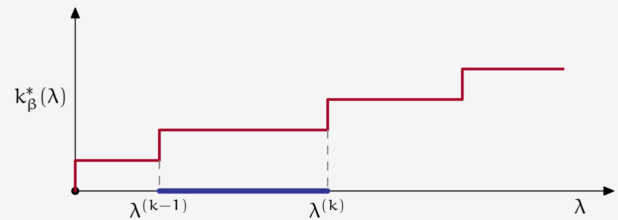

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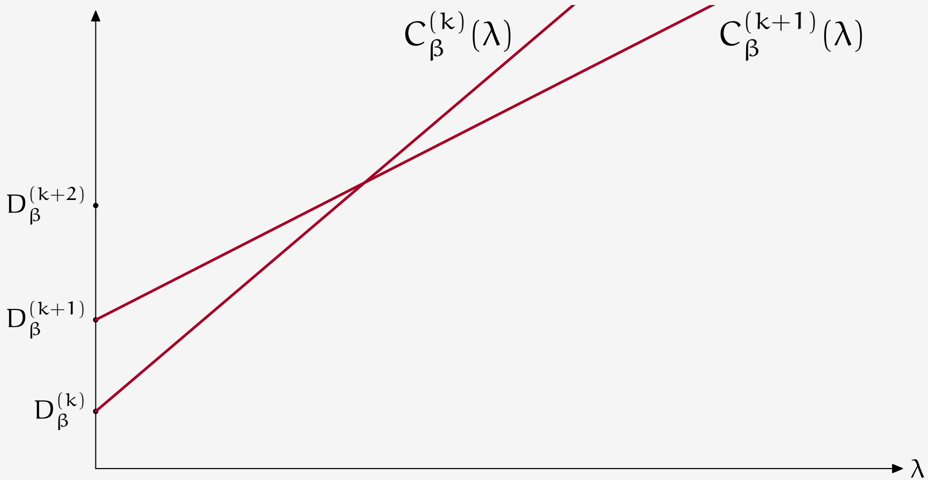

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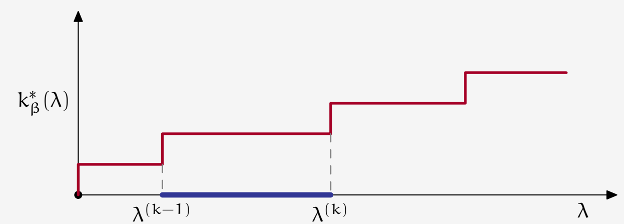

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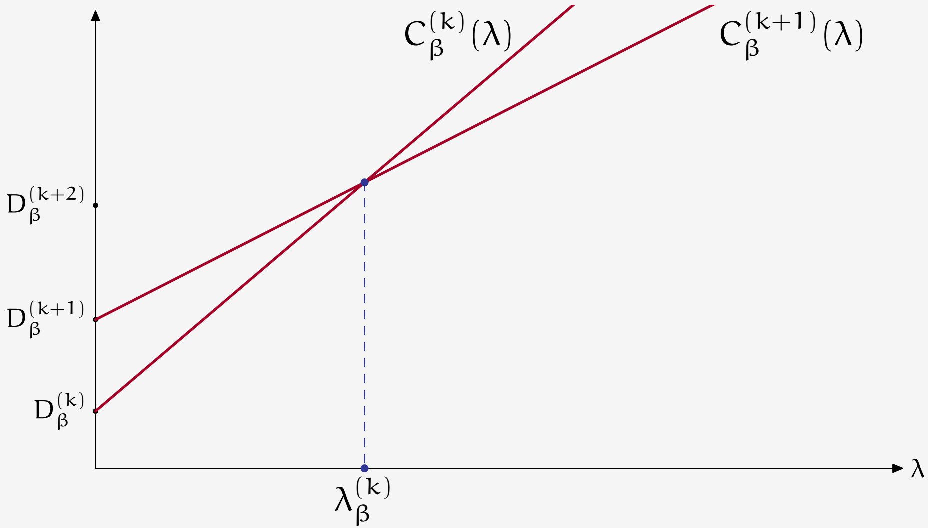

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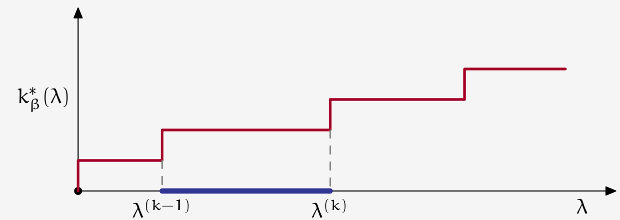

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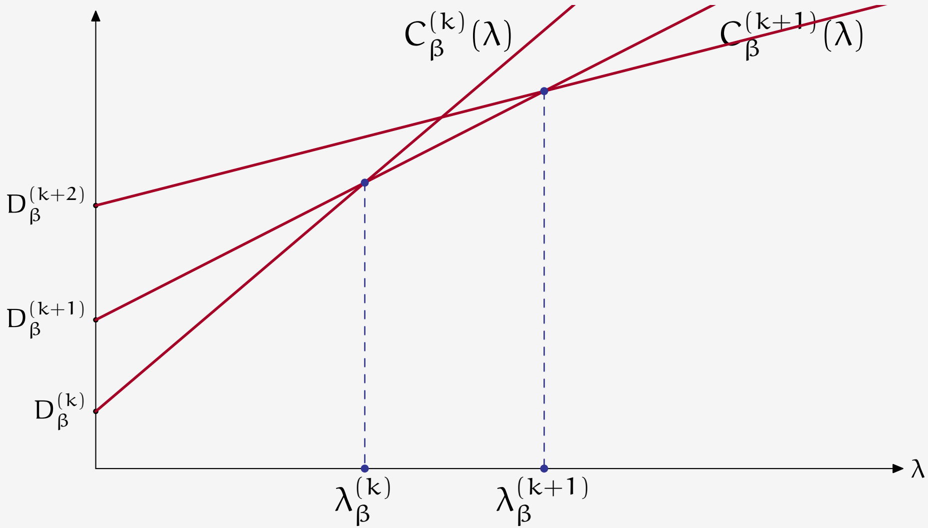

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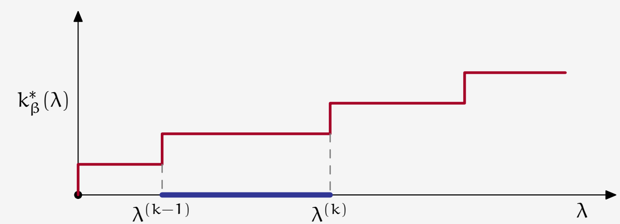

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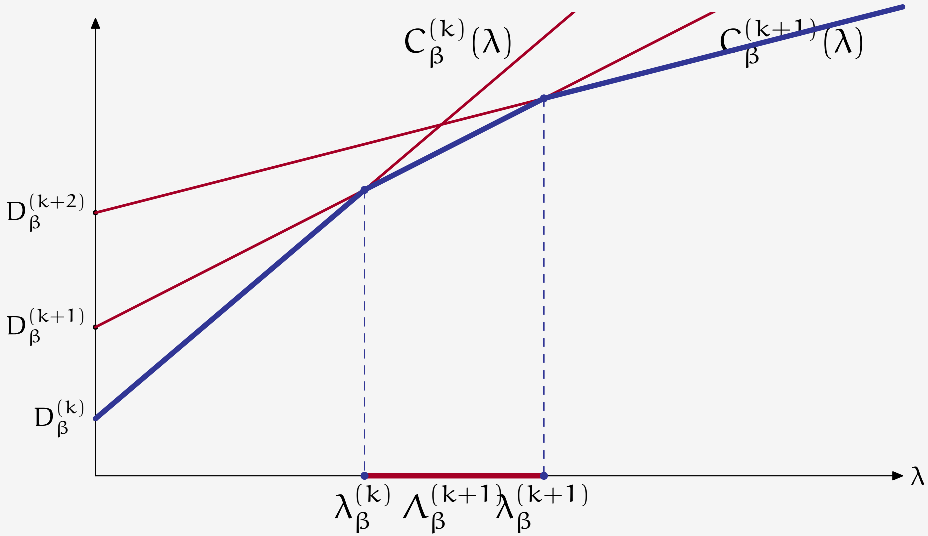

