Remote state estimation over erasure channels: structure of optimal strategies and fundamental limits

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Joint work with Jhelum Chakravorty and Jayakumar Subramanian

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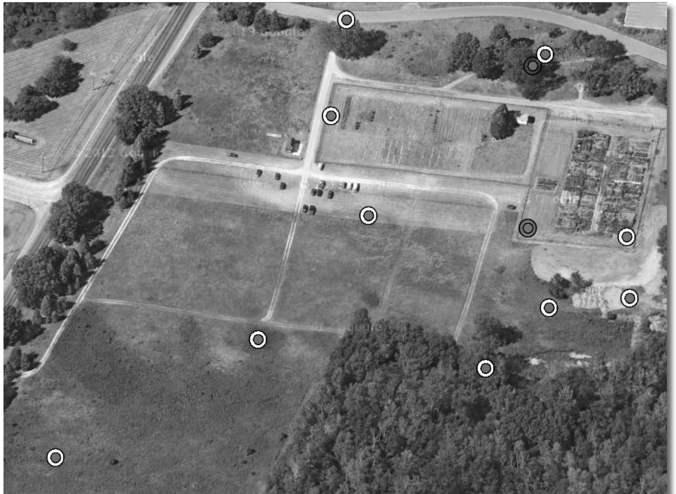
There is a need to revisit rate distortion theory to take network access into account.

Many applications require:

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction



Sensor Networks

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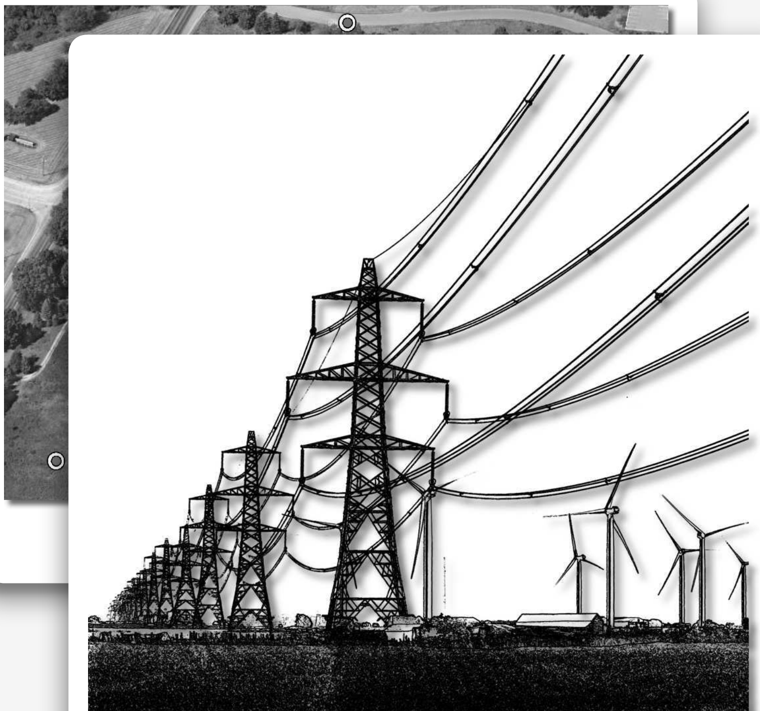

Smart Grids

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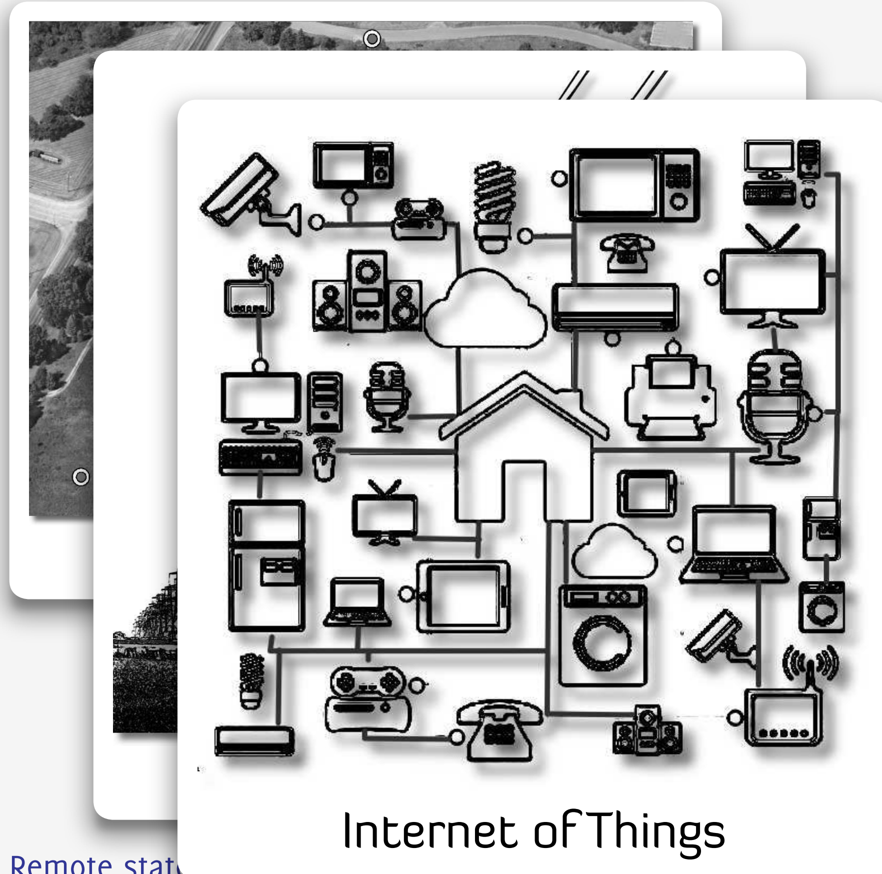

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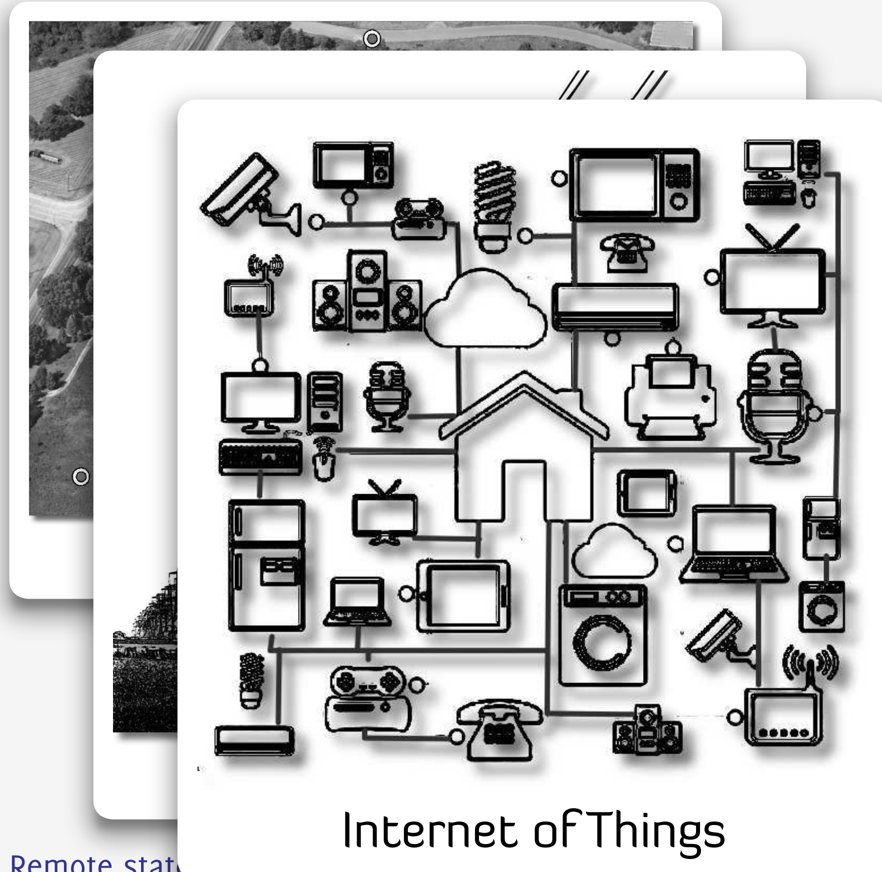

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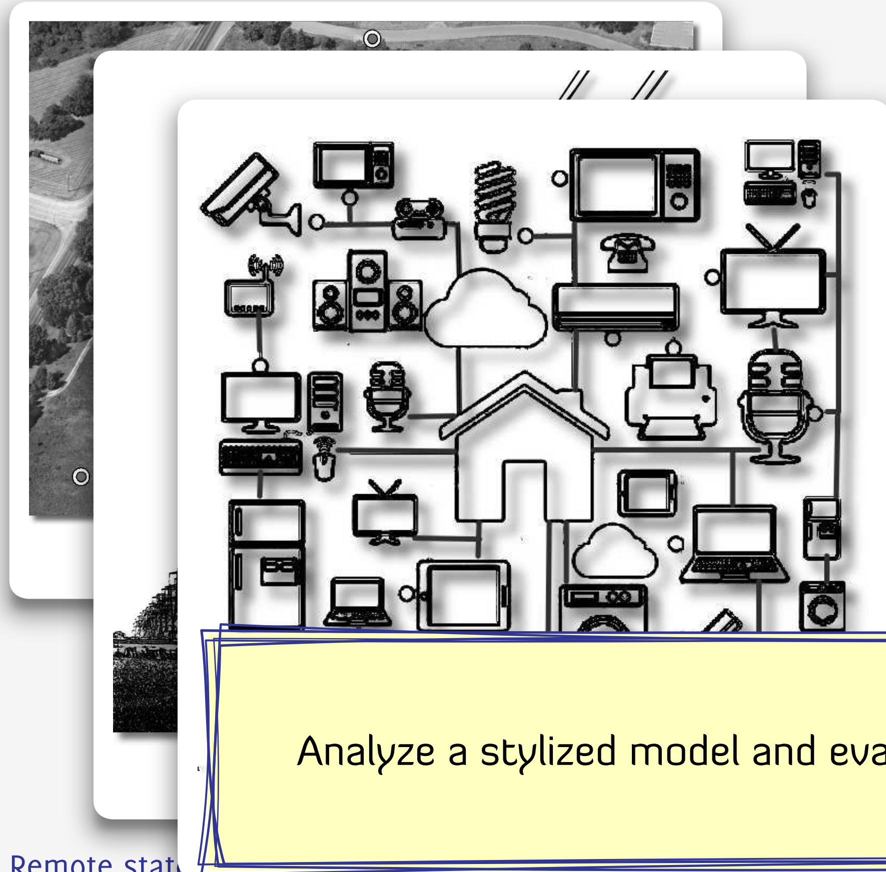
Salient features

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical



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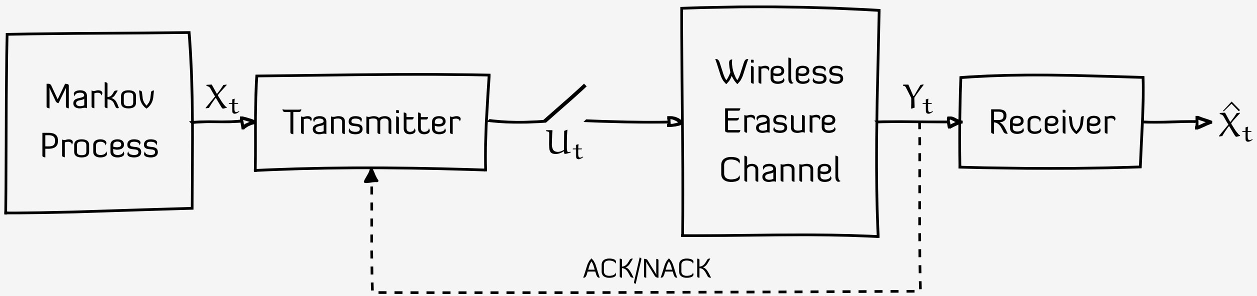
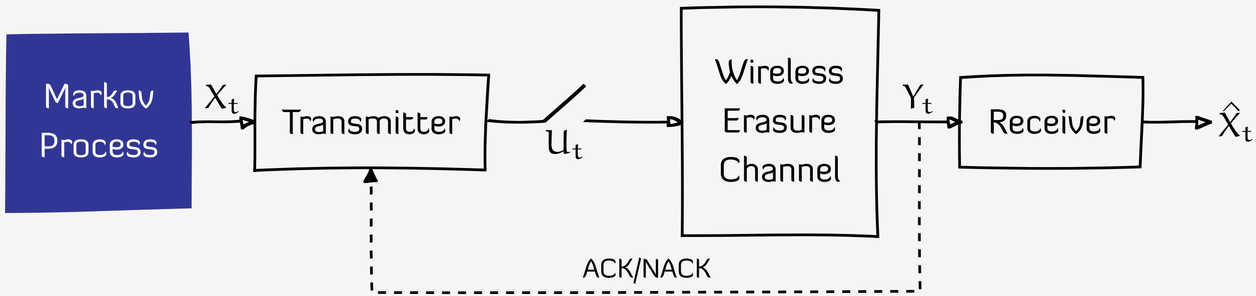
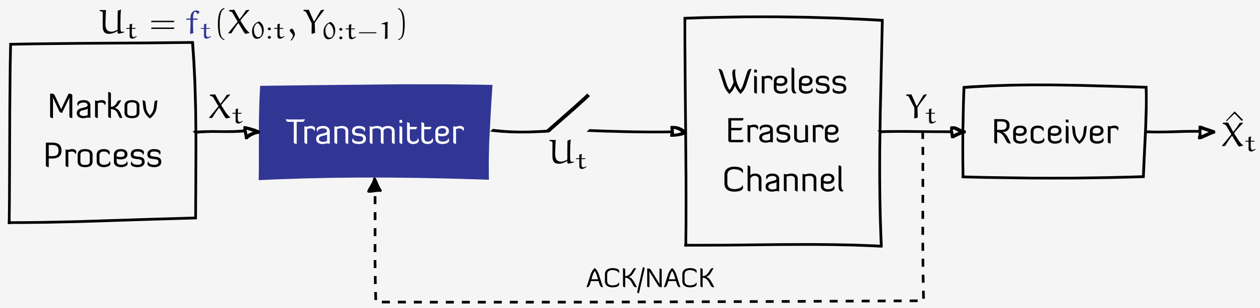
Analyze a stylized model and evaluate fundamental trade-offs



Markov Process
X_t
Transmitter
U_t
Wireless Erasure Channel
Y_t
Receiver
$\hat{X}_t$
ACK/NACK



- First order time-homogeneous Markov process
- The transmitter decides whether or not to transmit the current state
- The transmitted symbol is sent over an erasure channel (with acknowledgments)
- The receiver generates an estimate based on received symbol



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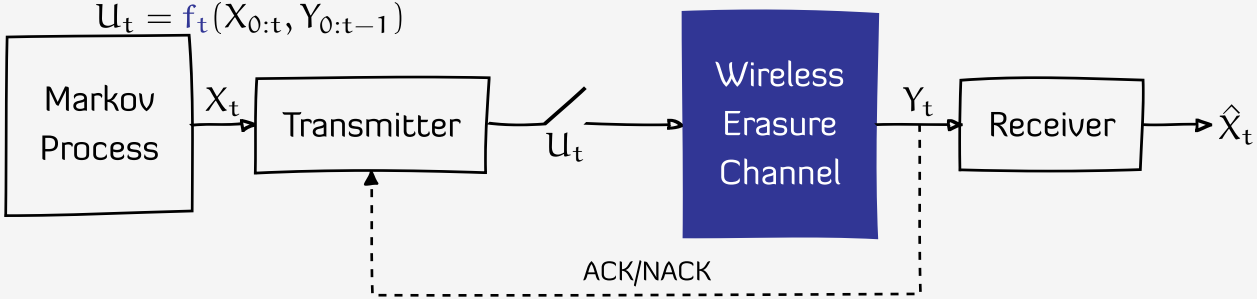
$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \text{ and } S_t = \text{ON} \\ \mathfrak{E}, & \text{if } U_t = 0 \text{ or } S_t = \text{OFF} \end{cases}$$

$$\mathbb{P}(S_t = \text{OFF}) = \varepsilon$$

$$U_t = f_t(X_{0:t}, Y_{0:t-1})$$



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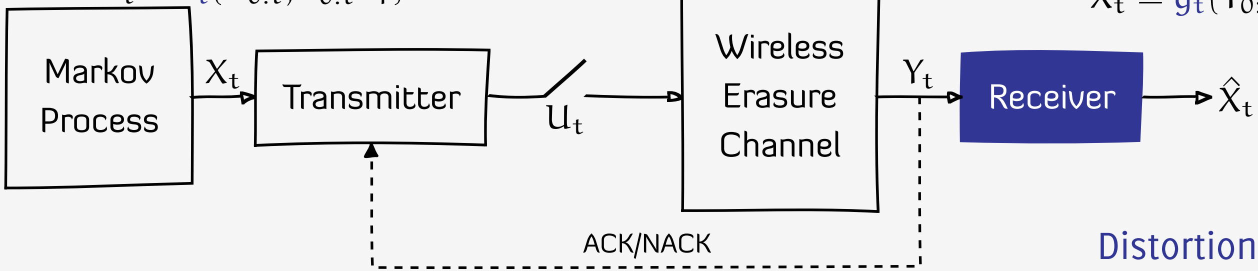

Distortion

$d(X_t - \hat{X}_t)$

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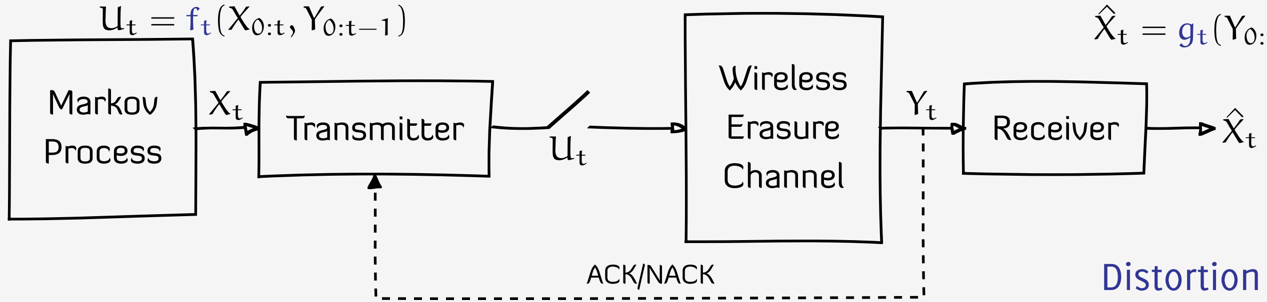

Distortion

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Communication Strategies

- **Transmission strategy** $f = \{f_t\}_{t=0}^{\infty}$.
- **Estimation strategy** $g = \{g_t\}_{t=0}^{\infty}$.

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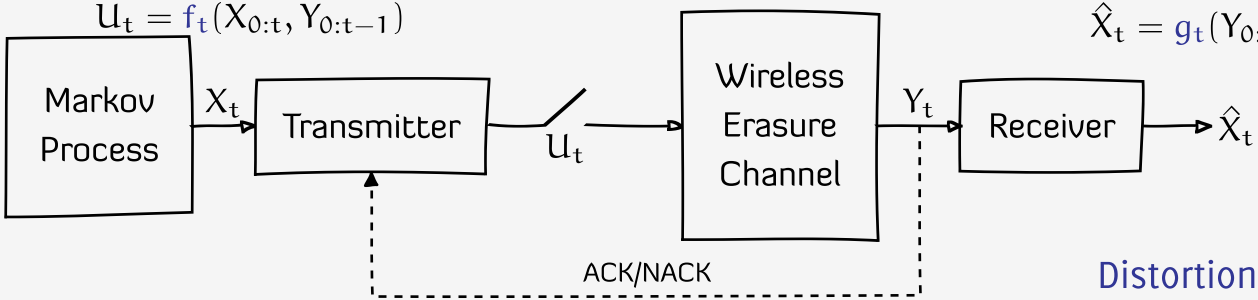

Distortion

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1. Discounted setup, $\beta \in (0, 1)$

$$D_\beta(f, g) = (1-\beta)\, \mathbb{E}_0^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\right]; \qquad N_\beta(f, g) = (1-\beta)\, \mathbb{E}_0^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t U_t\right]$$

2. Average cost setup, $\beta = 1$

$$D_1(f, g) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_0^{(f,g)}\left[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\right]; \qquad N_1(f, g) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_0^{(f,g)}\left[\sum_{t=0}^{T-1} U_t\right]$$

Optimization problems

Constrained communication

$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \left\{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \right\}$$

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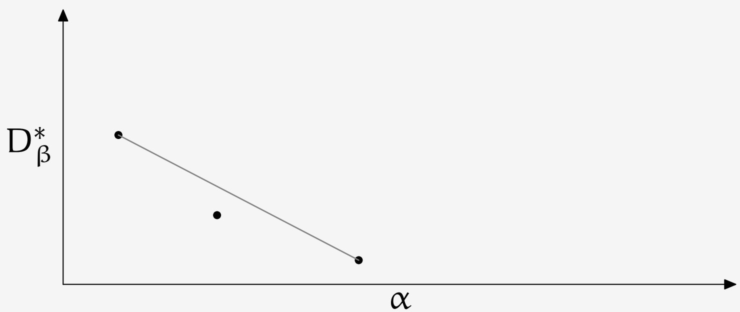

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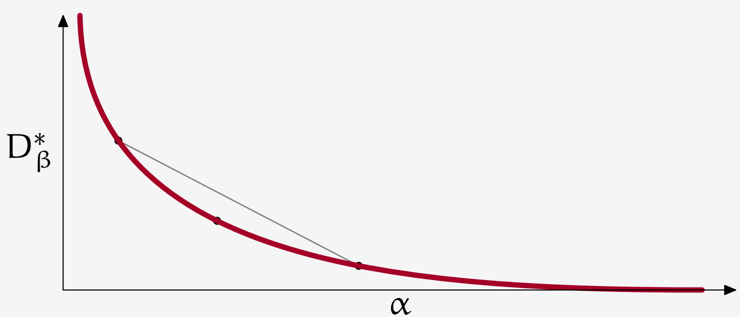

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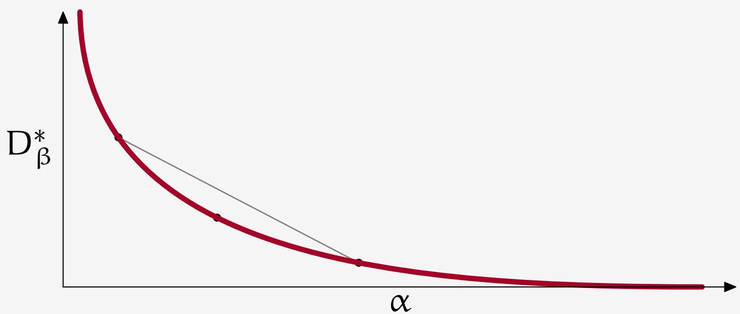
Costly communication (Lagrange relaxation)

$$\text{For } \lambda \in \mathbb{R}_{>0}, \quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) + \lambda N_\beta(f,g) \}$$



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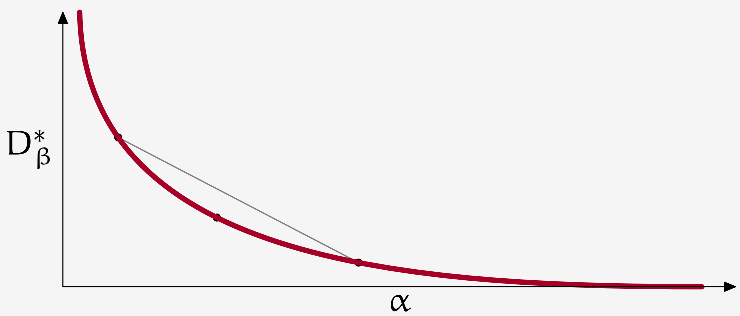

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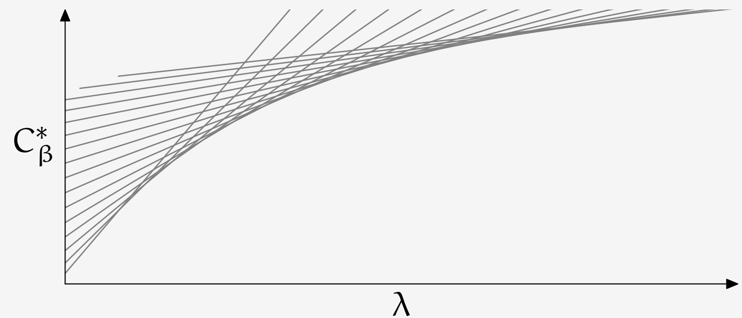

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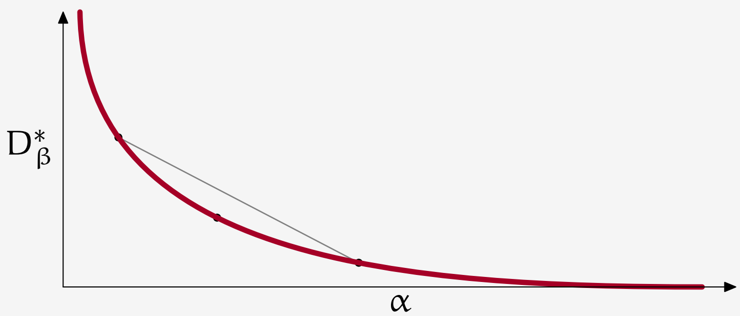

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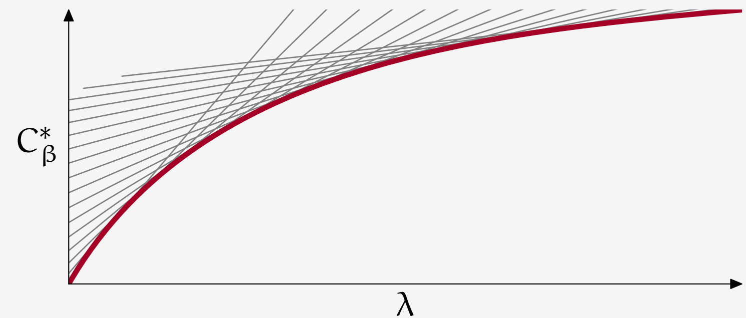

C^*_β is cts, inc, and concave

Optimization problems

Constrained communication

Costly c

Our result: Provide computable expressions for these trade-offs and identify optimal strategies that achieve them.

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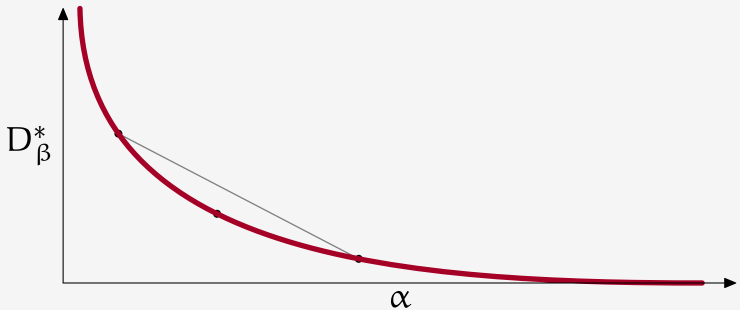

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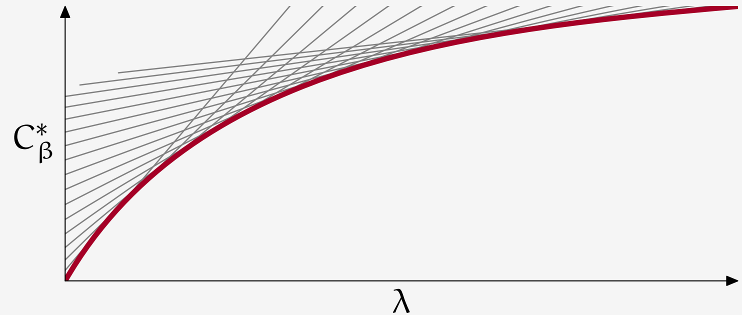

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Comparison to Information Theory

- Costly communication is analogous to communication under power constraint.
- Constrained communication is analogous to distortion-rate function.
  So, we call it distortion-transmission function.
- Due to zero-delay reconstruction, information theoretic approaches do not apply.

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Previous work on remote-state estimation

- [Marshak 1954] Static (one-shot) problem with arbitrary source distribution
- [Kushner 1964] Off-line choice of measurement times
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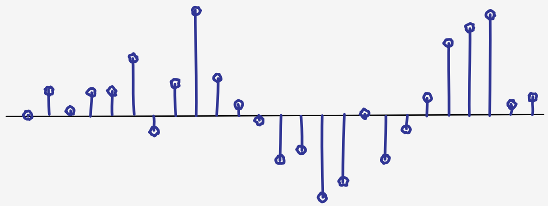
Other related work

- Event-based estimation . . .
- Censoring sensors . . .
- Sensor sleep scheduling . . .
- Age of Information . . .

An illustrative example

$X_{t+1} = X_t + W_t$, $W_t \sim \mathcal{N}(0, 1)$. Perfect channel



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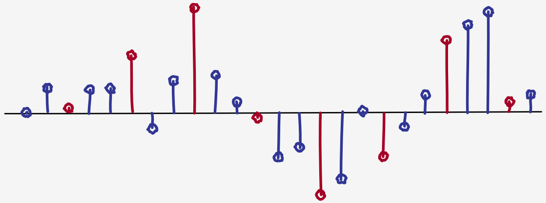
Periodic Transmission Strategy



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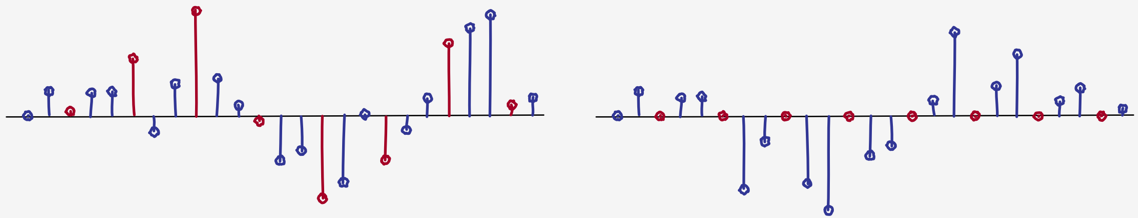

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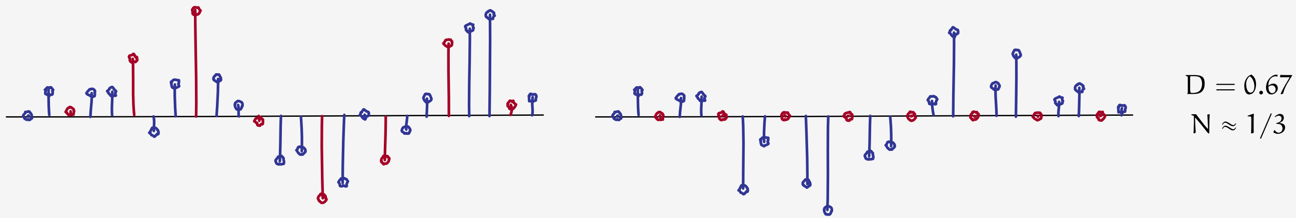

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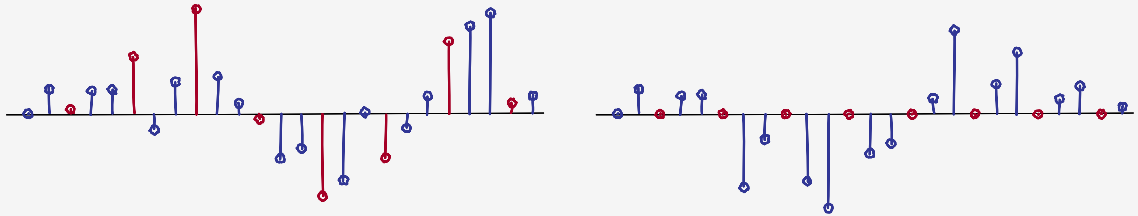
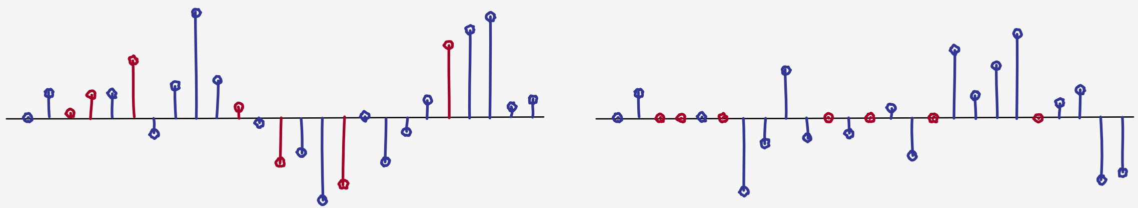


$D = 0.67$
$N \approx 1/3$

Randomized Transmission Strategy



$D = 2.00$
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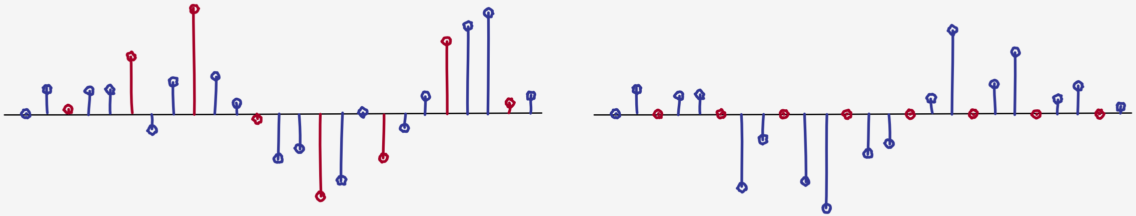

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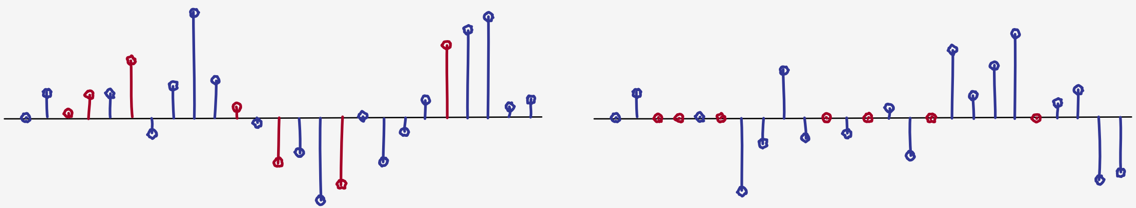

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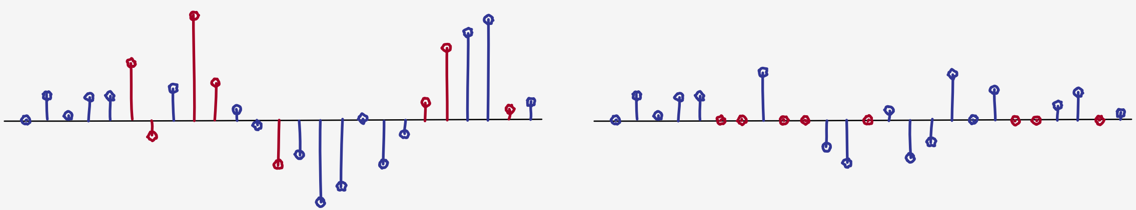
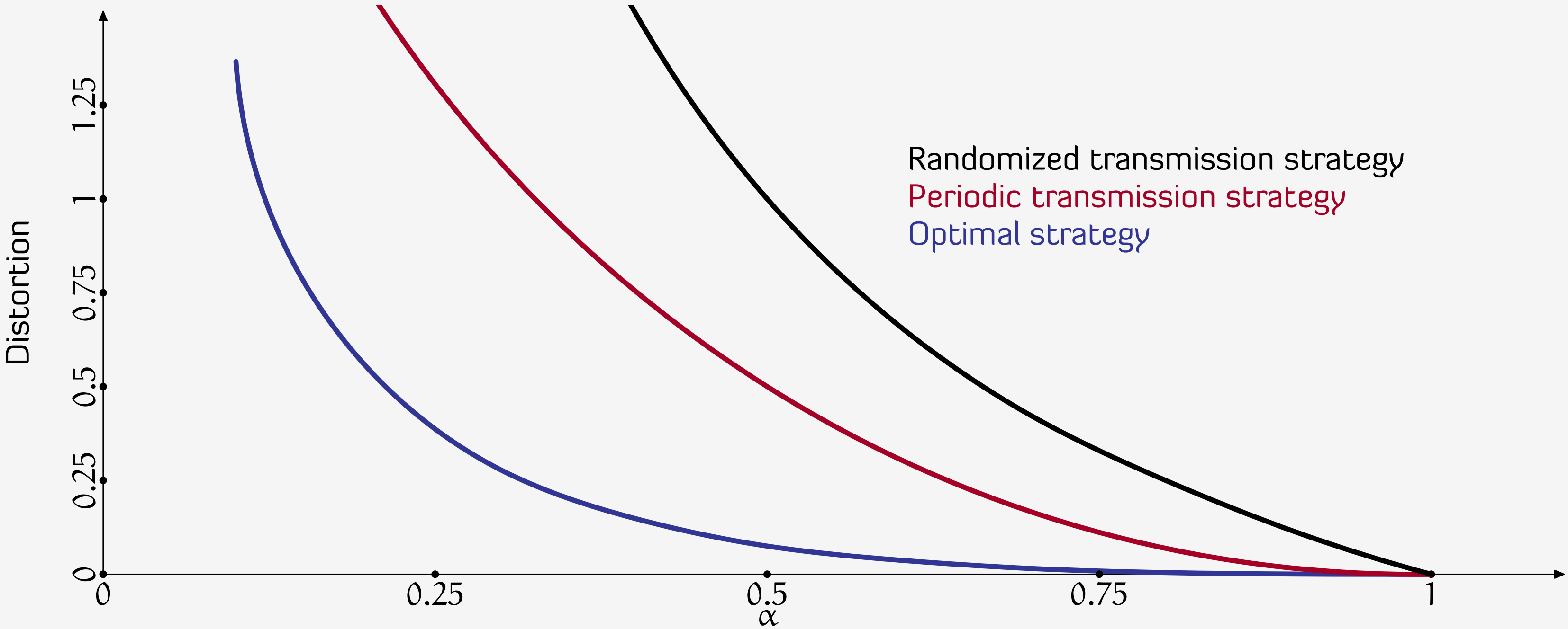
Optimal Transmission Strategy



$D = 0.24$
$N \approx 1/3$

Distortion-transmission trade-off: Perfect channel



What’s the conceptual difficulty?