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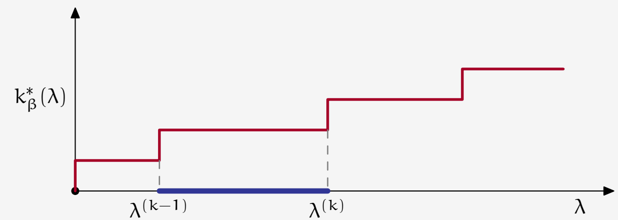
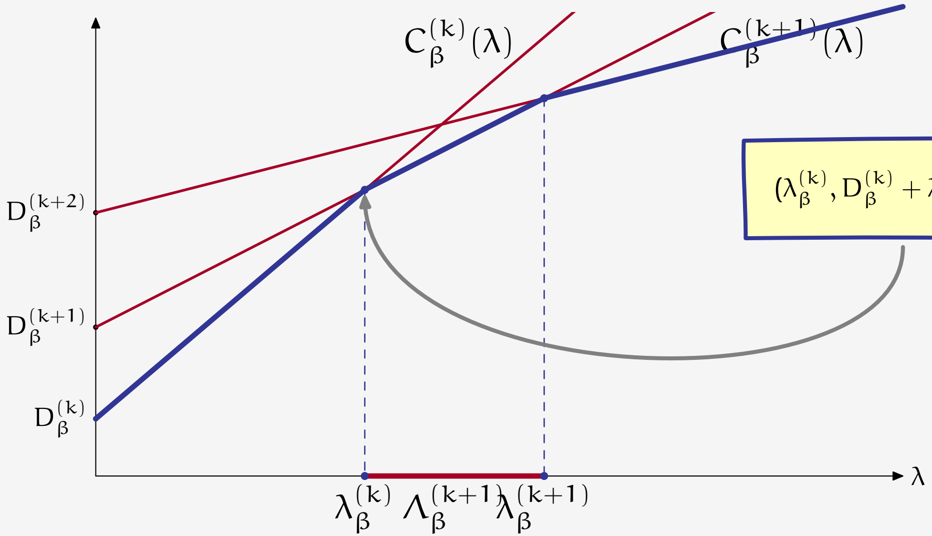
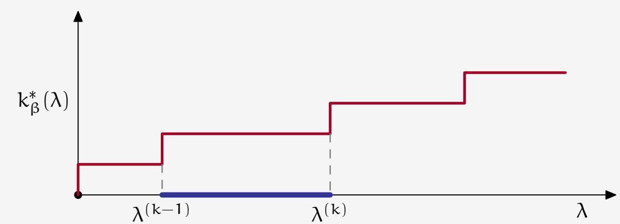


$$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$$
$$\implies \lambda_\beta^{(k)} = \big(D_\beta^{(k+1)} - D_\beta^{(k)}\big) / \big(N_\beta^{(k)} - N_\beta^{(k+1)}\big).$$

Step 3 Solution to costly comm. for discrete sources



Define $\Lambda_\beta^{(k)} := \{\lambda \in \mathbb{R}_{\geqslant 0} : k_\beta^*(\lambda) = k\}$
$= [\lambda_\beta^{(k-1)}, \lambda_\beta^{(k)}].$



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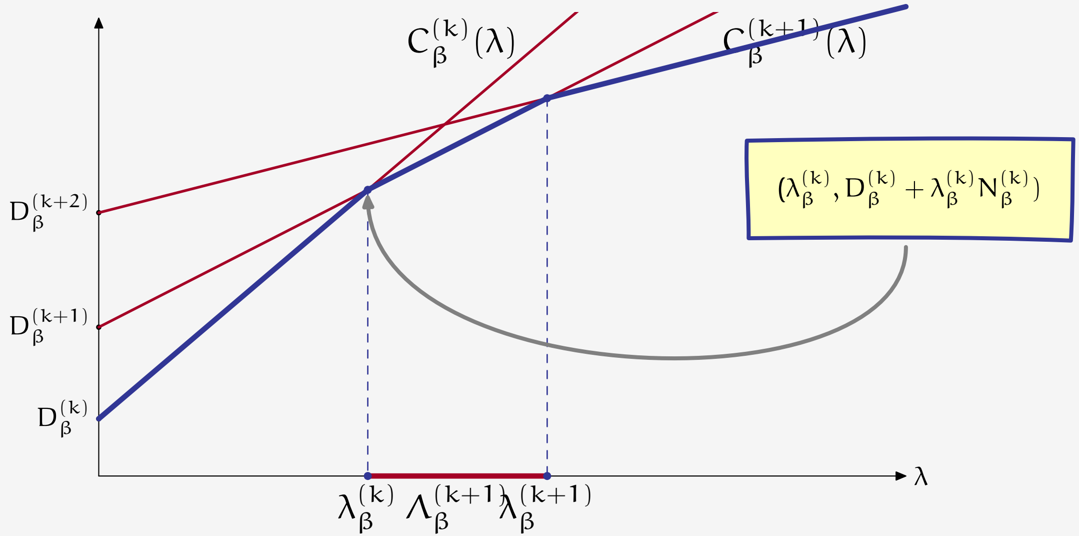

Theorem

Strategy $f^{(k+1)}$ is optimal for $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$.

$C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

Step 4 Solution to constrained comm. for discrete sources

Sufficient condition for optimality

A strategy (f°, g°) is optimal for the constrained problem if

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$

(C2) There exists $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for the Lagrange relaxation with parameter λ°.

Step 4 Solution to constrained comm. for discrete sources

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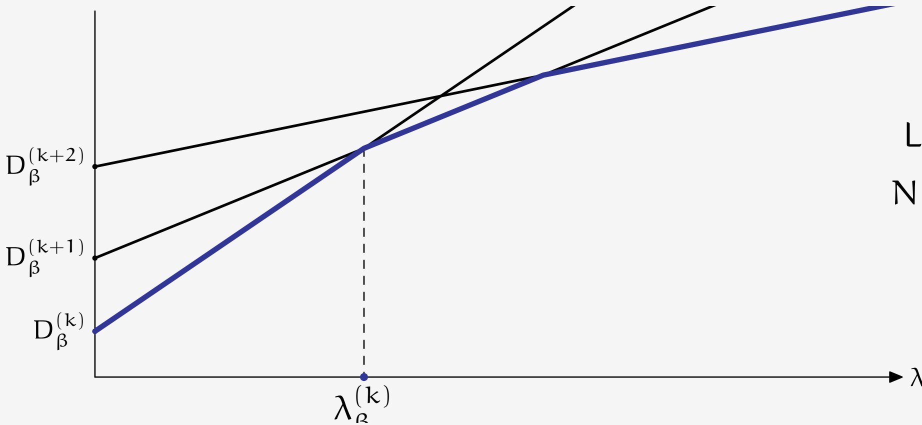

Let k_β^* be such that
$N_\beta^{(k_\beta^*)} > \alpha > N_\beta^{(k_\beta^*+1)}$

Solution to constrained comm. for discrete sources

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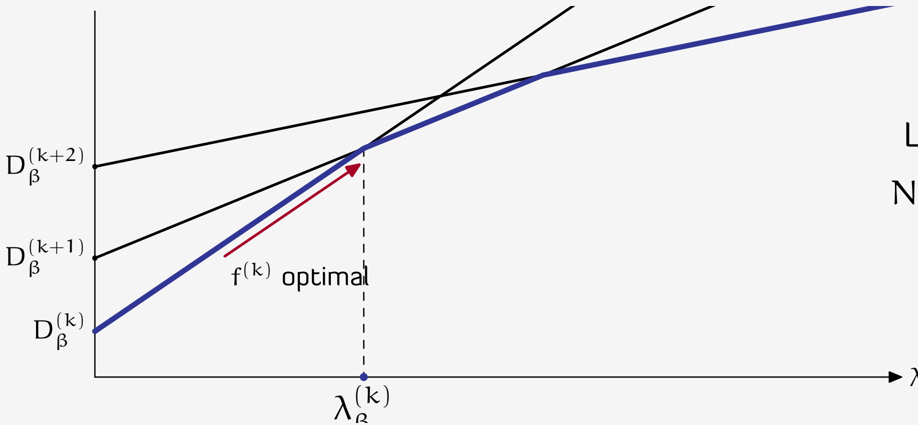

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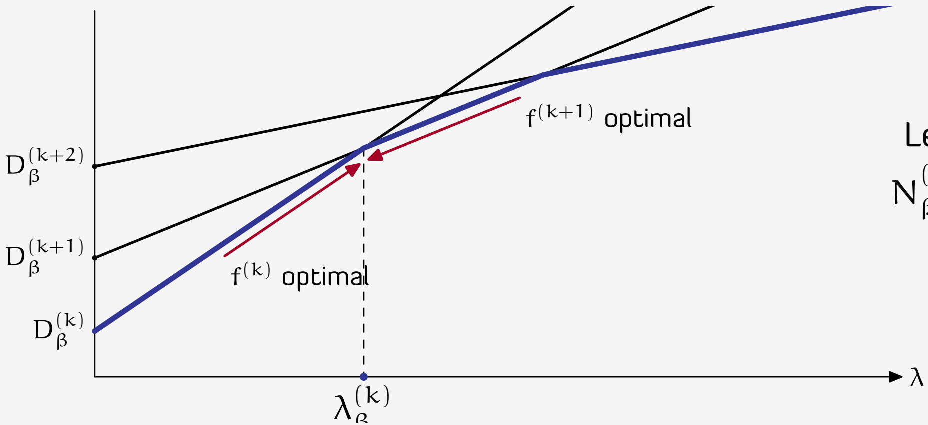

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Step 4 Solution to constrained comm. for discrete sources

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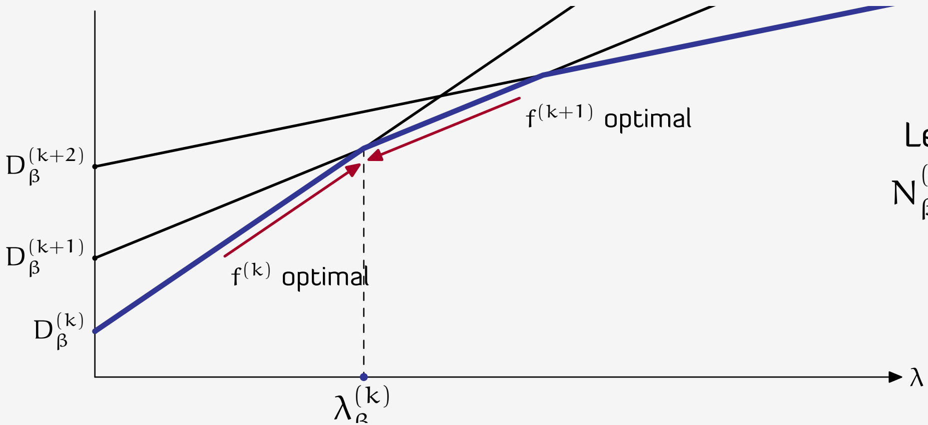
Randomized strategy $(\theta^*, f^{(k)}, f^{k+1})$ is optimal where

$$\theta^* N_\beta^{(k)} + (1 - \theta^*) N_\beta^{(k+1)} = \alpha$$



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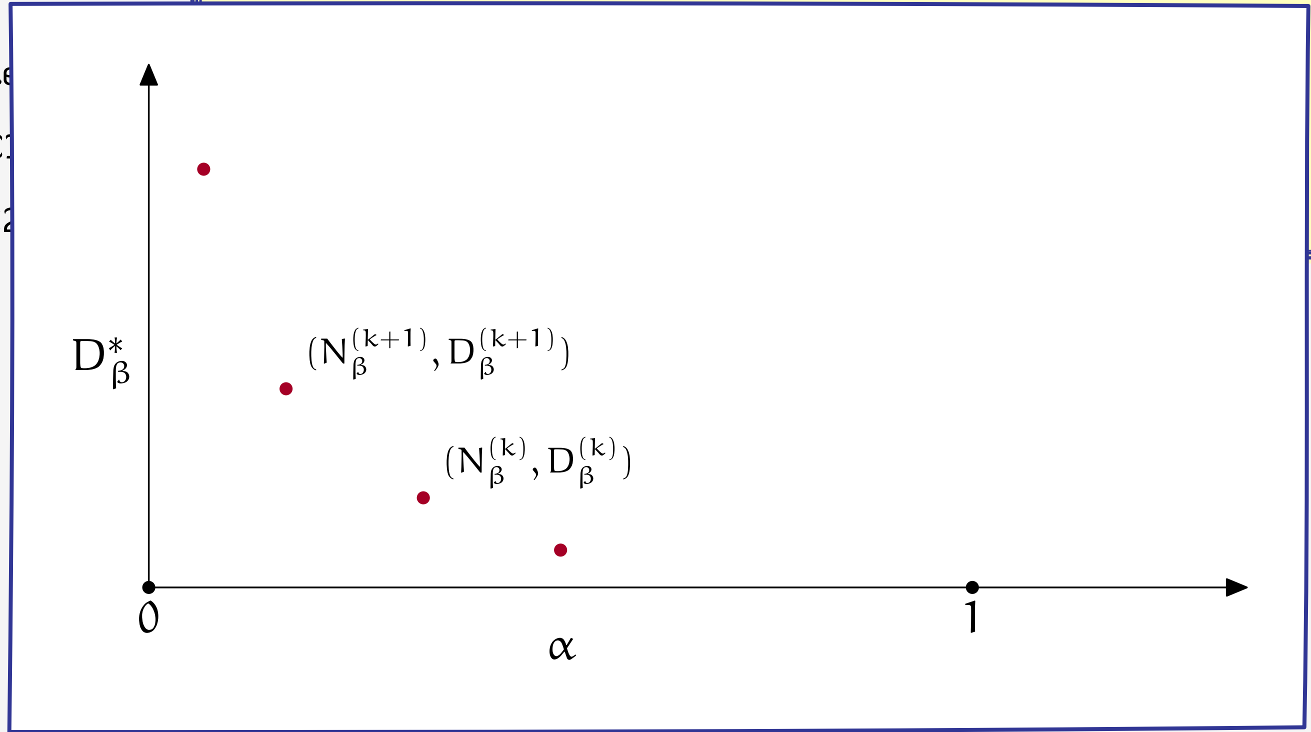

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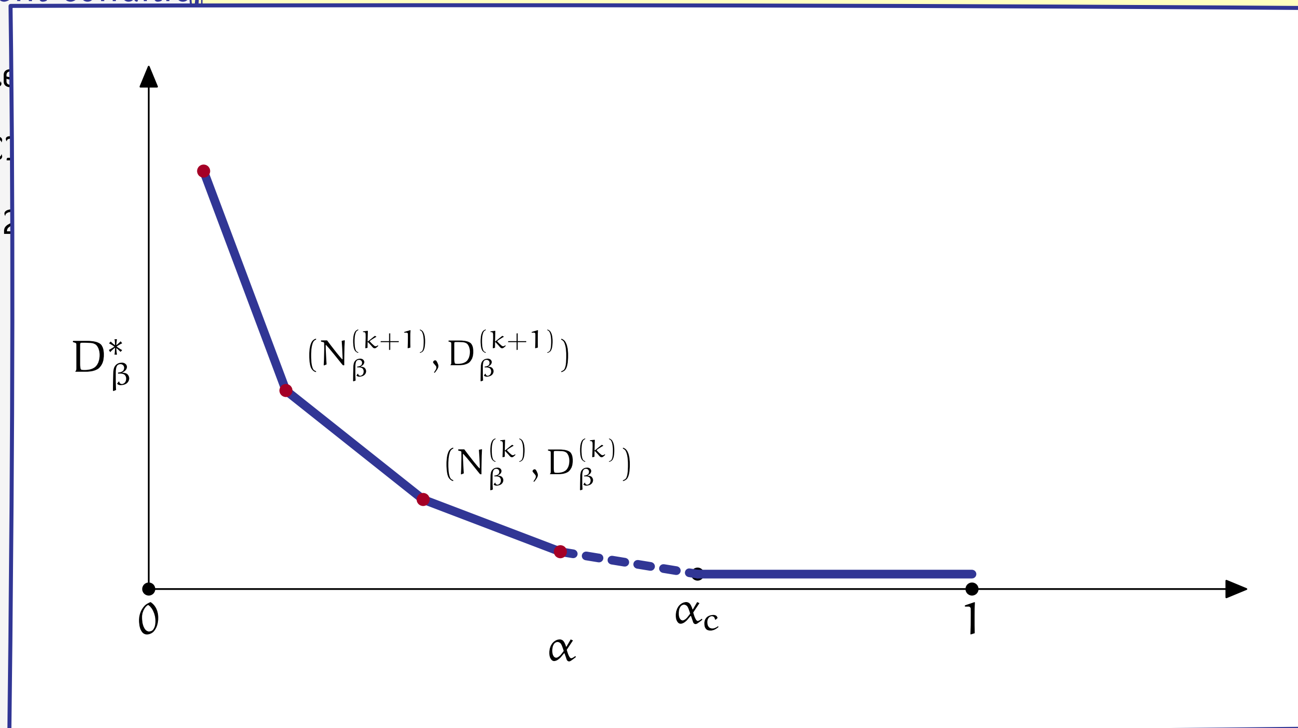

$\lambda_\beta^{(k)}$

Step 3 Solution to costly communication for continuous sources

Proposition As in the case of discrete sources:

- $C^{(k)}_\beta(\lambda) \coloneqq D^{(k)}_\beta + \lambda N^{(k)}_\beta$ is submodular in (k, λ).
- Hence, $k^*_\beta(\lambda) \coloneqq \arg\min_{k \geqslant 0} C^{(k)}_\beta(\lambda)$ is increasing in λ

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Theorem

If the pair (λ, k) satisfies

$$\lambda = -\frac{\partial_k D^{(k)}_\beta}{\partial_k N^{(k)}_\beta} \quad (\text{i.e., } \partial_k D^{(k)}_\beta + \lambda \partial_k N^{(k)}_\beta = 0)$$

then the strategy $(f^{(k)}, g^*)$ is optimal for the costly communication with cost λ.

The optimal performance $C^*_\beta(\lambda)$ is continuous, concave and increasing function of λ.