Static (one-shot) problem

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$\mathcal{S} \subset \mathcal{X}$ is the silence set

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$\mathcal{X}$

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Cost when $x \in \mathcal{S}$

$d(x - \hat{x})$

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Total expected cost

$$c(\hat{x}, \mathcal{S}) \coloneqq \lambda\, \mathbb{P}(X \notin \mathcal{S}) + \varepsilon \sum_{x \notin \mathcal{S}} \mathbb{P}(X = x) d(x - \hat{x}) + \sum_{x \in \mathcal{S}} \mathbb{P}(X = x) d(x - \hat{x})$$

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Choose $(\hat{x}, \mathcal{S})$ to minimize $c(\hat{x}, \mathcal{S})$.

Set-valued (or combinatorial) optimization problem.

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If a packet is received

$\mathcal{S}^1_2(x_1) \subset \mathcal{X}$ is the silence set

$\hat{x}^1_2$ is the estimate when no packet is received

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If a packet is not received

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Sequential optimization problem where the optimization problem at each step is a set-valued optimization problem that depends on a history of previously chosen sets!.

Exhaustive search complexity: $\left(|\mathcal{X}|2^{|\mathcal{X}|}\right)^{(2^{|\mathcal{X}|})^T}$

Main results

Distortion transmission function for auto-regressive sources

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

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Salient features

- The transmitter does not try to send information through timing events.
- The estimation strategy is the same to the one for intermittent observations and **does not depend on the choice of the threshold**

Distortion transmission function for auto-regressive sources

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

Optimal transmission strategy

$$U_t = \begin{cases} 1, & \text{if } |X_t - a\hat{X}_{t-1}| \geq k \\ 0, & \text{otherwise} \end{cases}$$

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Performance of threshold based strategies

- $K_\beta^{(k)}$: Expected discounted number of transmissions until first successful reception.
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Then, $D_\beta^{(k)} = \dfrac{L_\beta^{(k)}}{M_\beta^{(k)}}$ and $N_\beta^{(k)} = \dfrac{K_\beta^{(k)}}{M_\beta^{(k)}}$.

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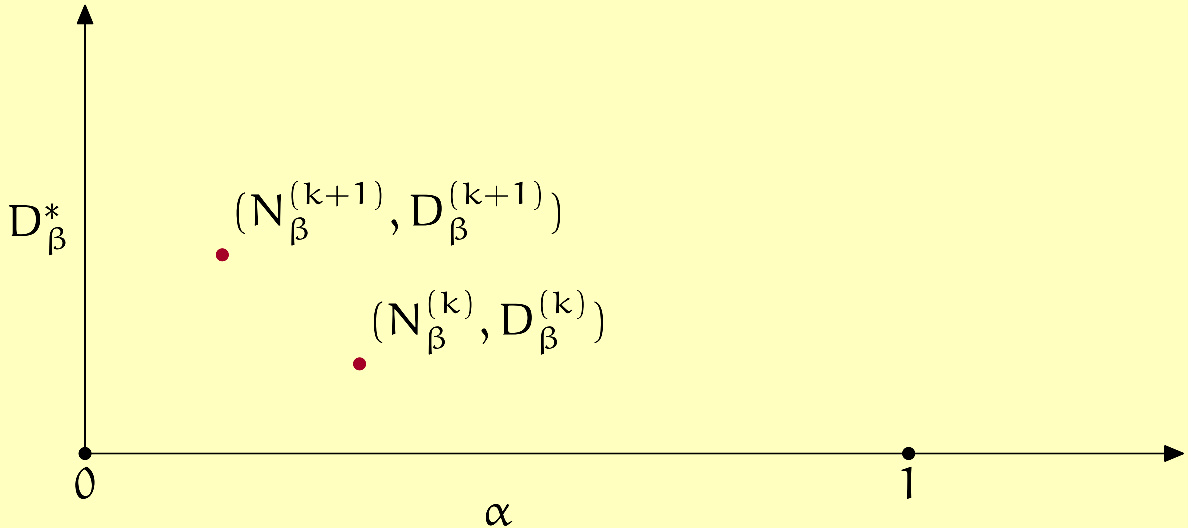

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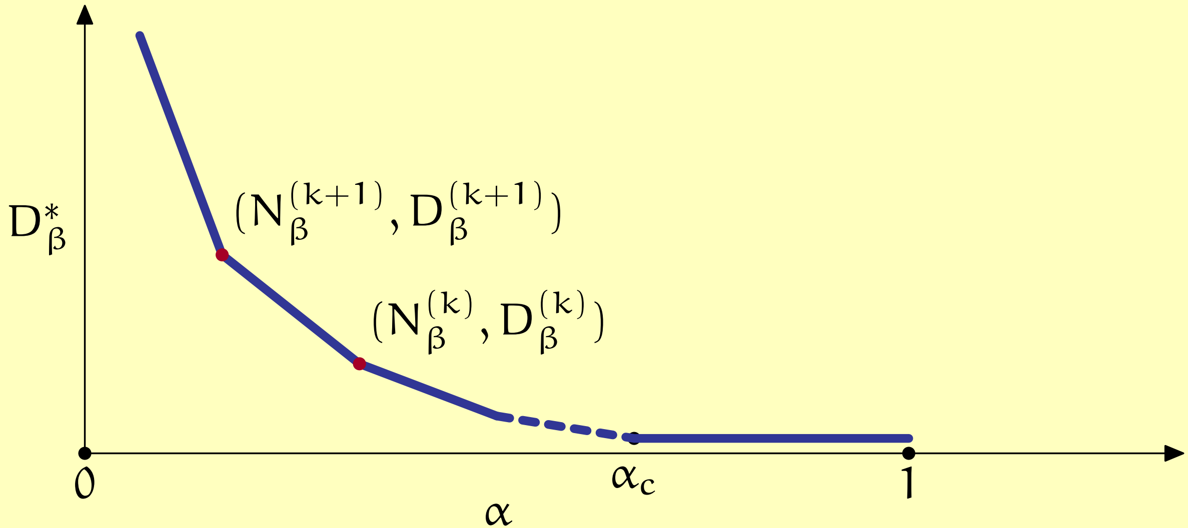

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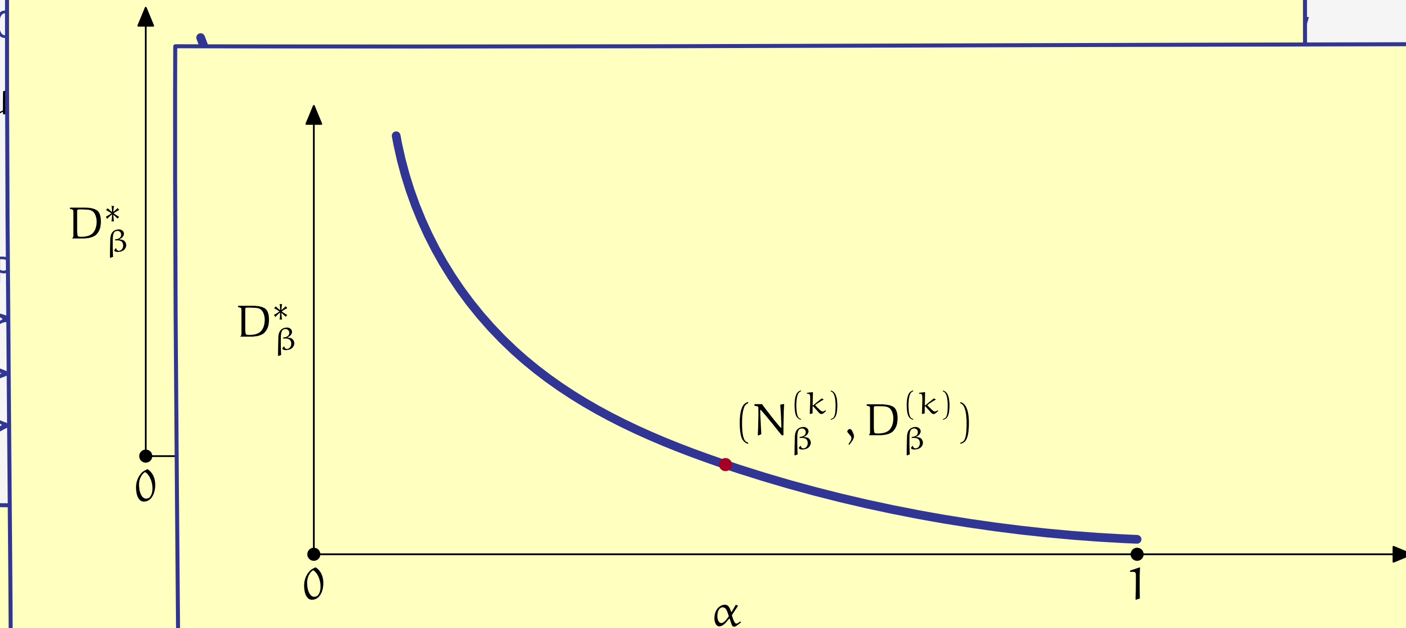

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Identify strategies that achieve the optimal trade-off

Simple and intuitive threshold based strategies are optimal.

Provide computable expressions for distortion-transmission function

Based on simple matrix calculations for discrete Markov processes

Based on solving Fredholm integral equations for continuous Markov processes

Provide simulation-based algorithms to compute optimal thresholds

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Beautiful example of stochastics and optimization

Decentralized stochastic control (or team theory) and POMDPs

Stochastic orders and majorization

Markov chain analysis, stopping times, and renewal theory

Constrained MDPs and Lagrangian relaxations

Stochastic approximation and simulation based optimization

Solution methodology

Standard technique

- **Achievability**: Identify a good strategy and evaluate its performance.
- **Converse**: Determine a lower bound on distortion.

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Our approach

- Model the optimization problem as a decentralized stochastic control problem.
  [Witsenhausen 1979, Walrand-Varaiya 1983, Teneketzis 2006, Mahajan-Teneketzis 2009, Kaspi-Merhav 2012, Asnani-Weissman 2013, Yüksel 2013 . . .]
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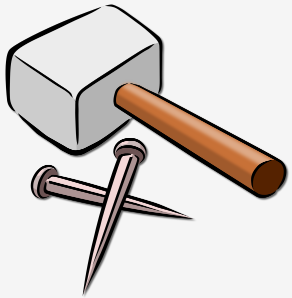
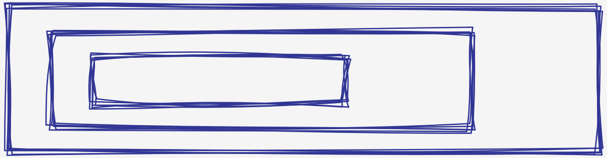
- The system has two decision makers: the transmitter and the estimator, that have access to different information.
- Identify qualitative properties of optimal strategies
- Identify a dynamic programming decomposition
- Determine optimal strategies based on the dynamic program.

So how do we start?

Decentralized stochastic control



Dealing with non-classical information structure



Classical info. struct.

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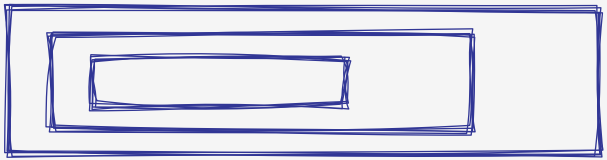

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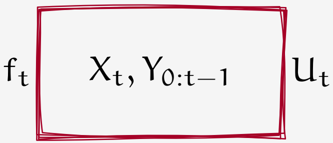

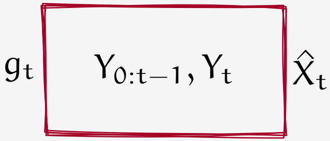

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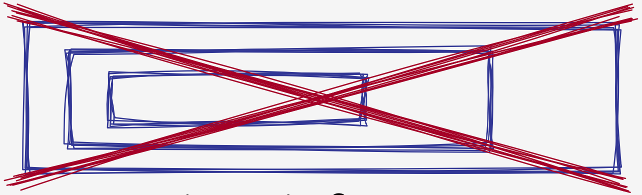

Non-Classical info. struct.

f_t [$X_t, Y_{0:t-1}$] U_t

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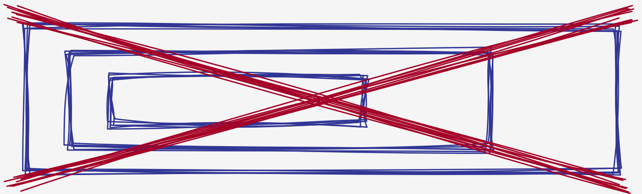
Dealing with non-classical information structure



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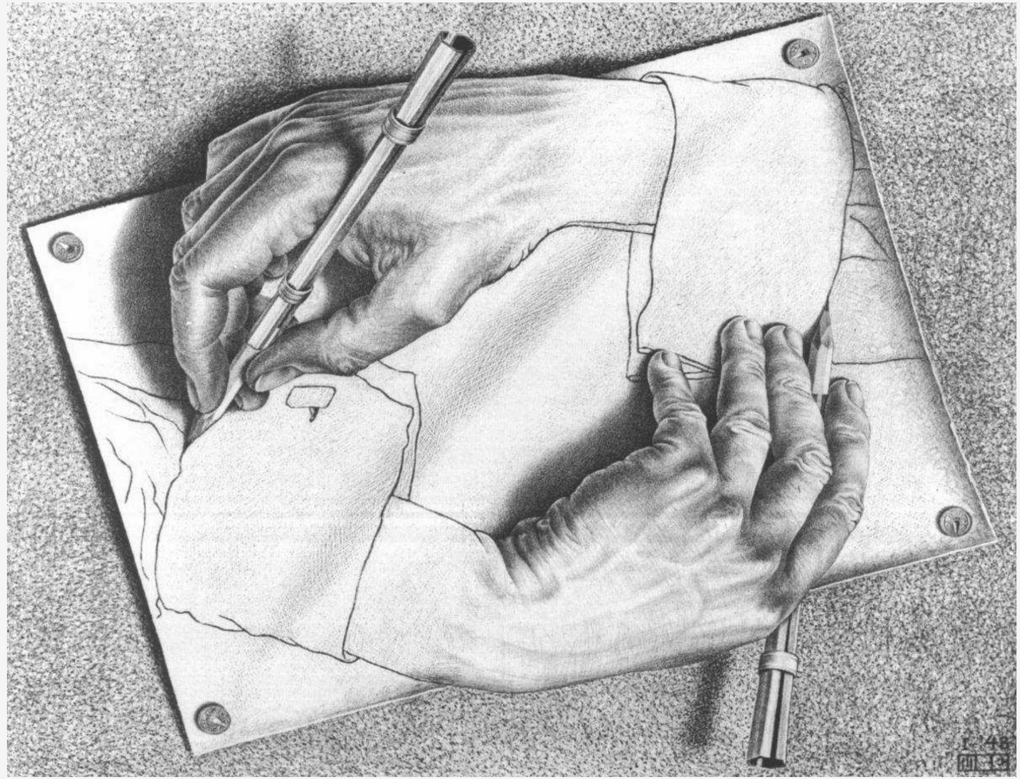
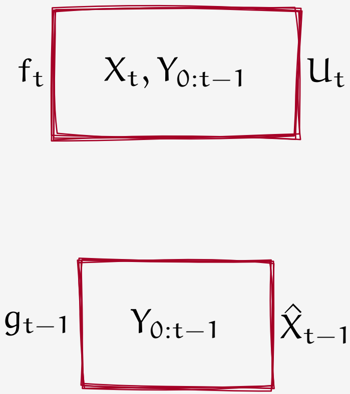


The common information approach (Nayyar, Mahajan, Teneketzis 2013)