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Scaling with variance for Gaussian noise

$$C^*_{\beta,\sigma}(\lambda) = \sigma^2 C^*_{\beta,1}\left(\frac{\lambda}{\sigma^2}\right).$$

Step 4 Solution to constrained communication for continuous sources

Theorem For any $\beta \in (0, 1]$ and $\alpha \in (0, 1)$, let $k^*_\beta(\alpha)$ be such that

$$N_\beta^{(k^*_\beta(\alpha))} = \alpha.$$

Such a $k^*_\beta(\alpha)$ always exists and we have the following:

- The strategy $(f^{(k^*_\beta(\alpha))}, g^*)$ is optimal for the constrained optimization problem with constraint α

  (For the Markov packet drop case, we need to check additional KKT conditions)

- The distortion transmission function $D^*_\beta(\alpha)$ is continuous, convex, and decreasing in α and is given by

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- Pick a threshold k and use strategy $f^{(k)}$ until first successful reception.
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Computation of optimal thresholds

Kiefer-Wolfowitz Algorithm

Robbins-Monro Algorithm

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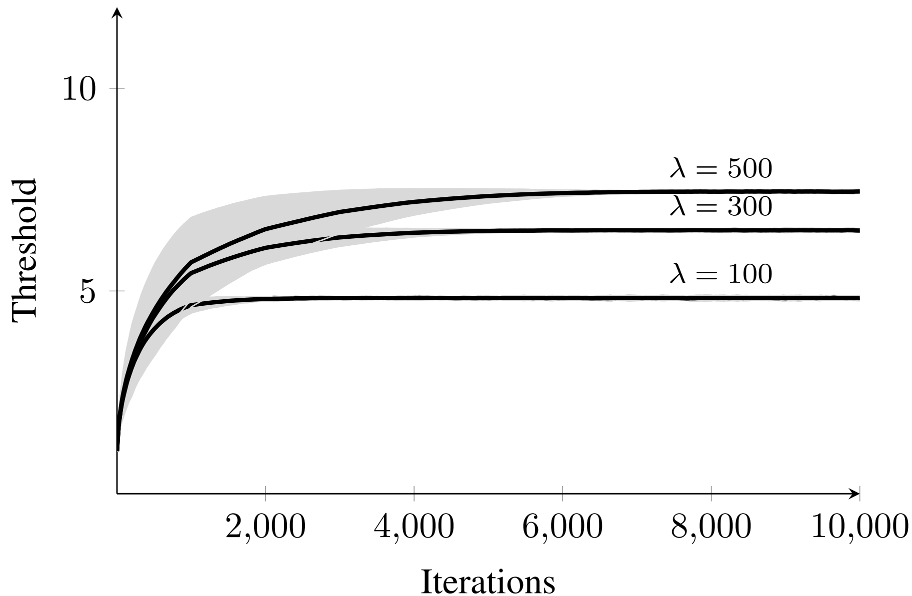

Costly communication ($\beta = 1.0$, i.i.d. packet drop $\varepsilon = 0.3$)

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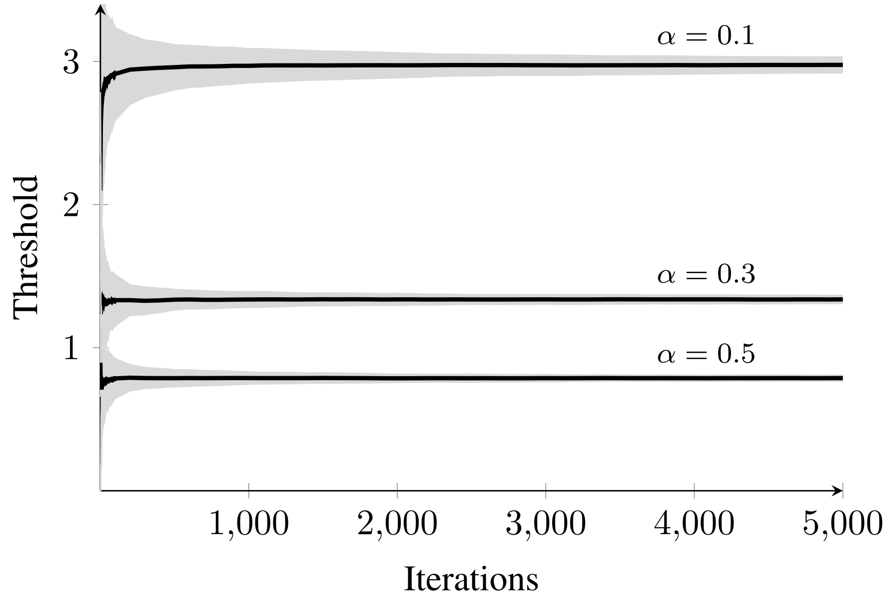
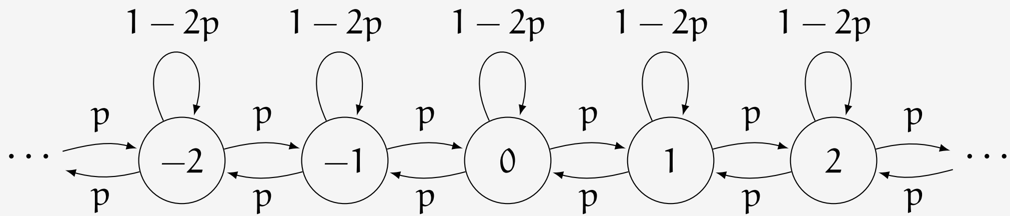


Constrained communication ($\beta = 1.0$, i.i.d. packet drop $\varepsilon = 0.3$)

Examples: Birth-death Markov chain and Gauss-Markov process

Example: Symmetric birth-death Markov chain (perfect channel)

$$P_{ij} = \begin{cases} p, & \text{if } |i-j| = 1; \\ 1-2p, & \text{if } i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{2}), \qquad d(e) = |e|$$



Example Symmetric birth-death Markov chain (perfect channel)

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Discounted cost

Let $K_\beta = -2 - (1-\beta)/\beta p$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$.

$$D_\beta^{(k)} = \frac{\sinh(k m_\beta) - k \sinh(m_\beta)}{2 \sinh^2(k m_\beta/2) \sinh(m_\beta)}$$

$$N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2) \cosh(k m_\beta)}{\sinh^2(k m_\beta/2)} - (1-\beta)$$

Average cost

$$D_1^{(k)} = \frac{k^2-1}{3k} \quad \text{and} \quad N_1^{(k)} = \frac{2p}{k^2}$$

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$\lambda_\beta^{(k)}$ can be computed in terms of $D_\beta^{(k)}$ and $N_\beta^{(k)}$.

Average cost

$D_1^{(k)} = \dfrac{k^2-1}{3k}$ and $N_1^{(k)} = \dfrac{2p}{k^2}$

$$\lambda_1^{(k)} = \frac{k(k+1)(k^2+k+1)}{6p(2k+1)}$$

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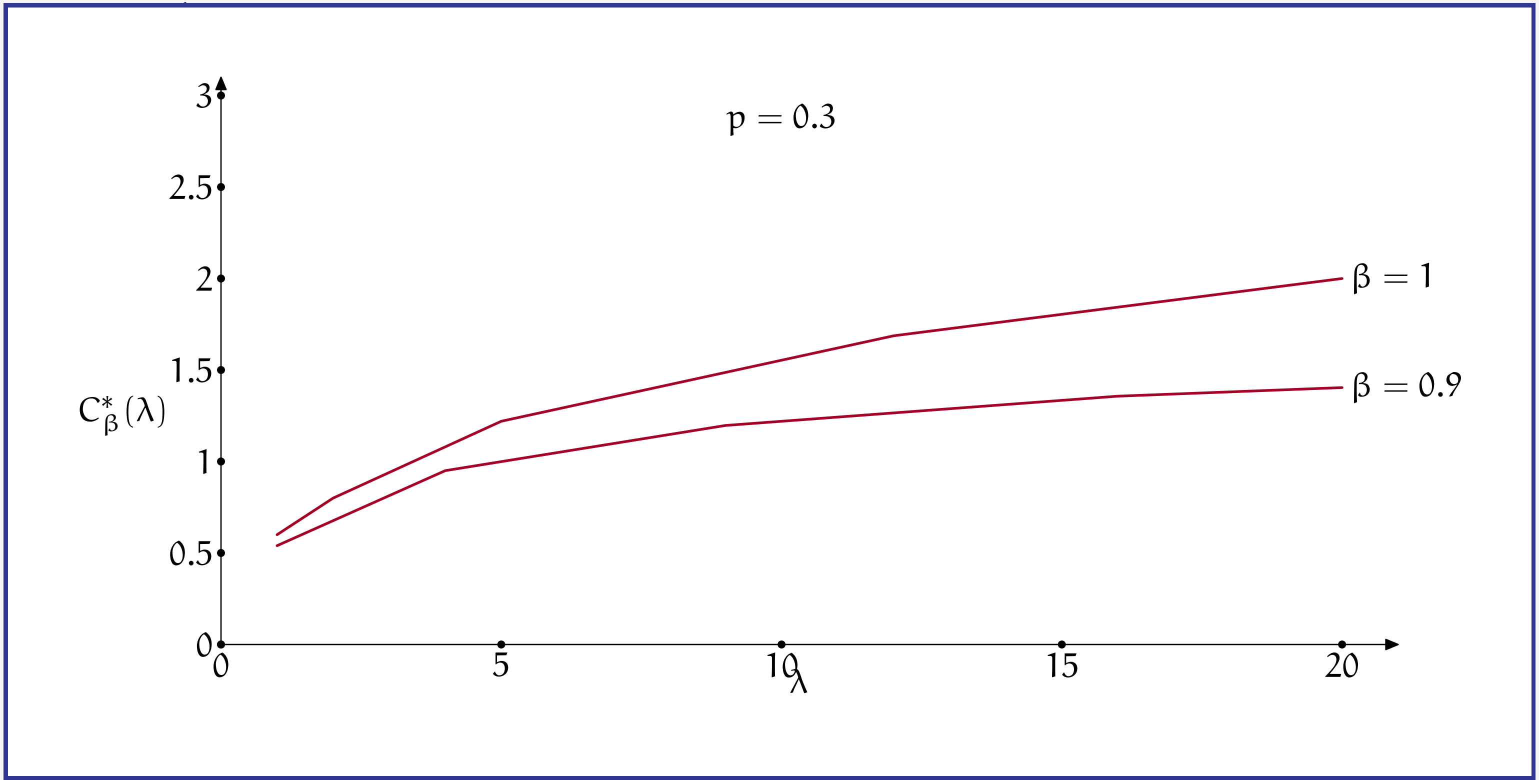

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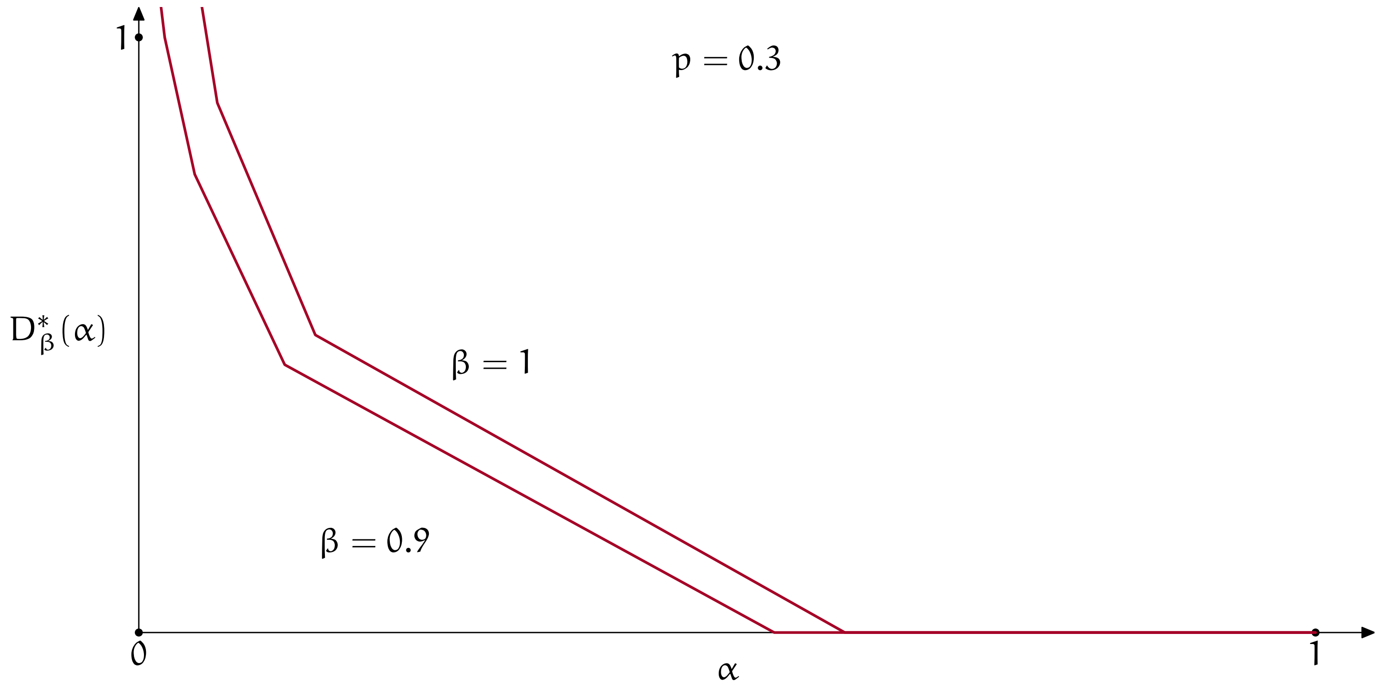
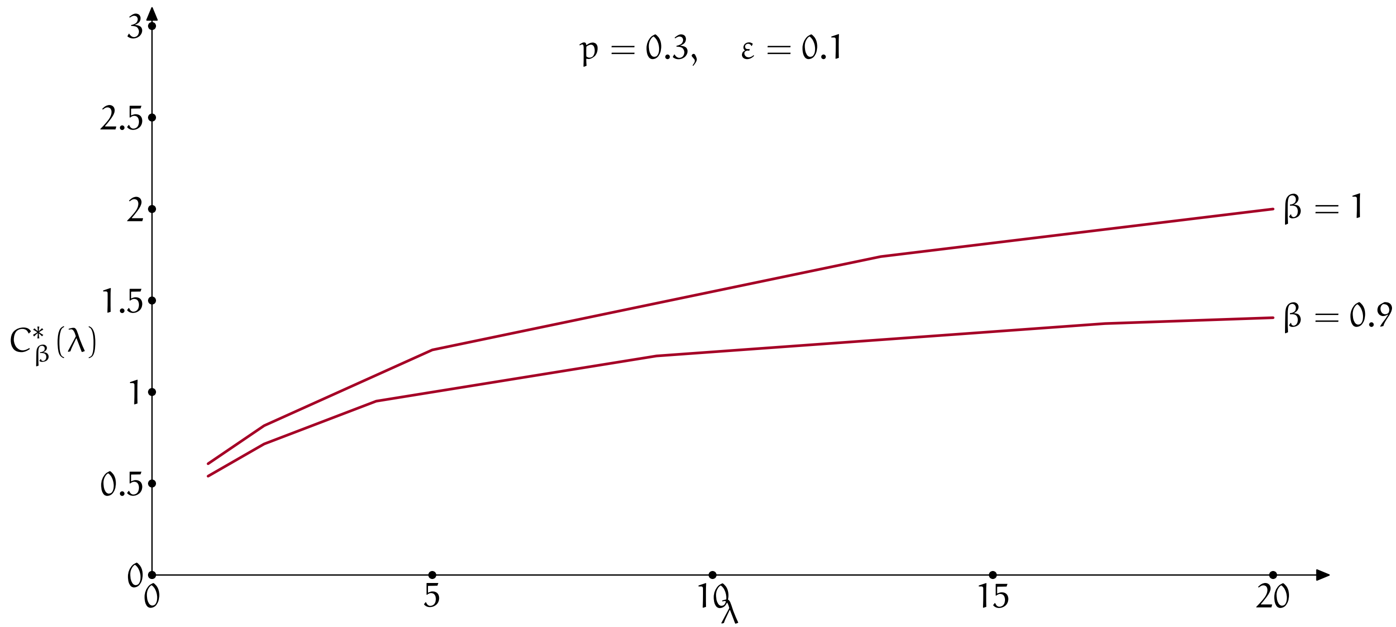


Example Symmetric birth-death Markov chain (i.i.d. packet drops)