Original system



▷ Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

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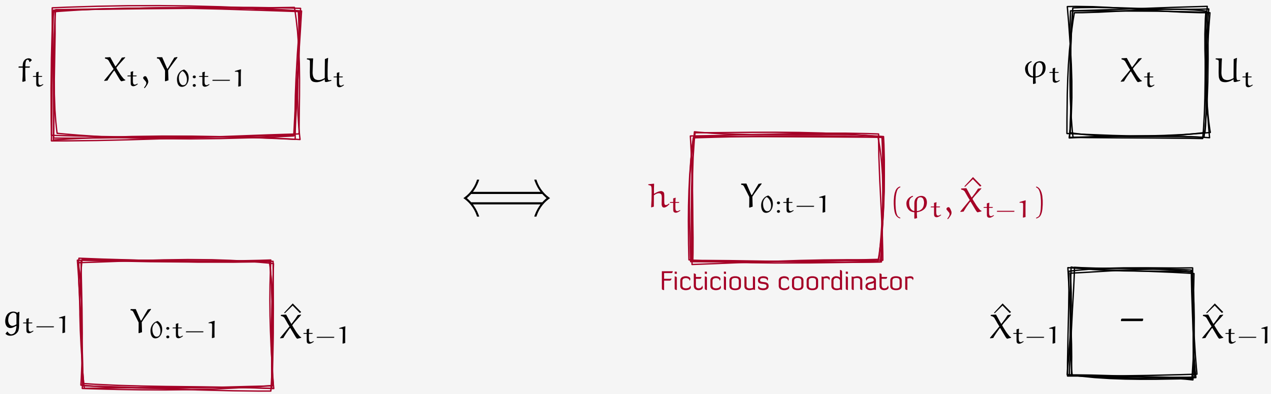
Original system

Coordinated system



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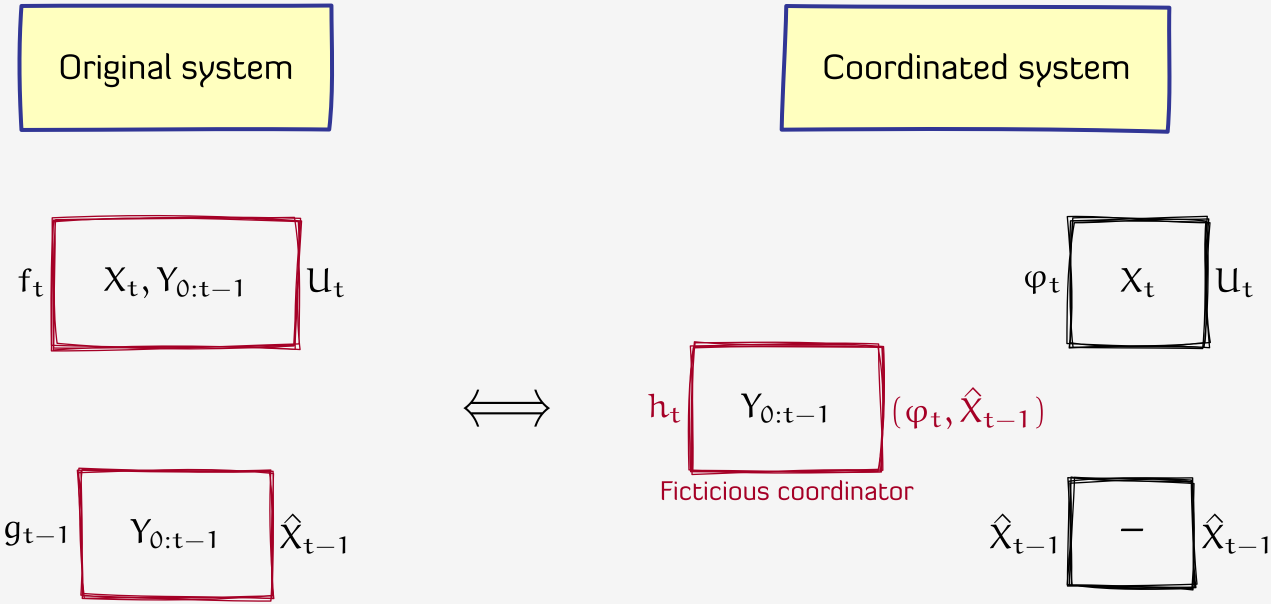

- The coordinated system is equivalent to the original system.

$$f_t(x, y_{0:t-1}) = h_t^1(y_{0:t-1})(x).$$

- The coordinated system is centralized. Belief state $\mathbb{P}(X_t \mid Y_{0:t-1})$.

Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Information states and dynamic program

Information states

Pre-transmission belief : $\Pi_{t|t-1}(x) = \mathbb{P}(X_t = x \mid Y_{0:t-1})$.

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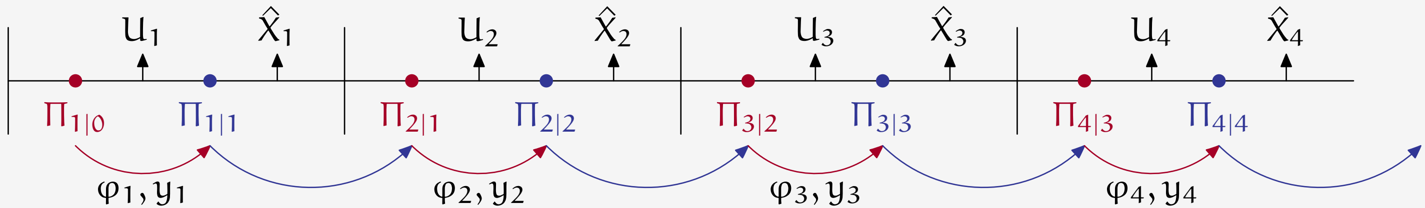

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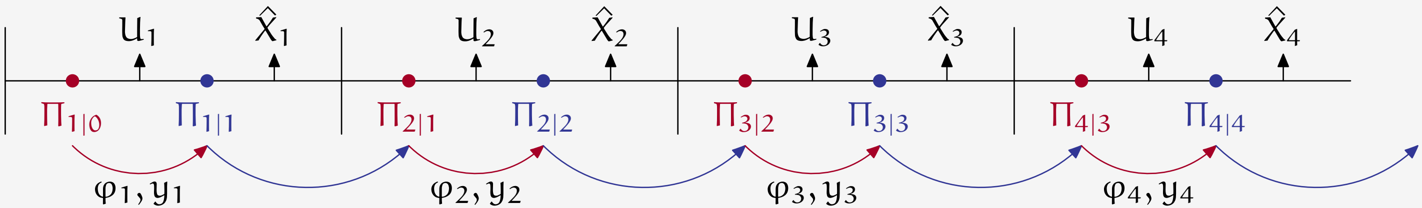

Information state update

- $\pi_{t+1|t} = \tilde{Q}(\pi_{t|t})$.
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In particular,
$$\pi_{t|t}(x) \coloneqq \begin{cases} \dfrac{\pi_{t|t-1}(x)\big[\varepsilon\varphi_t(x) + (1-\varphi_t(x))\big]}{\sum_{x' \in \mathcal{X}} \pi_{t|t-1}(x')\big[\varepsilon\varphi_t(x') + (1-\varphi_t(x'))\big]}, & \text{if } y_t = \mathfrak{E} \\ \delta_{y_t}, & \text{if } y_t \neq \mathfrak{E} \end{cases}$$

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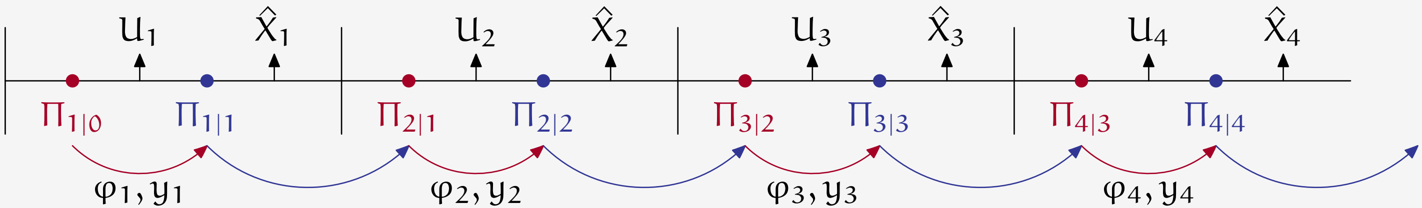

Structural results

There is no loss of optimality in using

$$U_t = f_t(X_t, \Pi_{t|t-1}) \quad \text{and} \quad \hat{X}_t = g_t(\Pi_{t|t}).$$

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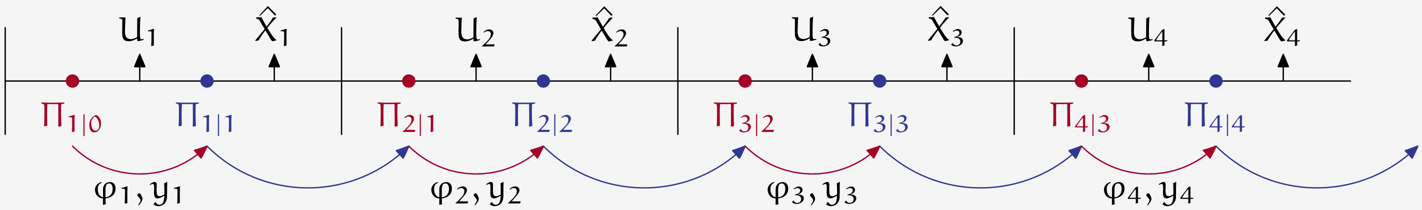

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Dynamic Program

$$V_{T+1|T}(\pi) = 0, \quad \text{and for } t = T, \dots, 0$$

$$V_{t|t}(\pi) = \min_{\hat{x} \in \mathcal{X}} \mathbb{E}[d(X_t - \hat{x}) + V_{t+1|t}(\Pi_{t+1}) \mid \Pi_{t|t} = \pi],$$

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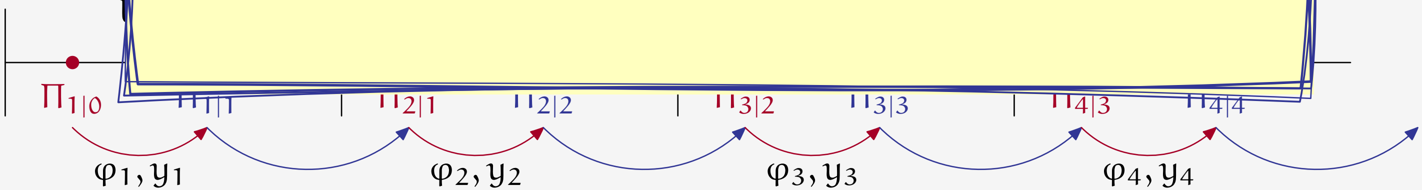
Information states and dynamic program

Information s

"Standard" POMDP. Optimal strategies can be computed numerically (at least, in principle).



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Can we use the DP to say something more about the optimal strategy?

Simplifying modeling assumptions

Markov process $X_{t+1} = aX_t + W_t$

- Discrete state process: $X_t, a, W_t \in \mathbb{Z}$
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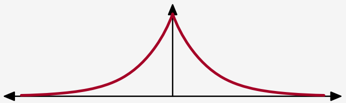
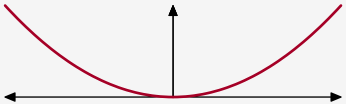
Noise Distribution Unimodal and symmetric



Distortion function Even and increasing



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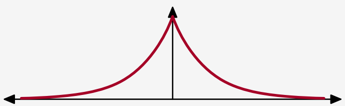

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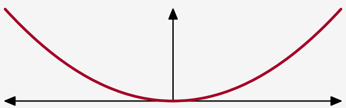

Proof outline

- **Step 1** Show that threshold-based strategies are optimal
- **Step 2** Find performance of arbitrary threshold based strategies
- **Step 3** Solution to the costly communication problem
- **Step 4** Solution to the constrained communication problem

Step 1 Preliminaries: Change of variables

Definition Let σ denote the last time a packet was received successfully. Define

$$E_t = X_t - a^{\sigma-t} X_\sigma$$

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We work with $\{E_t\}_{t \geqslant 0}$ rather than $\{X_t\}_{t \geqslant 0}$

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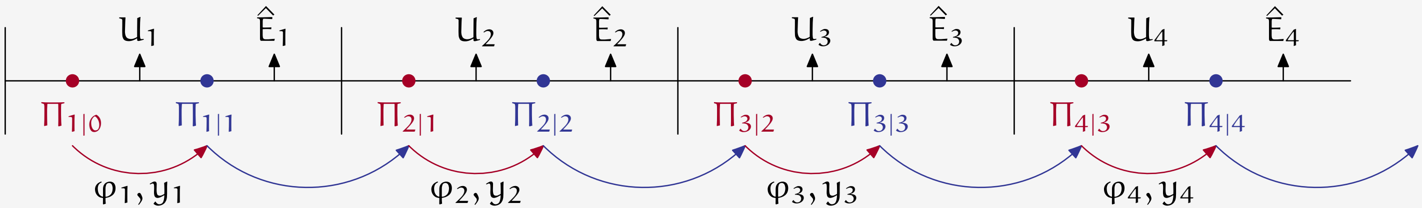

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Note that we can write $\pi_{t|t} = Q(\pi_{t|t-1}, \varphi_t, h_t)$, where $h_t = u_t s_t$

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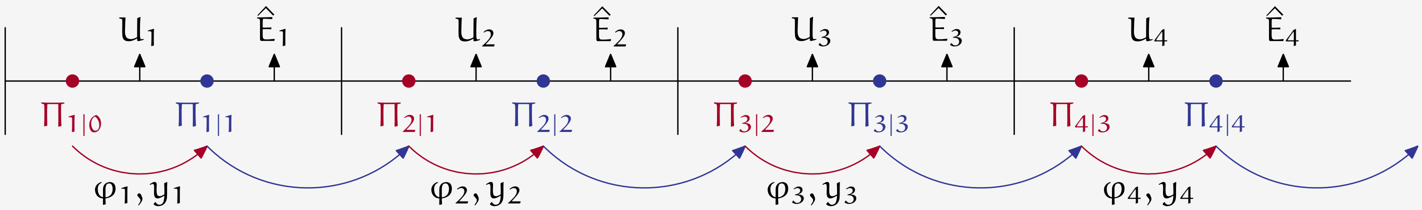
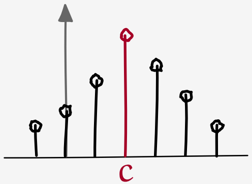


Dynamic Program Remains same as before

Step 1 Preliminaries: Majorization

[Hajek Mitzel Yang 2008, Lipsa Martins 2011, Nayyar et. al. 2013]

Almost uniform and unimodal (ASU) distribution about c



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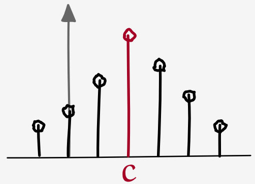

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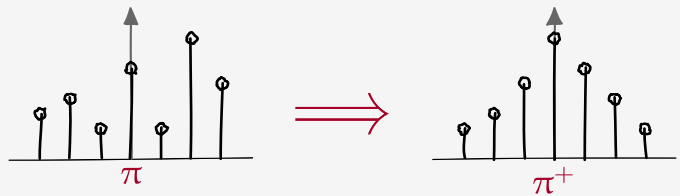
ASU Rearrangement



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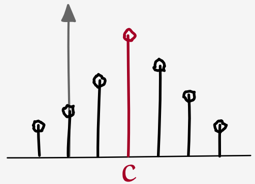

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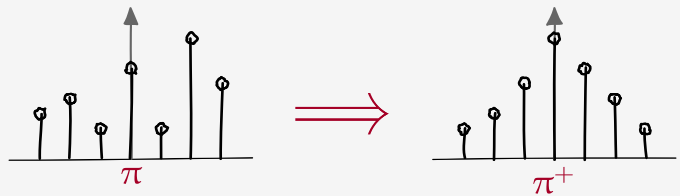
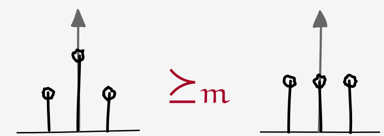


Majorization

$\xi \succeq_m \pi$ iff

$$\sum_{i=-n}^{n} \xi_i^+ \geqslant \sum_{i=-n}^{n} \pi_i^+ \quad \text{and} \quad \sum_{i=-n}^{n+1} \xi_i^+ \geqslant \sum_{i=-n}^{n+1} \pi_i^+$$



Invariant to permutations.

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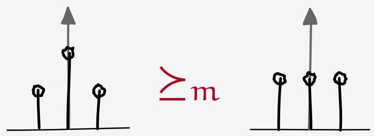

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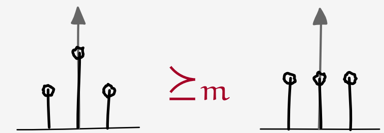
Step 1 Preliminaries: Majorization

Majorization $\xi \succeq_m \pi$ iff

$$\sum_{i=-n}^{n} \xi_i^+ \geqslant \sum_{i=-n}^{n} \pi_i^+ \quad \text{and} \quad \sum_{i=-n}^{n+1} \xi_i^+ \geqslant \sum_{i=-n}^{n+1} \pi_i^+$$



Invariant to permutations.

ASU Majorization $\xi \succeq_a \pi$ iff ξ is ASU and $\xi \succeq_m \pi$

Step 1 Properties of majorization

Threshold based strategies

Let $\mathcal{F}(c)$ denote the class of all threshold based strategies around c, i.e.,

$$\varphi \in \mathcal{F}(c) \quad \text{if } \exists k \text{ s.t.} \quad \varphi(e) = \begin{cases} 1 & \text{if } |e - ac| \geqslant k \\ 0 & \text{otherwise} \end{cases}$$

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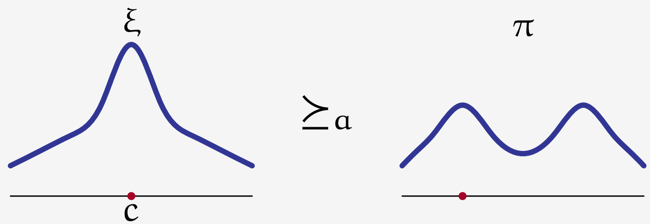
Property 1

For any $\xi \succeq_a \pi$ where ξ is ASU(c),



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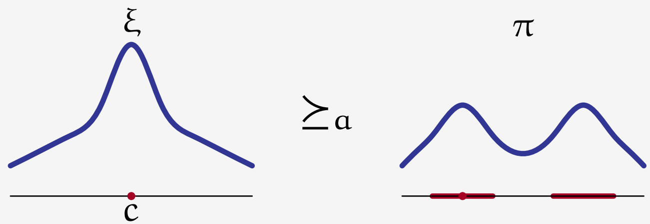

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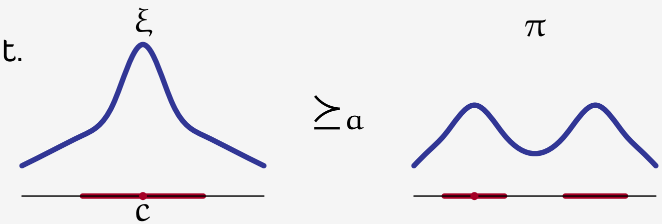

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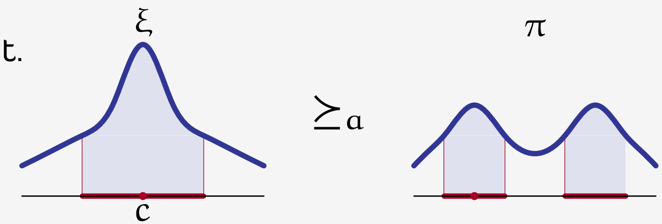
$$\sum_{e \in \mathcal{X}} \theta(e)\xi(e) = \sum_{e \in \mathcal{X}} \varphi(e)\pi(e).$$



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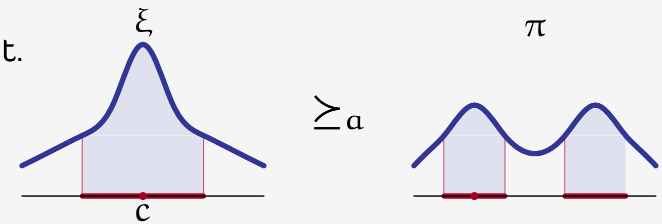
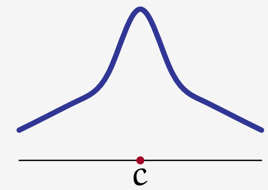


Moreover, for $h \in \{0, 1\}$ (recall $h = u \cdot s$), $Q(\xi, \theta, h) \succeq_a Q(\pi, \varphi, h)$.

Step 1 Properties of majorization

Property 2 If π is ASU(c), then $c \in \arg\min_{\hat{e} \in \mathcal{X}} \sum_{e \in \mathcal{X}} d(e - \hat{e})\pi(e)$



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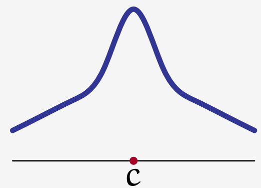

Property 3 if $\xi \succeq_a \pi$, then

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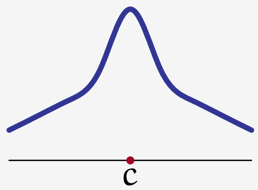

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Property 4 if $\xi \succeq_a \pi$, then $\tilde{Q}(\xi) \succeq_a \tilde{Q}(\pi)$

Step 1 An interchange argument to identify optimal strategies

Main theorem

The optimal estimation strategy is given as follows: $\hat{E}_0 = 0$ and for $t \geqslant 1$

$$\hat{E}_t = \begin{cases} 0, & \text{if } Y_t = \mathfrak{E} \\ E_t, & \text{if } Y_t \neq \mathfrak{E} \end{cases}$$

In addition, there exist thresholds $\{k_t\}_{t \geqslant 0}$ such that the following transmission strategy is optimal

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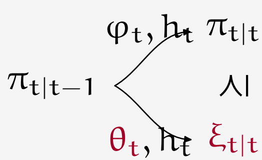

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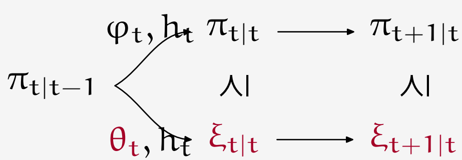

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$$\begin{array}{ccccc} & \varphi_t, h_t \nearrow & \pi_{t|t} & \longrightarrow & \pi_{t+1|t} \longrightarrow \\ \pi_{t|t-1} & & \succeq & & \succeq \\ & \theta_t, h_t \searrow & \xi_{t|t} & \longrightarrow & \xi_{t+1|t} \longrightarrow \end{array}$$

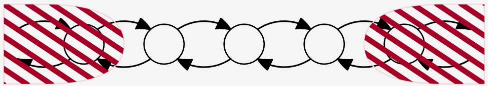
For infinite-horizon setup time-homogeneous **threshold-based strategies are optimal.**

How do we find the optimal threshold-based strategy?

Step 2 Performance of threshold-based strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } |e| \geq k \\ 0 & \text{otherwise} \end{cases}$$



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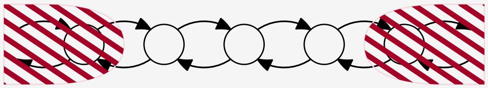
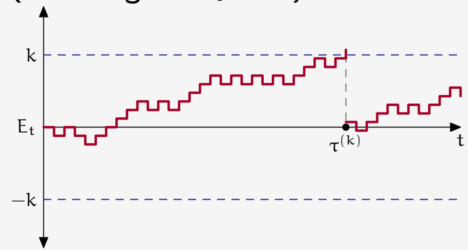


Let $\tau^{(k)}$ denote the stopping time of first reception (starting at $E_0 = 0$).



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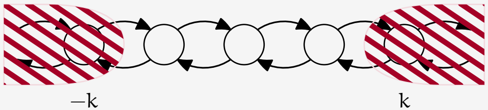

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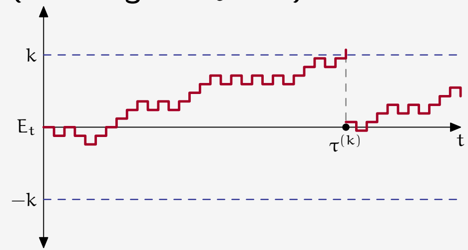

Define

$$L_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \,\Big|\, E_0 = e \Big].$$

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$$K_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t U_t \,\Big|\, E_0 = e \Big].$$

Step 2 Performance of threshold-based strategies

Proposition $\{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal relationships, we have:

$$D_\beta^{(k)} \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)}$$

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Computing $L_\beta^{(k)}$, $M_\beta^{(k)}$, $K_\beta^{(k)}$ is sufficient to compute the performance of $f^{(k)}$ (i.e., to compute $D_\beta^{(k)}$ and $N_\beta^{(k)}$).

$$K_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0} \beta^t U_t \Big| E_0 = e\Big].$$

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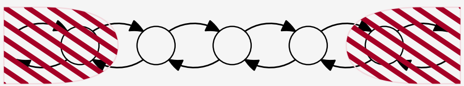

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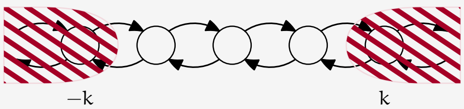

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$L_\beta^{(k)} = [I - \beta h^{(k)} \odot P]^{-1} h^{(k)} \odot d$
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$$M_\beta^{(k)}(e) = \begin{cases} 1 + \beta \sum_{n \in \mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n), & \text{if } |e| < k \\ \varepsilon\Big[1 + \beta \sum_{n \in \mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n)\Big], & \text{if } |e| \geq k \end{cases}$$

$$K_\beta^{(k)} = [I - \beta h^{(k)} \odot P]^{-1} \bar{h}^{(k)}$$

$$K_\beta^{(k)}(e) = \begin{cases} \beta \sum_{n \in \mathbb{Z}} p_{n-ae} K_\beta^{(k)}(n), & \text{if } |e| < k \\ 1 + \varepsilon\beta \sum_{n \in \mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n), & \text{if } |e| \geq k \end{cases}$$

Step 2 Computing $L_\beta^{(k)}$, $M_\beta^{(k)}$, and $K_\beta^{(k)}$: The discrete case

$L_\beta^{(k)} = [I - \beta h^{(k)} \odot P]^{-1} h^{(k)} \odot d$
where $h^{(k)} \odot P$ is substochastic.

$$L_\beta^{(k)}(e) = \begin{cases} d(e) + \beta \sum_{n \in \mathbb{Z}} p_{n-ae} L_\beta^{(k)}(n), & \text{if } |e| < k \\ \ldots \end{cases}$$

$M_\beta^{(k)} = [I - \beta h^{(k)} \odot P]^{-1} h^{(k)}$

$D_\beta^{(k)}$ and $N_\beta^{(k)}$ can be computed using these expressions.

$K_\beta^{(k)} = [I - \beta h^{(k)} \odot P]^{-1} \bar{h}^{(k)}$

$$K_\beta^{(k)}(e) = \begin{cases} \beta \sum_{n \in \mathbb{Z}} p_{n-ae} K_\beta^{(k)}(n), & \text{if } |e| < k \\ 1 + \varepsilon\beta \sum_{n \in \mathbb{Z}} p_{n-ae} M_\beta^{(k)}(n), & \text{if } |e| \geq k \end{cases}$$

Step 2 Computing $L_\beta^{(k)}$, $M_\beta^{(k)}$, and $K_\beta^{(k)}$: The continuous case

- The expressions are similar to the discrete case.
- $h^{(k)} \odot P$ is a contraction operator
- The equations for $L_\beta^{(k)}$, etc. are Fredholm integral equations of the second kind. Numerical solution can be obtained by using Picard's iteration and Nystrom interpolation.

We will later provide a simulation based approach to compute $C_\beta^*(\lambda)$ and $D_\beta^*(\alpha)$ that does not need an exact computation of $L_\beta^{(k)}$, etc.

Optimal trade-offs for costly and constrained communication for discrete sources

Step 3 Solution to costly optimization problem

Proposition

- $C_\beta^{(k)}(\lambda) \coloneqq D_\beta^{(k)} + \lambda N_\beta^{(k)}$ is submodular in (k, λ).
- Hence, $k_\beta^*(\lambda) \coloneqq \arg\min_{k \geq 0} C_\beta^{(k)}(\lambda)$ is increasing in λ

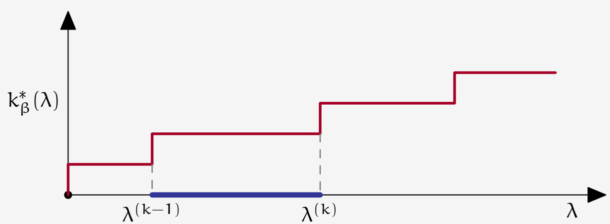
Step 3 Solution to costly optimization problem

Proposition

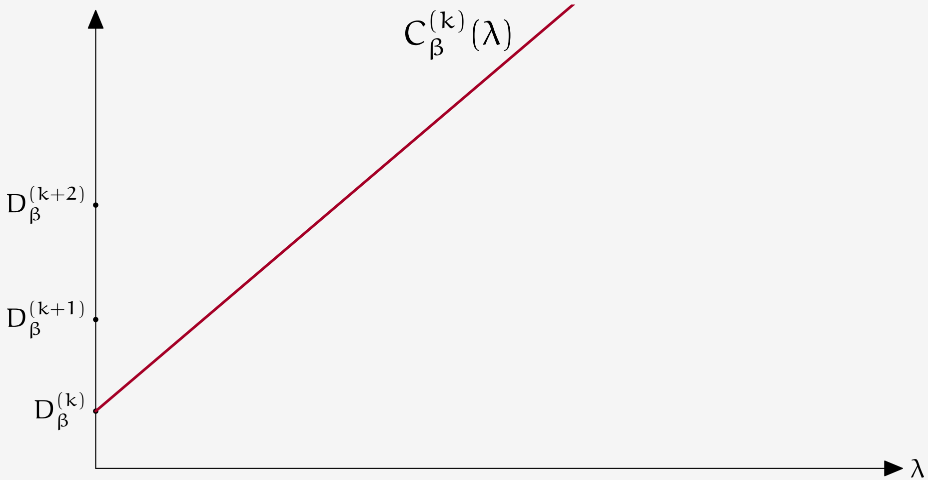
- $C^{(k)}_\beta(\lambda) \coloneqq D^{(k)}_\beta + \lambda N^{(k)}_\beta$ is submodular in (k, λ).
- Hence, $k^*_\beta(\lambda) \coloneqq \arg\min_{k \geqslant 0} C^{(k)}_\beta(\lambda)$ is increasing in λ

Define $\Lambda^{(k)}_\beta \coloneqq \{\lambda \in \mathbb{R}_{\geqslant 0} : k^*_\beta(\lambda) = k\}$
$= [\lambda^{(k-1)}_\beta, \lambda^{(k)}_\beta]$.

$$C^{(k)}_\beta(\lambda^{(k)}_\beta) = C^{(k+1)}_\beta(\lambda^{(k)}_\beta)$$



Step 3 Solution to costly optimization problem



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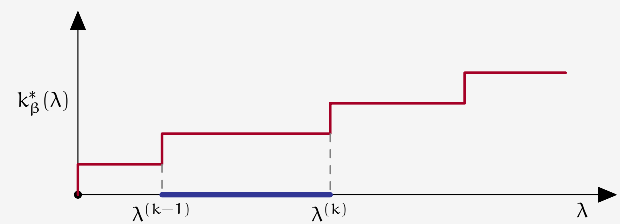
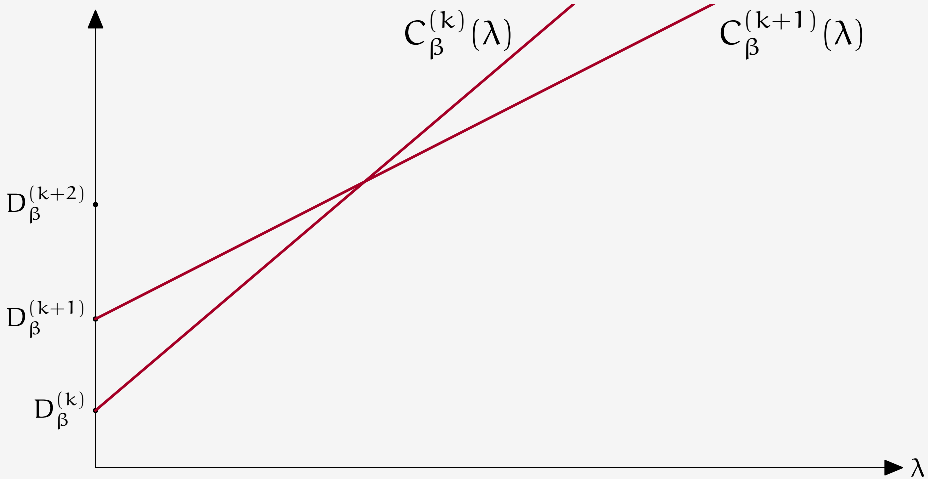


$$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$$

Step 3 Solution to costly optimization problem



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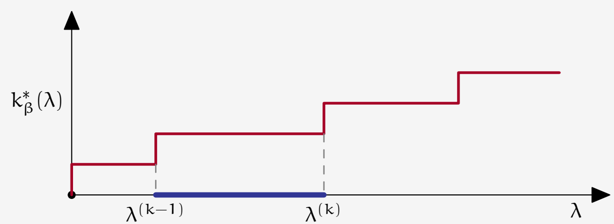

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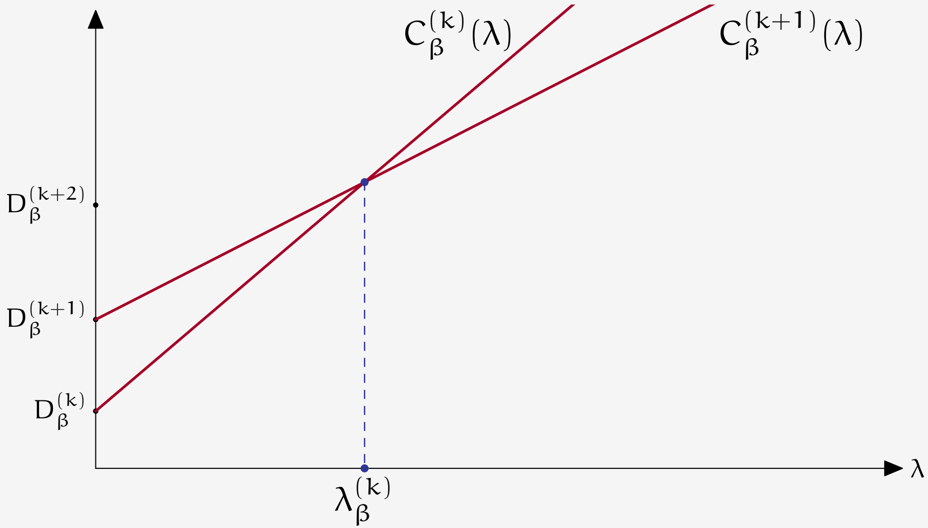

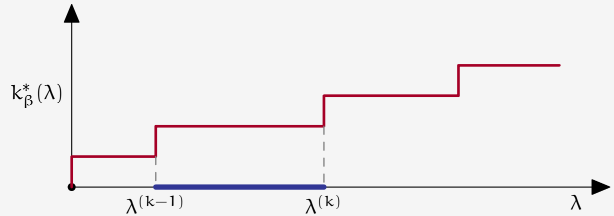

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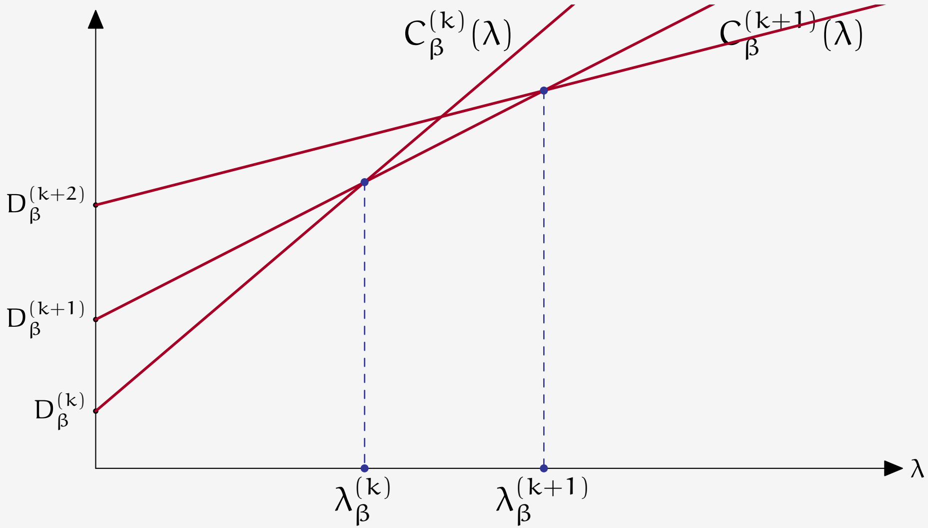