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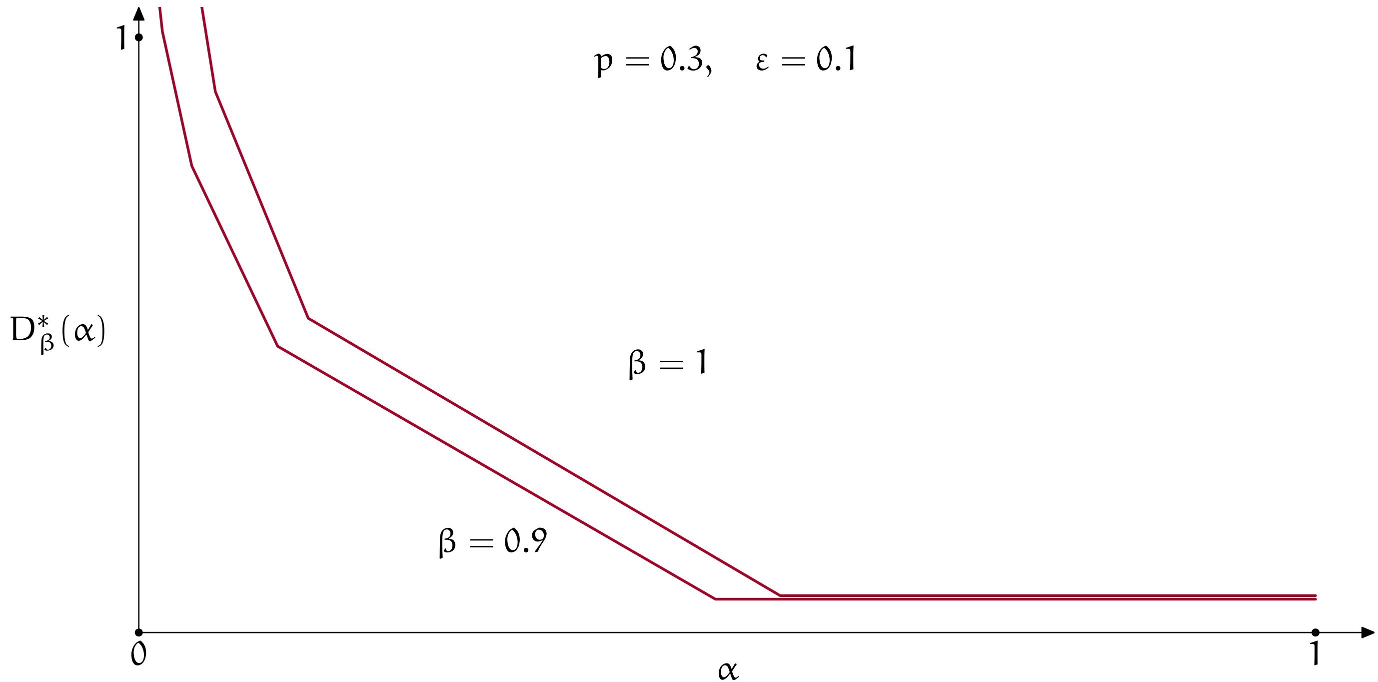
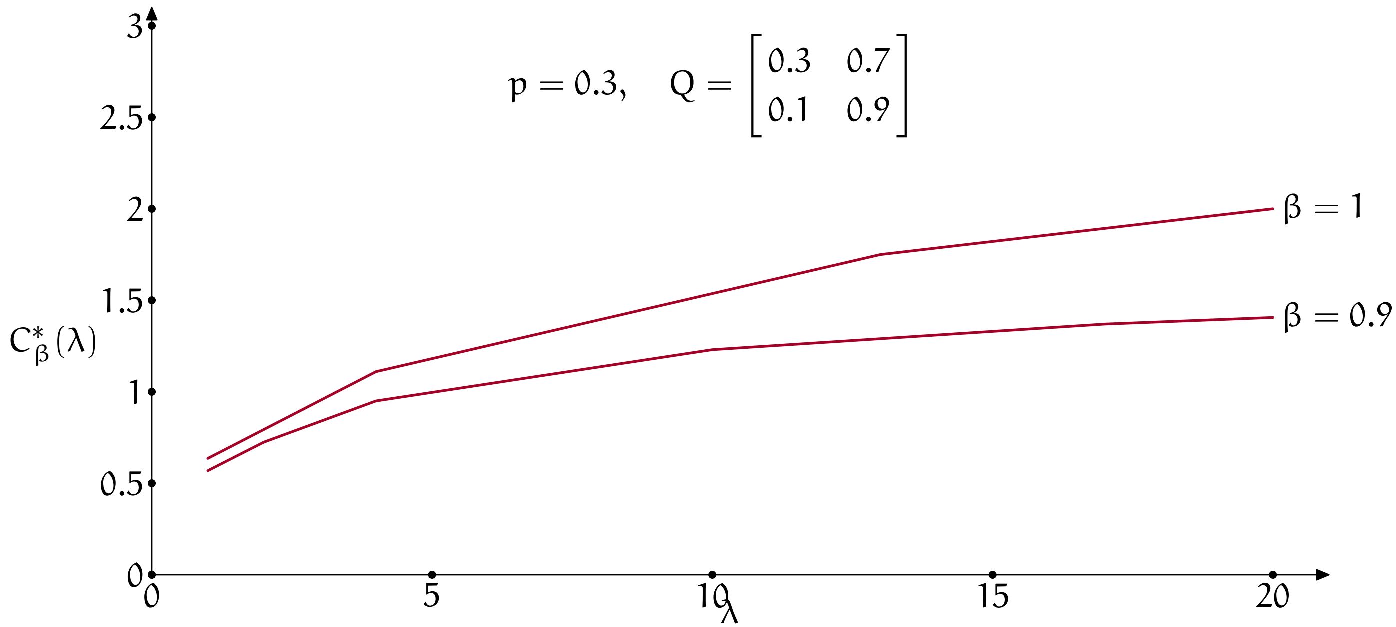
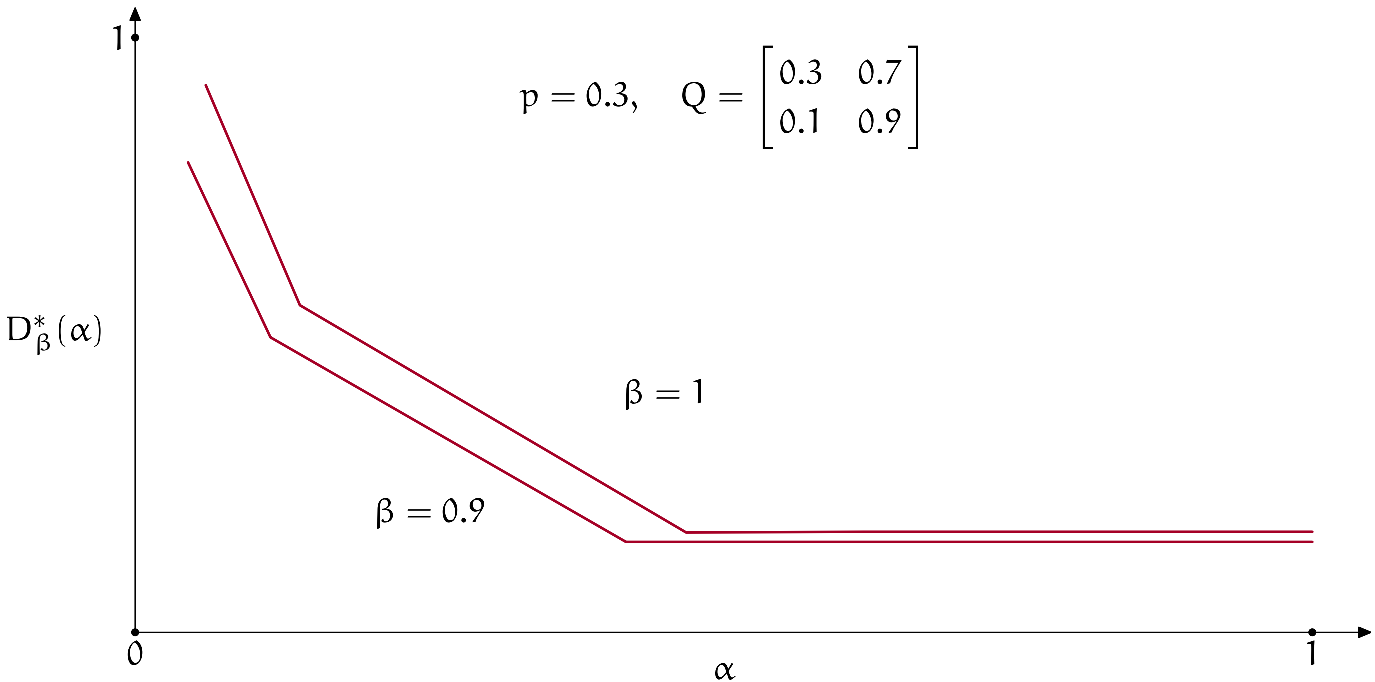


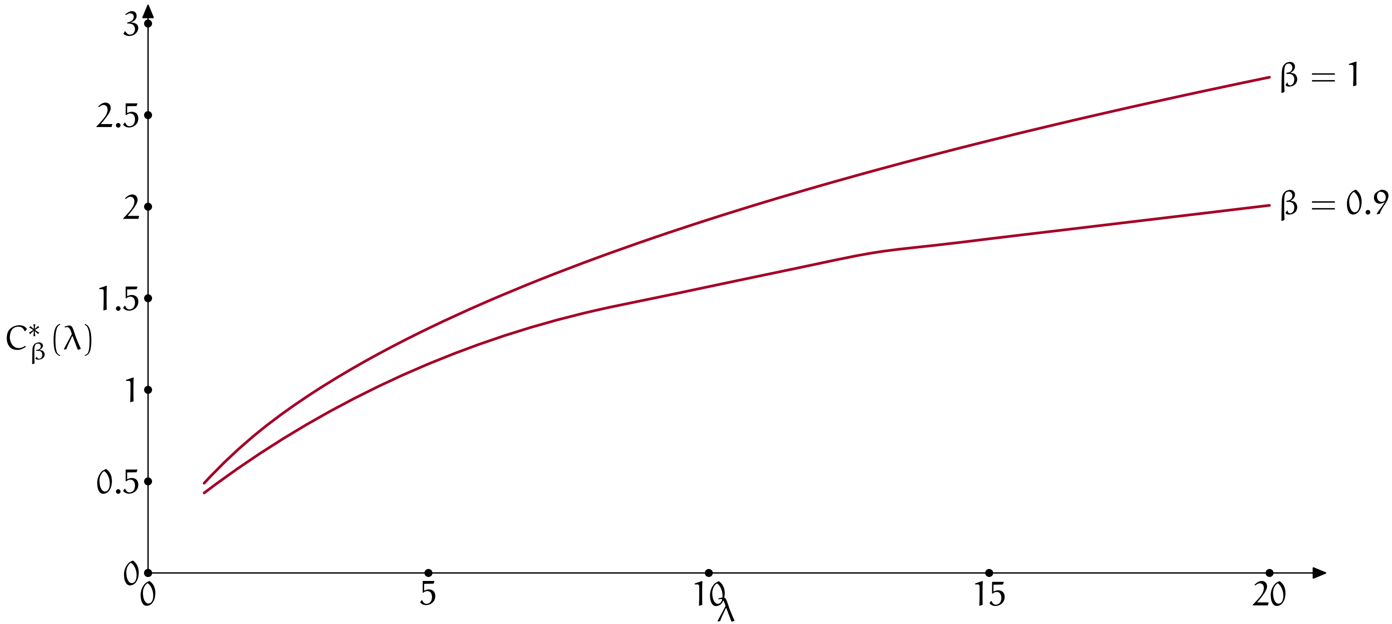
Example Symmetric birth-death Markov chain (Markov packet drops)