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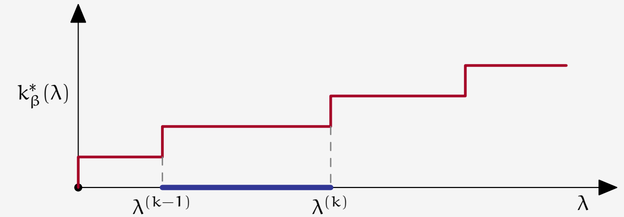
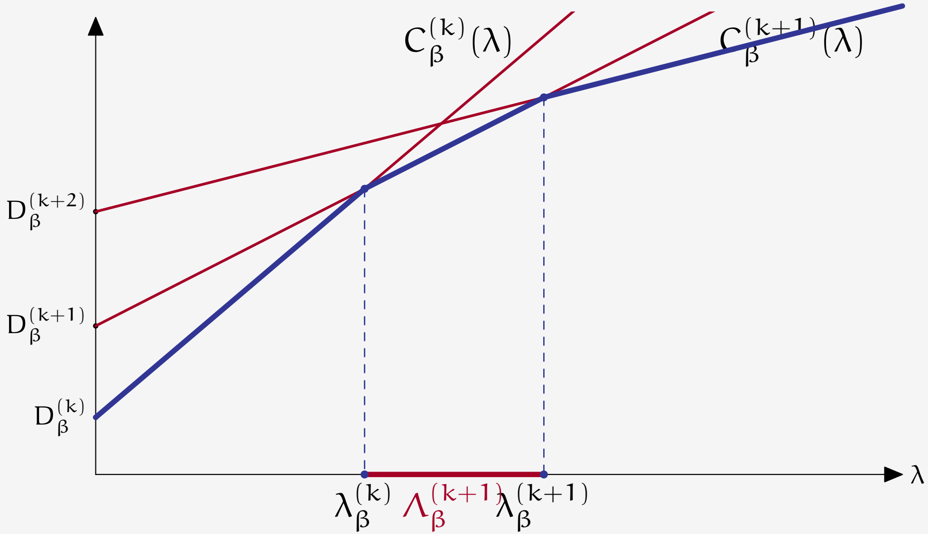


Step 3 Solution to costly optimization problem



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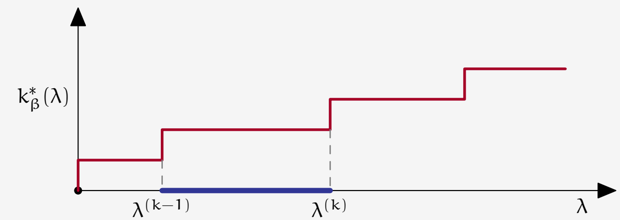

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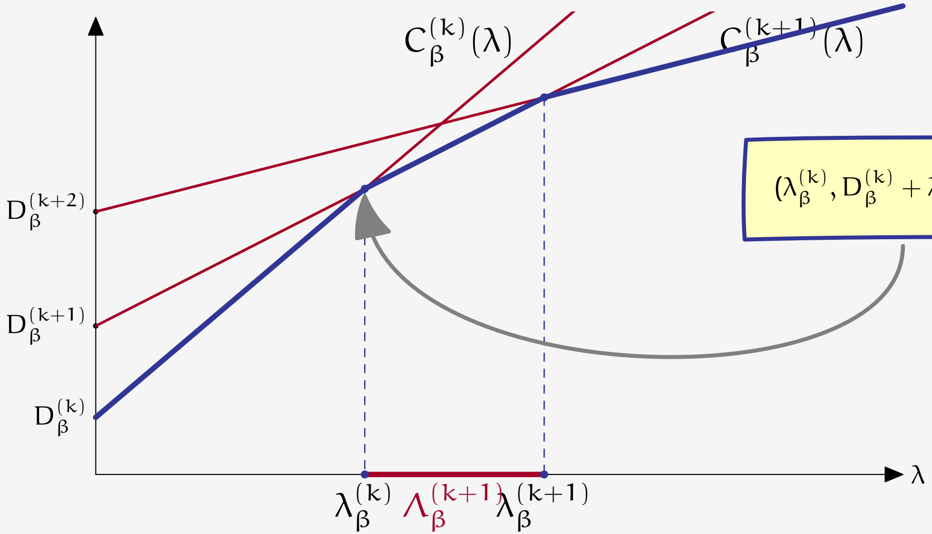
Step 3 Solution to costly optimization problem



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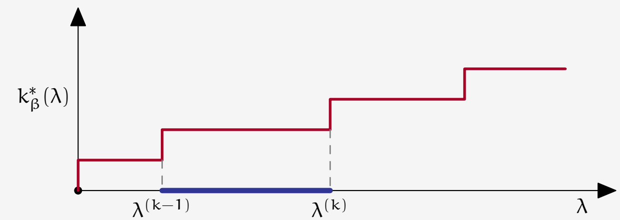

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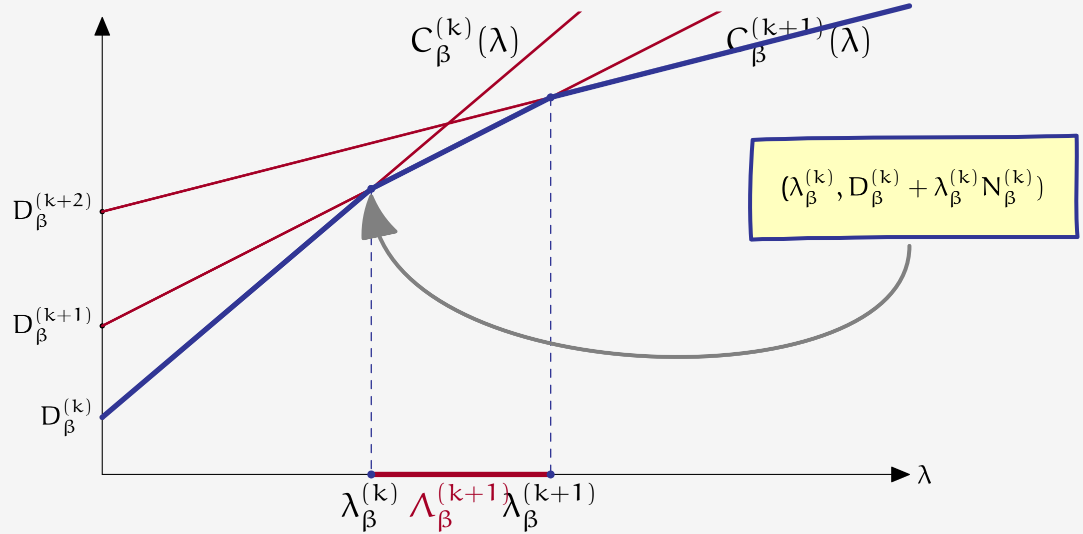

Theorem

Strategy $f^{(k+1)}$ is optimal for $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$.

$C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}_{\geq 0}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

Step 4 Solution to constrained communication problem

Sufficient condition for optimality

A strategy (f°, g°) is optimal for the constrained problem if

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$

(C2) There exists $\lambda^\circ \geqslant 0$ such that (f°, g°) is optimal for the Lagrange relaxation with parameter λ°.

Solution to constrained communication problem

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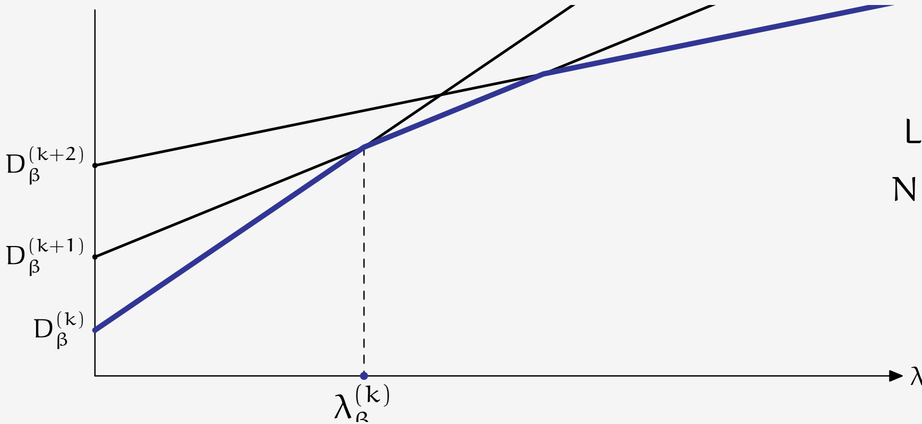

Let k_β^* be such that

$$N_\beta^{(k_\beta^*)} > \alpha > N_\beta^{(k_\beta^*+1)}$$

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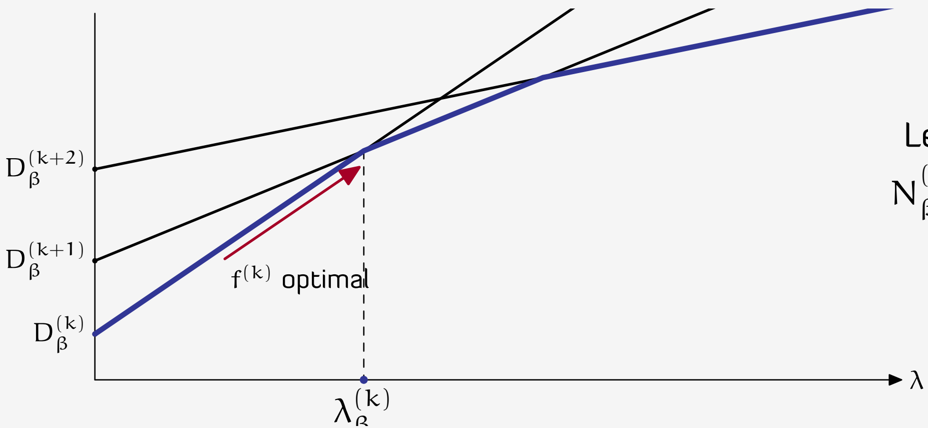

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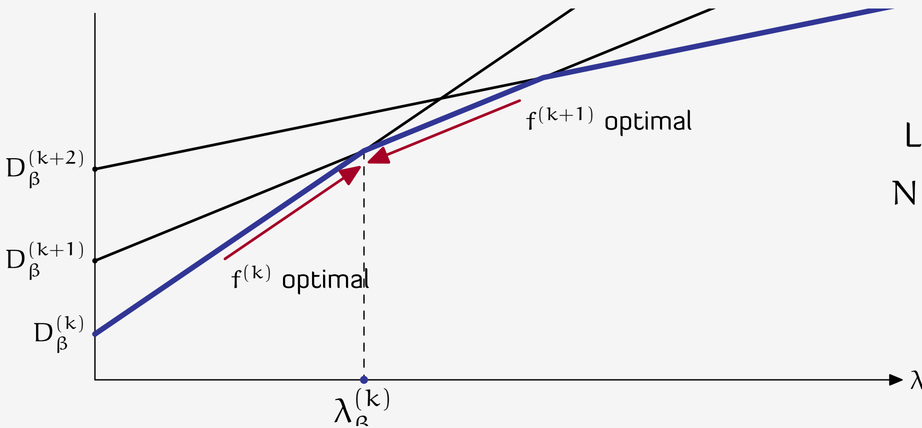

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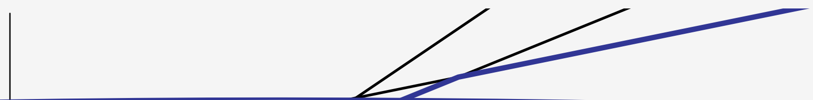

such that

$> N_\beta^{(k^*_\beta+1)}$

Randomized strategy $(\theta^*, f^{(k)}, f^{k+1})$ is optimal where

$$\theta^* N_\beta^{(k)} + (1-\theta^*) N_\beta^{(k+1)} = \alpha$$

Sufficient condition for optimality

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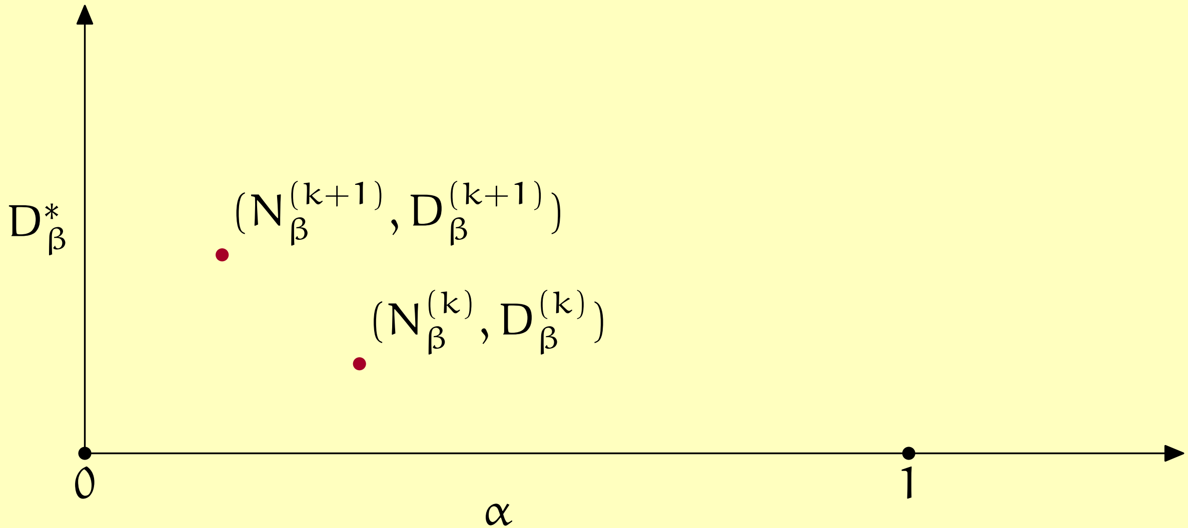

D^*_β is PWL, dec, and convex

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Sufficient condition for optimality

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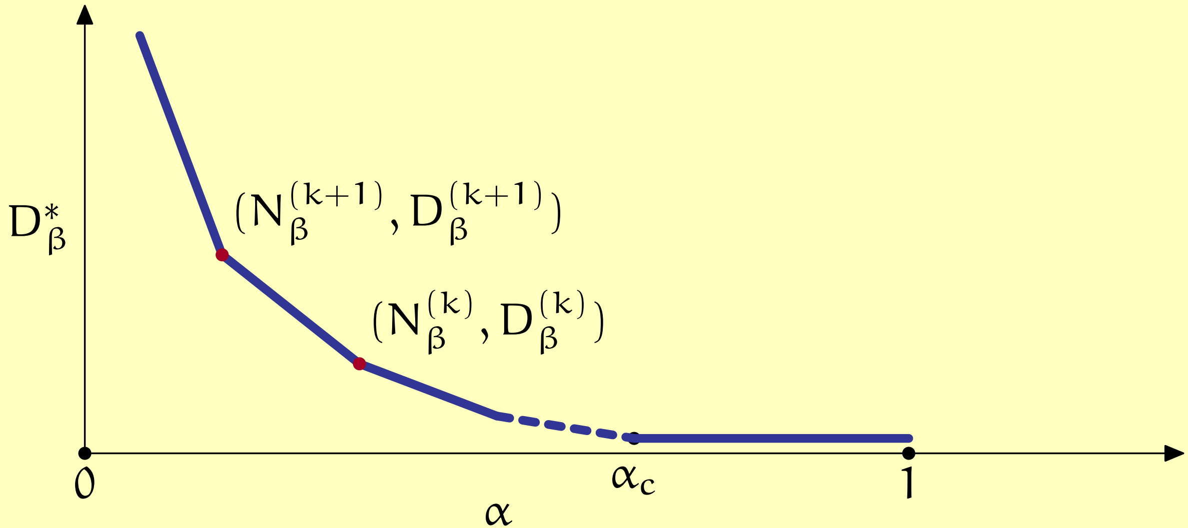
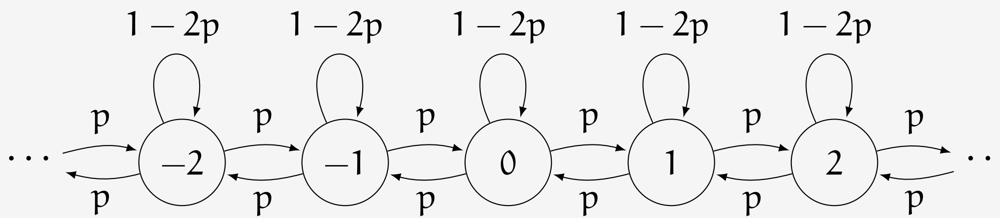


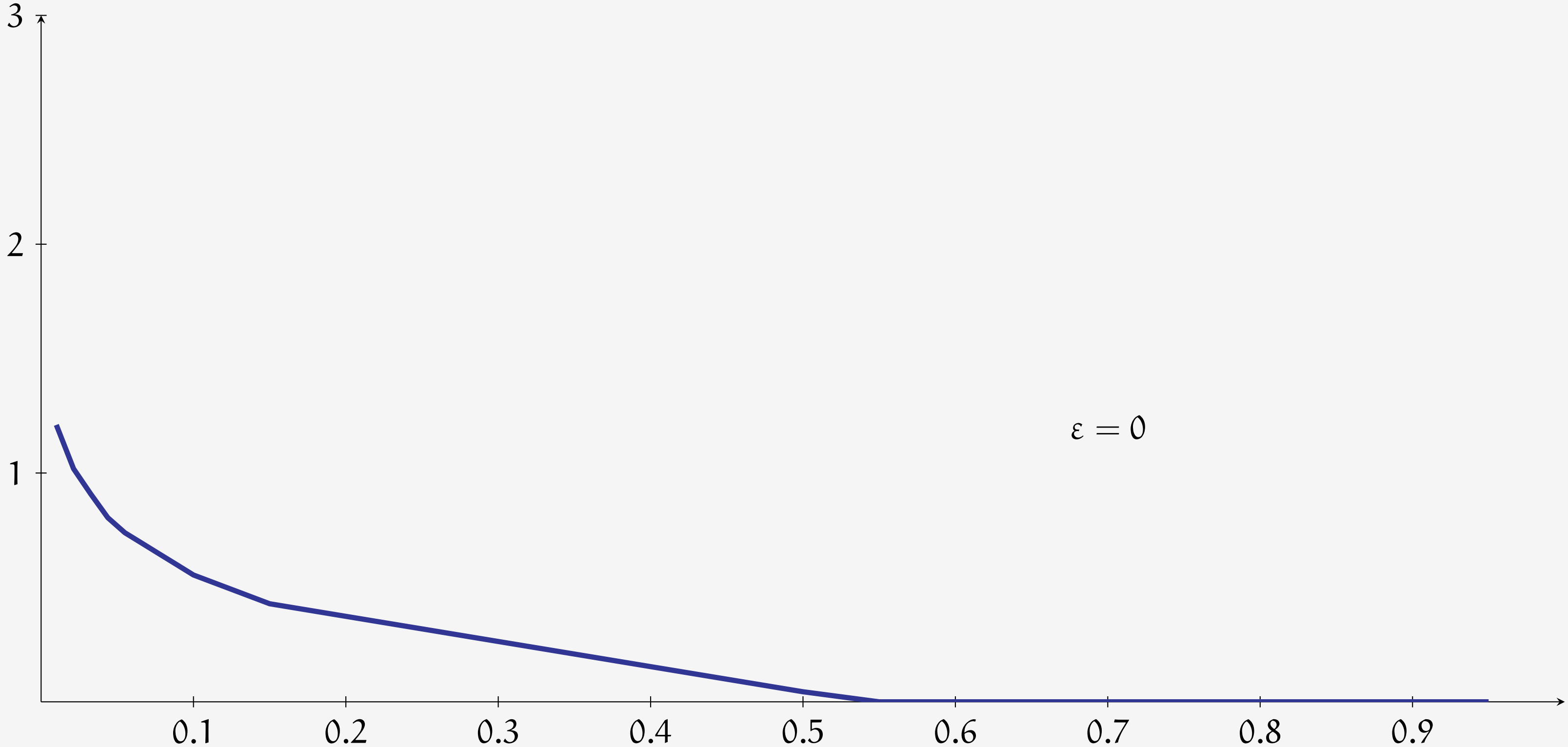
D^*_β is PWL, dec, and convex

Example Symmetric birth-death Markov chain

$$p_n = \begin{cases} p, & \text{if } |n| = 1; \\ 1-2p, & \text{if } n = 0; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{where } p \in (0, \tfrac{1}{3}), \qquad d(e) = |e|$$



Example Symmetric birth-death Markov chain $(p = 0.3, \beta = 0.9)$



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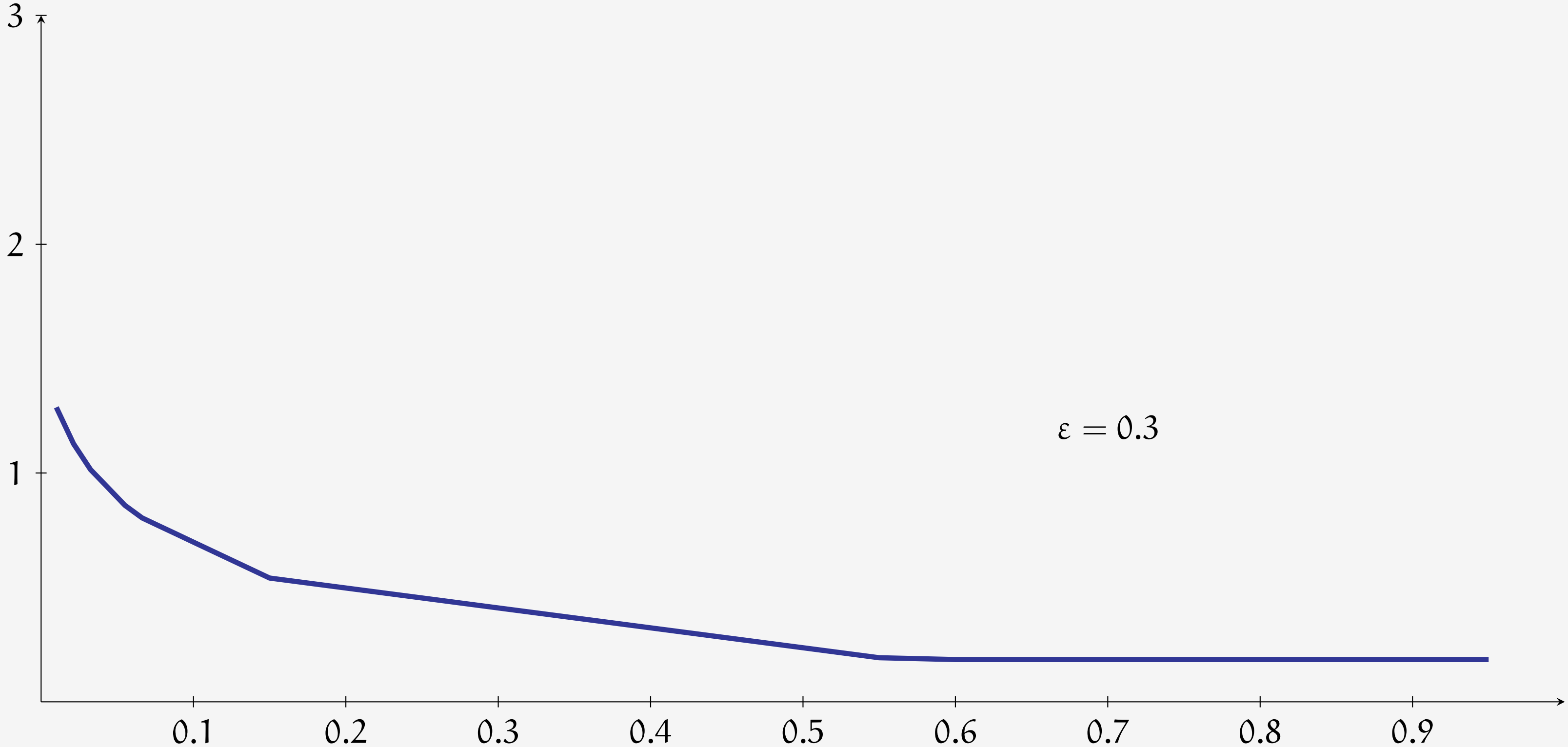

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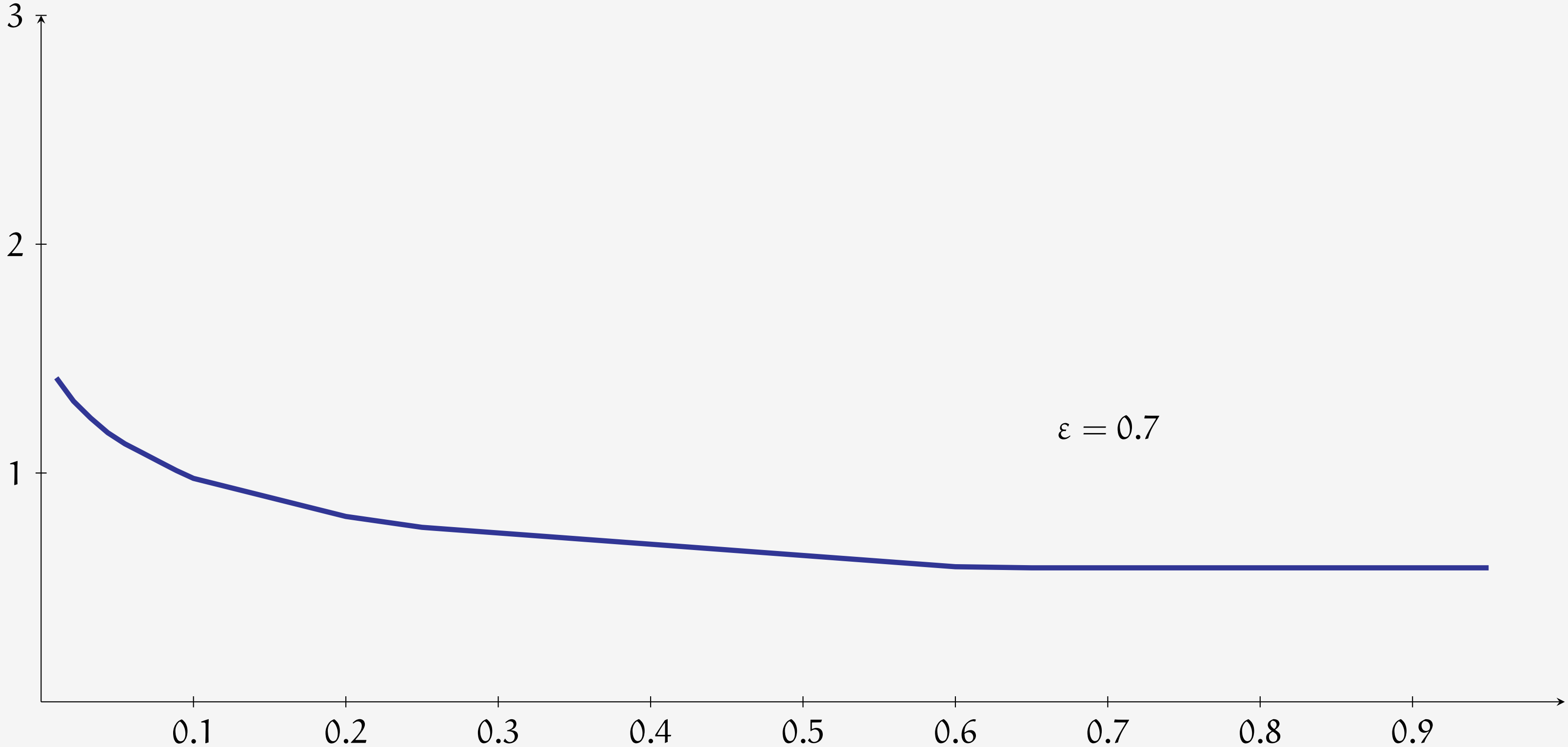

Optimal trade-offs for costly and constrained communication for continuous sources

Step 3 Solution to costly optimization problem

Proposition As in the case of discrete sources:

- $C^{(k)}_\beta(\lambda) \coloneqq D^{(k)}_\beta + \lambda N^{(k)}_\beta$ is submodular in (k, λ).
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Theorem If the pair (λ, k) satisfies

$$\lambda = -\frac{\partial_k D_\beta^{(k)}}{\partial_k N_\beta^{(k)}} \quad (\text{i.e., } \partial_k D_\beta^{(k)} + \lambda \partial_k N_\beta^{(k)} = 0)$$

then the strategy $(f^{(k)}, g^*)$ is optimal for the costly communication with cost λ.

The optimal performance $C_\beta^*(\lambda)$ is continuous, concave and increasing function of λ.

Step 3 Solution to costly optimization problem

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The optimal performance $C^*_\beta(\lambda)$ is continuous, concave and increasing function of λ.

Scaling with variance for Gaussian noise

$$C^*_{\beta,\sigma}(\lambda) = \sigma^2 C^*_{\beta,1}\left(\frac{\lambda}{\sigma^2}\right).$$

Step 4 Solution to constrained optimization problem

Theorem For any $\beta \in (0,1]$ and $\alpha \in (0,1)$, let $k^*_\beta(\alpha)$ be such that

$$N_\beta^{(k^*_\beta(\alpha))} = \alpha.$$

Such a $k^*_\beta(\alpha)$ always exists and we have the following:

- The strategy $(f^{(k^*_\beta(\alpha))}, g^*)$ is optimal for the constrained optimization problem with constraint α
- The distortion transmission function $D^*_\beta(\alpha)$ is continuous, convex, and decreasing in α and is given by

$$D^*_\beta(\alpha) = D_\beta^{(k^*_\beta(\alpha))}$$

Step 4 Solution to constrained optimization problem

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Scaling with variance for Gaussian noise

$$k^*_{\beta,\sigma}(\alpha) = k^*_{\beta,1}(\alpha) \text{ and } D^*_{\beta,\sigma}(\alpha) = \sigma^2 D^*_{\beta,1}(\alpha).$$

Computation of optimal thresholds

Costly communication Given λ, find k such that $\partial_k(D_\beta^{(k)} + \lambda N_\beta^{(k)}) = 0$.

Constrained communication Given α, find k such that $N_\beta^{(k)} = \alpha$.

Computation of optimal thresholds

Costly communication Given λ, find k such that $\partial_k(D^{(k)}_\beta + \lambda N^{(k)}_\beta) = 0$.

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Main idea

- Pick a threshold k and use strategy $f^{(k)}$ until first successful reception.
- The sample path values of L, M, and K may be viewed as a "noisy" observation of true $L^{(k)}_\beta$, $M^{(k)}_\beta$, and $K^{(k)}_\beta$.
- Use stochastic approximation to find optimal thresholds.

Computation of optimal thresholds

Kiefer-Wolfowitz Algorithm

Robbins-Monro Algorithm

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Computation of optimal thresholds

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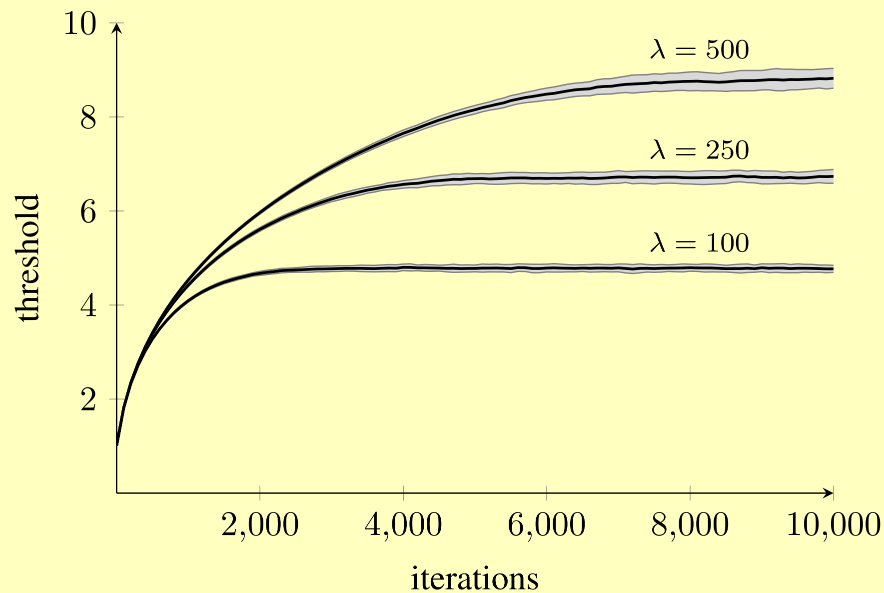

Costly communication ($\beta = 0.9$, $\varepsilon = 0.3$)

Computation of optimal thresholds

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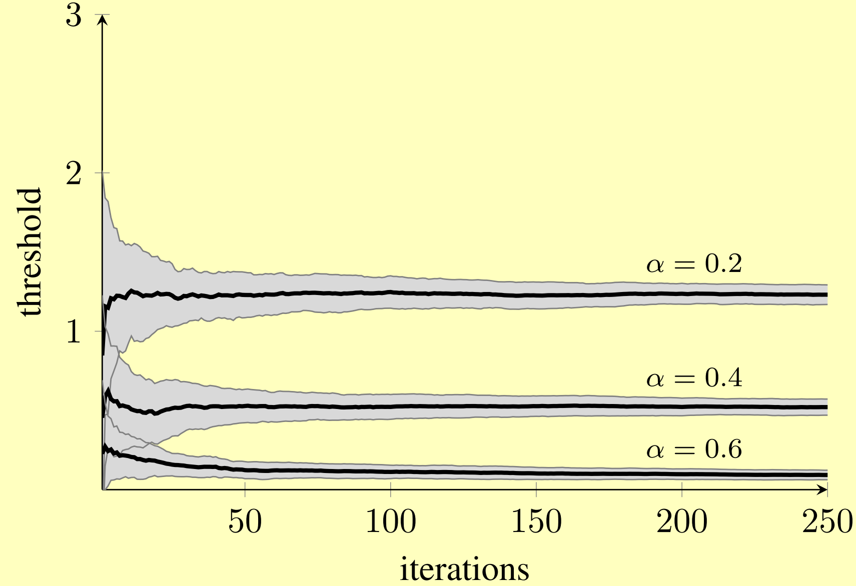
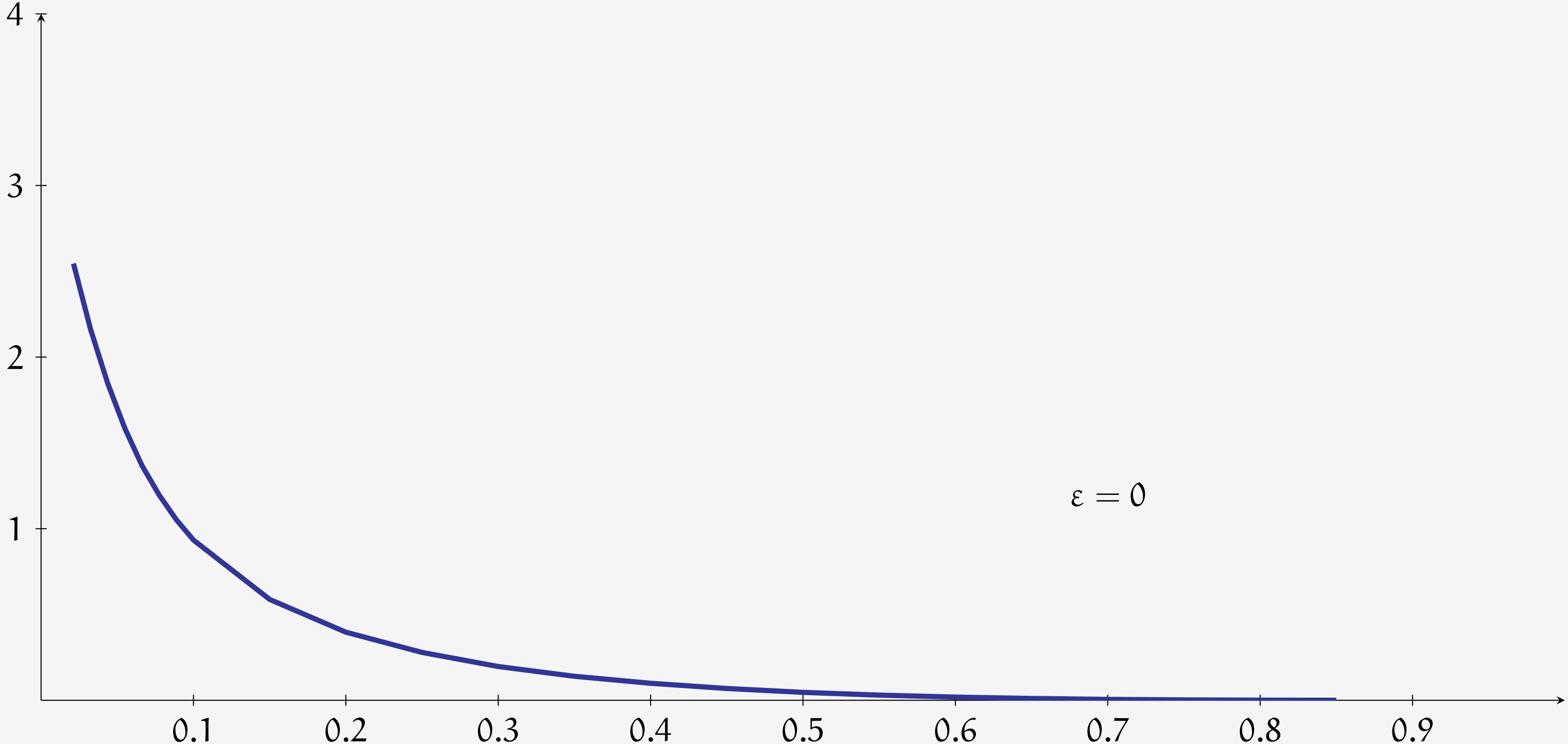