Example Symmetric birth-death Markov chain (Markov packet drops)



Gauss-Markov process ($\alpha = 1$, $\sigma^2 = 1$)



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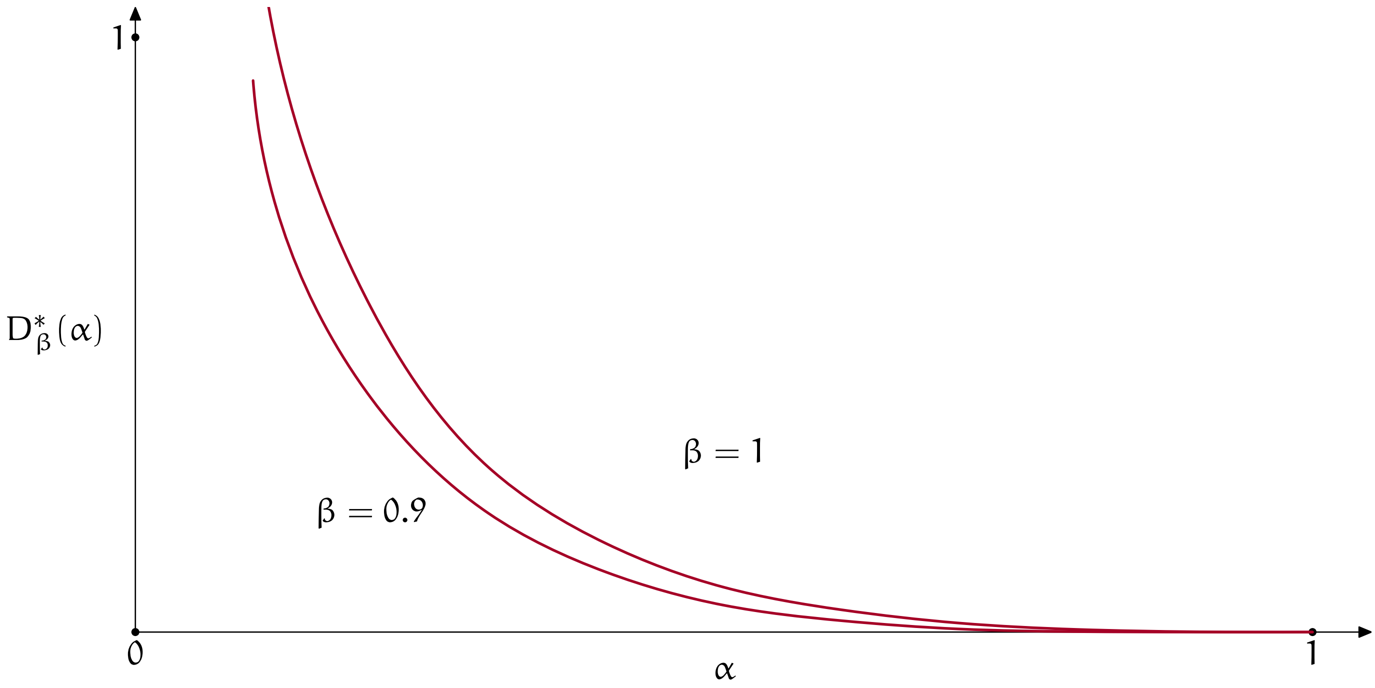
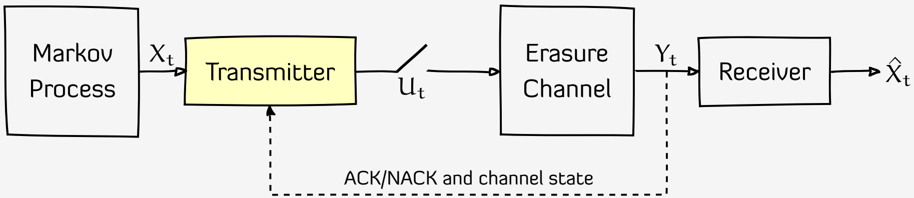


Summary

Communication system (cont.)



Feedback The receiver sends two bits of feedback: ACK/NACK and channel state.

Transmitter Decides whether to transmit or not. Denoted by $U_t \in \{0, 1\}$.
If $U_t = 0$, $\bar{X}_t = \mathfrak{E}$. If $U_t = 1$, $\bar{X}_t = X_t$.

$$U_t = f_t(X_{1:t}, Y_{1:t-1}, S_{1:t-1})$$

Summary

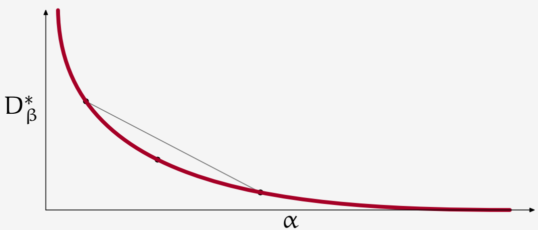
Optimization problems

Constrained communication

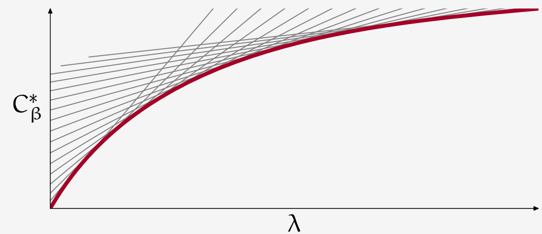
$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \}$$

Costly communication (Lagrange relaxation)

$$\text{For } \lambda \in \mathbb{R}_{>0}, \quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) + \lambda N_\beta(f,g) \}$$



D^*_β is cts, dec, and convex



C^*_β is cts, inc, and concave

Summary

Communication system (cont.)

Optimization problems

Optimal strategies and their performance

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

Distortion $d(x, \hat{x}) = d(x - \hat{x})$ where $d(\cdot)$ is symmetric and quasi-convex.

Optimal transmission strategy

$$U_t = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geqslant k(S_{t-1}) \\ 0, & \text{otherwise} \end{cases}$$

Optimal estimation strategy

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

Performance of threshold based strategies

- $K_\beta^{(k)}$: Expected discounted number of transmissions until first successful reception.
- $L_\beta^{(k)}$: Expected discounted distortion until first successful reception.
- $M_\beta^{(k)}$: Expected discounted time until first successful reception.

Then, $D_\beta^{(k)} = \dfrac{L_\beta^{(k)}}{M_\beta^{(k)}}$ and $N_\beta^{(k)} = \dfrac{K_\beta^{(k)}}{M_\beta^{(k)}}$. **(Renewal Relationships)**

Summary

Communication system (cont.)

Optimization problems

Optimal strategies and their performance

Dynamic program

$$V^1_{T+1}(s, \pi^1) = 0$$

and for $t \in \{T, \dots, 0\}$

$$V^1_t(s, \pi^1) = \min_{\varphi: \mathcal{X} \to \{0,1\}} \{\lambda \pi^1(B_1(\varphi)) + \pi^1(B_0(\varphi)) W^0_t(\pi^1, \varphi) + \sum_{x \in B_1(\varphi)} \pi^1(x) W^1_t(\pi^1, \varphi, x)\}$$

$$V^2_t(s, \pi^2) = \min_{\hat{x} \in \mathcal{X}} \sum_{x \in \mathcal{X}} \pi^2(x) d(x, \hat{x}) + V^1_{t+1}(s, \pi^2 P)$$

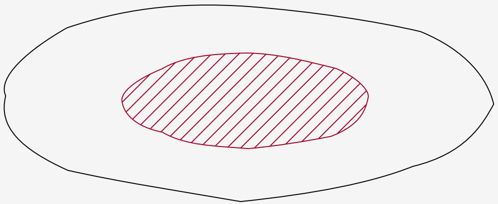
where $W^0_t(\pi^1, \varphi) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \pi^1|_\varphi)$

$$W^1_t(\pi^2, \varphi, x) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \delta_x)$$

Summary

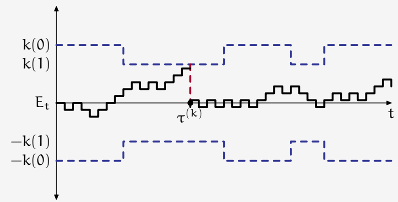
Communication system (cont.)
Optimization problems
Optimal strategies and their performance
Dynamic program

Step 1 Threshold strategies are optimal

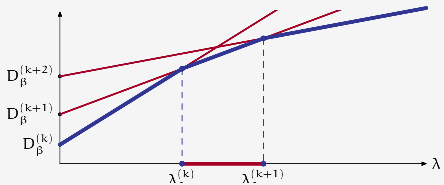


Search space of strategies (f, g)

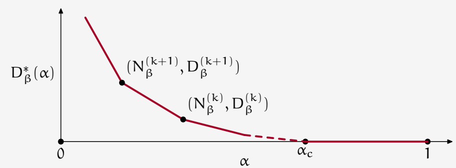
Step 2 Performance of threshold strategies



Step 3 Optimal costly communication



Step 4 Optimal constrained communication



Summary

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Kiefer-Wolfowitz Algorithm

Constrained communication: Given α, find k such that $N_\beta^{(k)} = \alpha$.

Robbins-Monro Algorithm

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Concluding Remarks

Generalization to vector sources

- Difficulty: If X_t is ASU, is $AX_t + W_t$ also ASU?

- Even if threshold policies are not optimal, the tools developed may be useful to identify best threshold-based strategy.

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Results are derived under idealized assumptions

Future directions

- Quantization . . .
- Power control . . .
- Scheduling multiple sources . . .
- Model network delays . . .

Beautiful example of stochastics and optimization

Decentralized control, POMDP, stochastic orders, majorization, Markov chains, constrained optimization, stochastic approximation

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References

Jhelum Chakravorty and Aditya Mahajan, “Fundamental limits of remote estimation of autoregressive Markov processes under communication constraints,” IEEE TAC, March 2017.

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Jhelum Chakravorty, Jayakumar Subramanian, and Aditya Mahajan, “Stochastic approximation based methods for computing the optimal thresholds in remote-state estimation with packet drops,” ACC 2017

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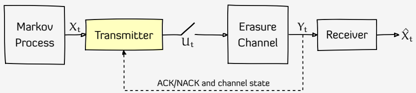
Communication system (cont.)



Feedback The receiver sends two bits of feedback: ACK/NACK and channel state.

Transmitter Decides whether to transmit or not. Denoted by $U_t \in \{0,1\}$.
If $U_t = 0$, $\bar{X}_t = \mathfrak{E}$. If $U_t = 1$, $\bar{X}_t = X_t$.
$U_t = f_t(X_{1:t}, Y_{1:t-1}, S_{1:t-1})$

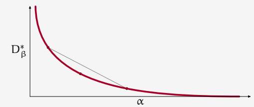
Optimization problems

Constrained communication

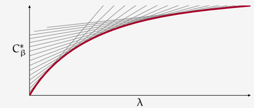
For $\alpha \in (0,1)$, $D^*_\beta(\alpha) := \inf_{(f,g)} \{D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha\}$

Costly communication (Lagrange relaxation)

For $\lambda \in \mathbb{R}_{>0}$, $C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) := \inf_{(f,g)} \{D_\beta(f,g) + \lambda N_\beta(f,g)\}$



D^*_β is cts, dec, and convex



C^*_β is cts, inc, and concave

Optimal strategies and their performance

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

Distortion $d(x,\hat{x}) = d(x - \hat{x})$ where $d(\cdot)$ is symmetric and quasi-convex.

Optimal transmission strategy

$$U_t = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geqslant k(S_{t-1}) \\ 0, & \text{otherwise} \end{cases}$$

Optimal estimation strategy

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$$

Performance of threshold based strategies

- $K^{(k)}_\beta$: Expected discounted number of transmissions until first successful reception.
- $L^{(k)}_\beta$: Expected discounted distortion until first successful reception.
- $M^{(k)}_\beta$: Expected discounted time until first successful reception.

Then, $D^{(k)}_\beta = \dfrac{L^{(k)}_\beta}{M^{(k)}_\beta}$ and $N^{(k)}_\beta = \dfrac{K^{(k)}_\beta}{M^{(k)}_\beta}$. (Renewal Relationships)

Dynamic program

$$V^1_{T+1}(s, \pi^1) = 0$$

and for $t \in \{T, \dots, 0\}$

$$V^1_t(s,\pi^1) = \min_{\varphi: \mathcal{X} \to \{0,1\}} \{\lambda \pi^1(B_1(\varphi)) + \pi^1(B_0(\varphi)) W^0_t(\pi^1, \varphi) + \sum_{x \in B_1(\varphi)} \pi^1(x) W^1_t(\pi^1, \varphi, x)\}$$

$$V^2_t(s,\pi^2) = \min_{\hat{x} \in \mathcal{X}} \sum_{x \in \mathcal{X}} \pi^2(x) d(x, \hat{x}) + V^1_{t+1}(s, \pi^2 P)$$

where $W^0_t(\pi^1, \varphi) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \pi^1|_\varphi)$

$$W^1_t(\pi^2, \varphi, x) = Q_{s0} V^2_t(0, \pi^1) + Q_{s1} V^2_t(1, \delta_x)$$

Step 1 A change of variables

Define $Z_0 = 0$ and $Z_t = \begin{cases} aZ_{t-1}, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ Y_t, & \text{if } Y_t \in \mathcal{X} \end{cases}$ (Observable at both Tx and Rx)

$$E_t = X_t - aZ_{t-1}, \quad E^+_t = X_t - Z_t, \quad \hat{E}_t = \hat{X}_t - Z_t$$

Thus, these are related as

$$E^+_t = \begin{cases} E_t, & \text{if } Y_t \in \{\mathfrak{E}_0, \mathfrak{E}_1\} \\ 0, & \text{if } Y_t \in \mathcal{X} \end{cases} \quad \text{and} \quad E_{t+1} = aE^+_t + W_t$$

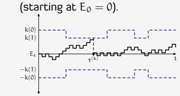
Note $X_t - \hat{X}_t = E^+_t - \hat{E}_t$ and hence $d(X_t - \hat{X}_t) = d(E^+_t - \hat{E}_t)$.

Step 2 Performance of threshold-based strategies

Consider a threshold-based strategy

$$f^{(k)}(e,s) = \begin{cases} 1 & \text{if } |e| \geqslant k(s) \\ 0 & \text{otherwise} \end{cases}$$

Let $\tau^{(k)}$ denote the stopping time of first reception (starting at $E_0 = 0$).

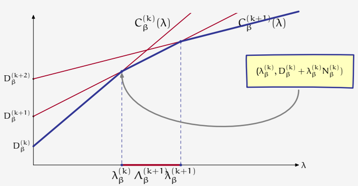


Define $L^{(k)}_\beta(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big]$. (Distortion until first reception)

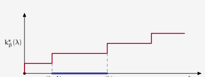
$M^{(k)}_\beta(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Big| E_0 = e\Big]$. (Time until the first reception)

$K^{(k)}_\beta(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}} \beta^t U_t \Big| E_0 = e\Big]$. (Transmissions until the first reception)

Step 3 Solution to costly comm. for discrete sources



Define $\Lambda^{(k)}_\beta := \{\lambda \in \mathbb{R}_{\geqslant 0} : k^*_\beta(\lambda) = k\}$
$= [\lambda^{(k-1)}_\beta, \lambda^{(k)}_\beta]$.



$C^{(k)}_\beta(\lambda^{(k)}_\beta) = C^{(k+1)}_\beta(\lambda^{(k)}_\beta)$
$\implies \lambda^{(k)}_\beta = (D^{(k+1)}_\beta - D^{(k)}_\beta)/(N^{(k)}_\beta - N^{(k+1)}_\beta)$.

Step 4 Solution to constrained comm. for discrete sources

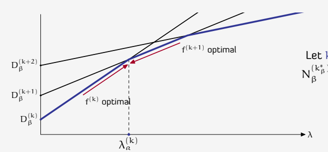
Sufficient conditi...

A strategy (f°, g°) i...

(C1) $N_\beta(f^\circ, g$...

(C2) There exi... ...meter λ°.

Randomized strategy $(\theta^*, f^{(k)}, f^{k+1})$ is optimal where
$\theta^* N^{(k)}_\beta + (1 - \theta^*) N^{(k+1)}_\beta = \alpha$



Let k^*_β be such that
$N^{(k^*_\beta)}_\beta > \alpha > N^{(k^*_\beta+1)}_\beta$

Computation of optimal thresholds

Costly communication Given λ, find k such that $\partial_k(D^{(k)}_\beta + \lambda N^{(k)}_\beta) = 0$. — Kiefer-Wolfowitz Algorithm

Constrained communication Given α, find k such that $N^{(k)}_\beta = \alpha$. — Robbins-Monro Algorithm

Main idea

- Pick a threshold k and use strategy $f^{(k)}$ until first successful reception.
- The sample path values of L, M, and K may be viewed as a "noisy" observation of true $L^{(k)}_\beta$, $M^{(k)}_\beta$, and $K^{(k)}_\beta$.
- Use stochastic approximation to find optimal thresholds.