Constrained communication ($\beta = 0.9$, $\varepsilon = 0.3$)

Distortion transmission fn for Gauss-Markov process $(\sigma^2 = 1, \beta = 0.9)$



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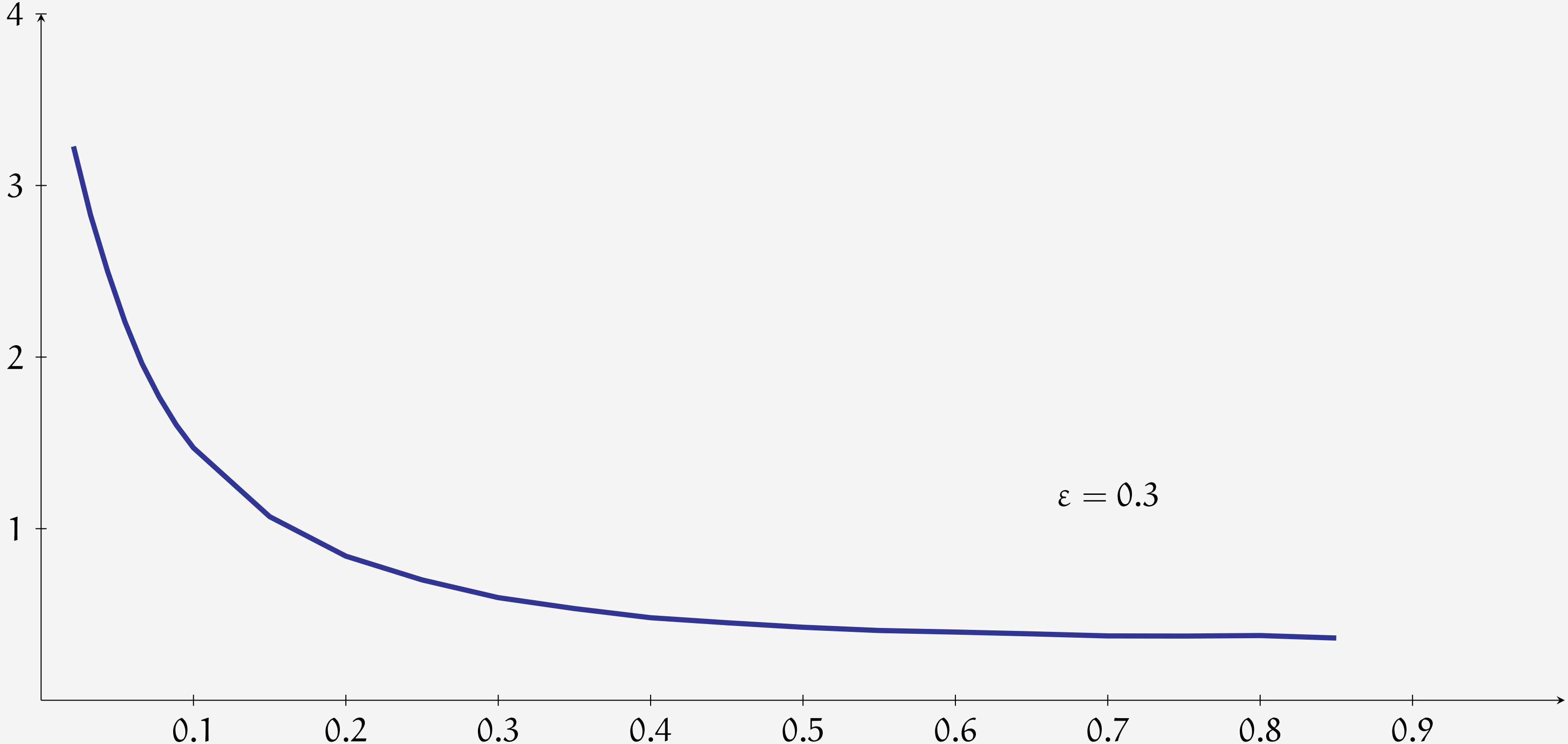

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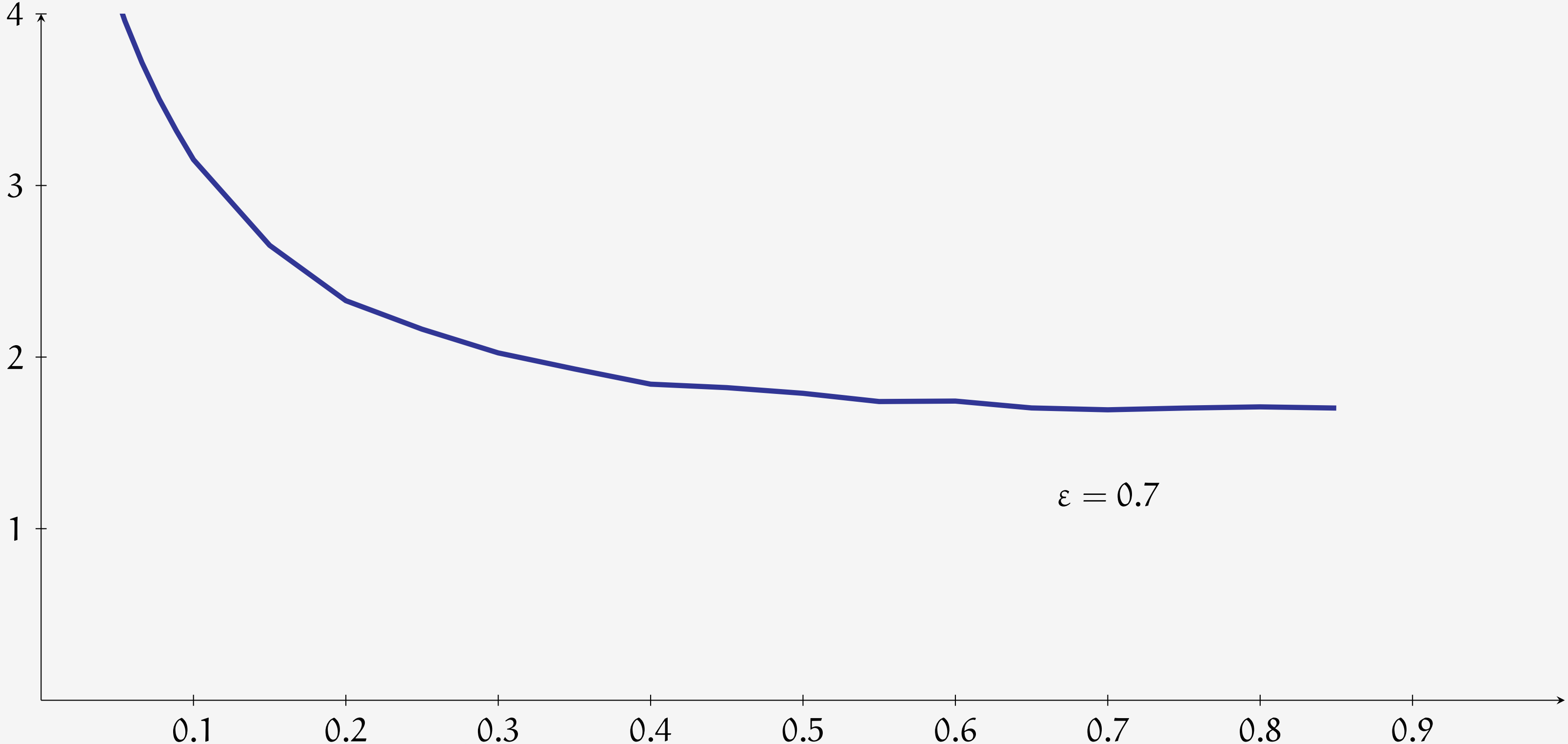
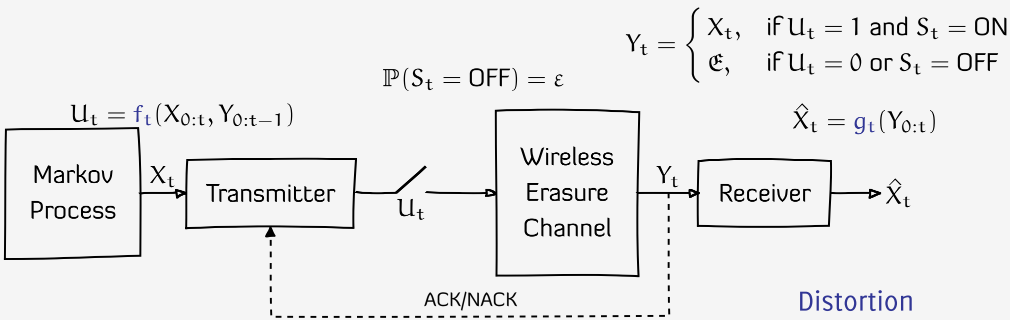


Summary



Distortion

$d(X_t - \hat{X}_t)$

1. Discounted setup, $\beta \in (0, 1)$

$$D_\beta(f, g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\Big]; \qquad N_\beta(f, g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f, g) = \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\Big]; \qquad N_1(f, g) = \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

Summary

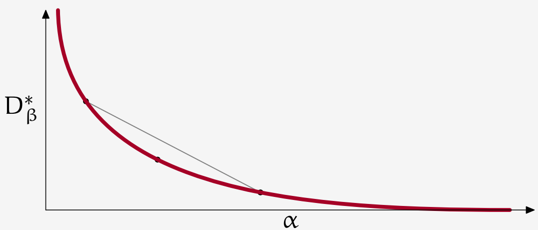
Optimization problems

Constrained communication

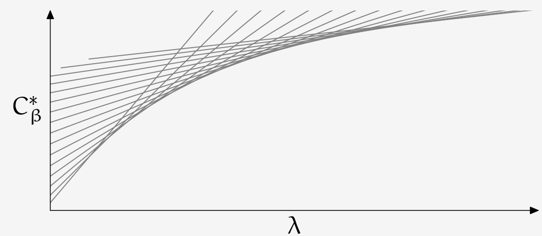
$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \}$$

Costly communication (Lagrange relaxation)

$$\text{For } \lambda \in \mathbb{R}_{>0}, \quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) + \lambda N_\beta(f,g) \}$$



D^*_β is cts, dec, and convex



Summary

Optimization problems

Distortion transmission function for auto-regressive sources

Source model $X_{t+1} = aX_t + W_t$, where W_t has symmetric and unimodal distribution. $X_t \in \mathbb{Z}/\mathbb{R}$.

Optimal transmission strategy

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- $K_\beta^{(k)}$: Expected discounted number of transmissions until first successful reception.
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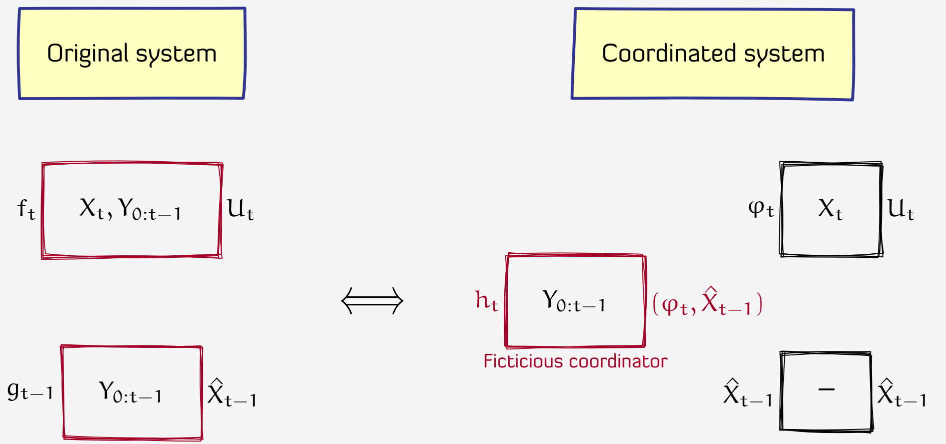
Summary

Optimization problems

Distortion transmission function for auto-regressive sources

The common information approach (Nayyar, Mahajan, Teneketzis 2013)



- The coordinated system is equivalent to the original system.

  $$f_t(x, y_{0:t-1}) = h_t^1(y_{0:t-1})(x).$$

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- Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Summary

Simplifying modeling assumptions

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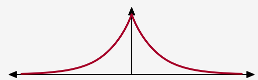

Noise Distribution: Unimodal and symmetric

Distortion function: Even and increasing

Proof outline

- **Step 1** Show that threshold-based strategies are optimal
- **Step 2** Find performance of arbitrary threshold based strategies
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Summary

Computation of optimal thresholds

Costly communication: Given λ, find k such that $\partial_k(D^{(k)}_\beta + \lambda N^{(k)}_\beta) = 0$.

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Constrained communication: Given α, find k such that $N^{(k)}_\beta = \alpha$.

Robbins-Monro Algorithm

Main idea

- Pick a threshold k and use strategy $f^{(k)}$ until first successful reception.
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Concluding Remarks

Generalization to vector sources

- Difficulty: If X_t is ASU, is $AX_t + W_t$ also ASU?
- Even if threshold policies are not optimal, the tools developed may be useful to identify best threshold-based strategy.

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Results are derived under idealized assumptions

Future directions

- Power or rate control ...
- Markovian or burst erasures ...
- Scheduling multiple sources ...
- Model network delays ...

References

Jhelum Chakravorty and Aditya Mahajan, "Fundamental limits of remote estimation of autoregressive Markov processes under communication constraints," IEEE TAC, Sep 2017 (to appear).

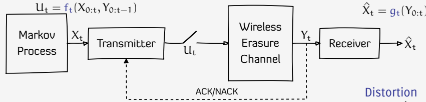
Jhelum Chakravorty and Aditya Mahajan, "Remote-state estimation with packet drop," IFAC Conference on Networked Control Systems (NecSys), Aug 2016. (Best Student Paper Award)

Jhelum Chakravorty, Jayakumar Subramanian, and Aditya Mahajan, "Stochastic approximation based methods for computing the optimal thresholds in remote-state estimation with packet drops," ACC 2017 (submitted)

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \text{ and } S_t = \text{ON} \\ \mathfrak{E}, & \text{if } U_t = 0 \text{ or } S_t = \text{OFF} \end{cases}$$

$\mathbb{P}(S_t = \text{OFF}) = \varepsilon$

$U_t = f_t(X_{0:t}, Y_{0:t-1})$ $\hat{X}_t = g_t(Y_{0:t})$



Distortion $d(X_t - \hat{X}_t)$

1. Discounted setup, $\beta \in (0,1)$

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\Big]; \qquad N_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\Big]; \qquad N_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

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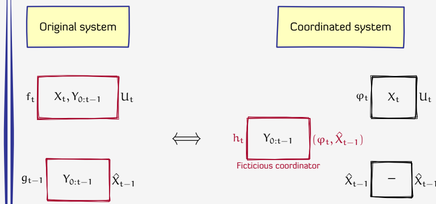

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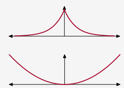

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Step 1 Properties of majorization

Threshold based strategies

Let $\mathcal{F}(c)$ denote the class of all threshold based strategies around c, i.e.,

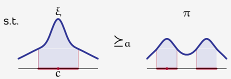
$$\varphi \in \mathcal{F}(c) \quad \text{if } \exists k \text{ s.t.} \quad \varphi(e) = \begin{cases} 1 & \text{if } |e - ac| \geq k \\ 0 & \text{otherwise} \end{cases}$$

Property 1

For any $\xi \succeq_a \pi$ where ξ is ASU(c),

and any φ, there exists a $\theta \in \mathcal{F}(c)$ s.t.

$$\sum_{e\in\mathcal{X}} \theta(e)\xi(e) = \sum_{e\in\mathcal{X}} \varphi(e)\pi(e).$$

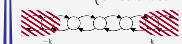


Moreover, for $h \in \{0,1\}$ (recall $h = u \cdot s$), $Q(\xi, \theta, h) \succeq_a Q(\pi, \varphi, h)$.

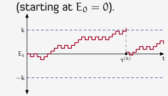
Step 2 Performance of threshold-based strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } |e| \geq k \\ 0 & \text{otherwise} \end{cases}$$



Let $\tau^{(k)}$ denote the stopping time of first reception (starting at $E_0 = 0$).



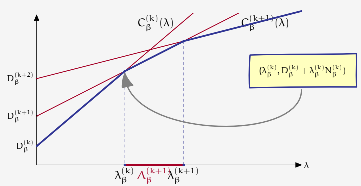
Define

$$L_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big].$$

$$M_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Big| E_0 = e\Big].$$

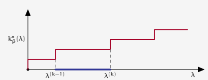
$$K_\beta^{(k)}(e) = \mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t U_t \Big| E_0 = e\Big].$$

Step 3 Solution to costly optimization problem



Define $\Lambda_\beta^{(k)} := \{\lambda \in \mathbb{R}_{\geq 0} : k_\beta^*(\lambda) = k\}$
$= [\lambda_\beta^{(k-1)}, \lambda_\beta^{(k)}]$.

$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$



Step 4 Solution to constrained communication problem

Sufficient condition for optimality

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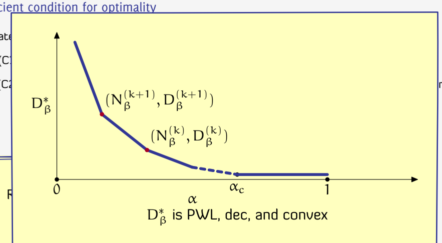

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