Fundamental limits of remote-estimation under communication constraints

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Joint work with Jhelum Chakravorty

Mathematical Cybernetics: Hybrid, Stochastic and Decentralized Systems
Carleton University, Ottawa, 28–29 May 2015

Motivation

Many applications require:

- Sequential transmission of data
- Zero- (or finite-) delay reconstruction

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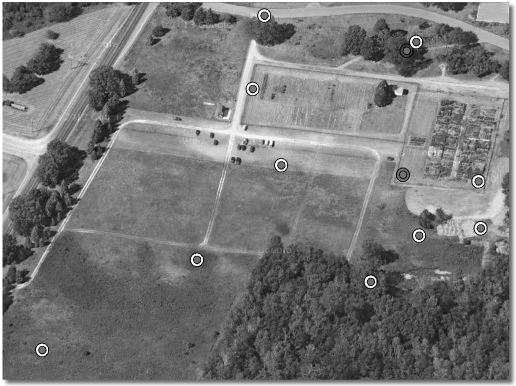

Sensor Networks

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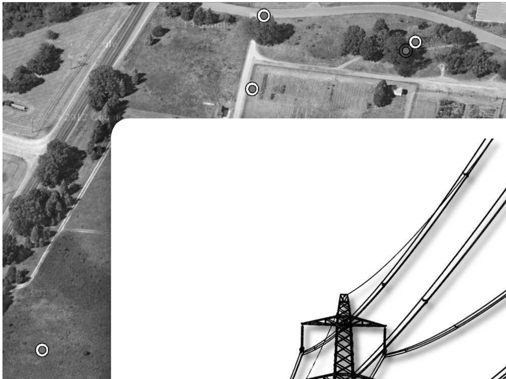

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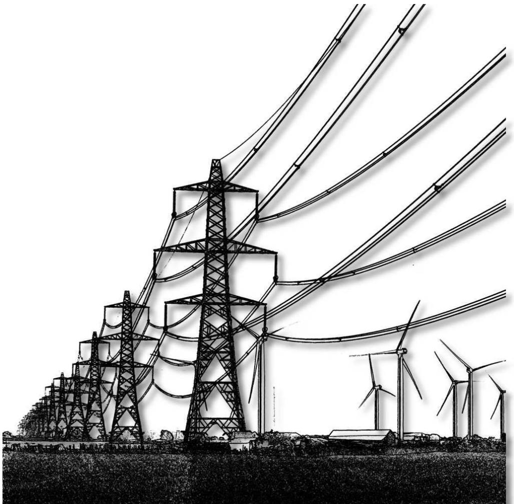

Smart Grids

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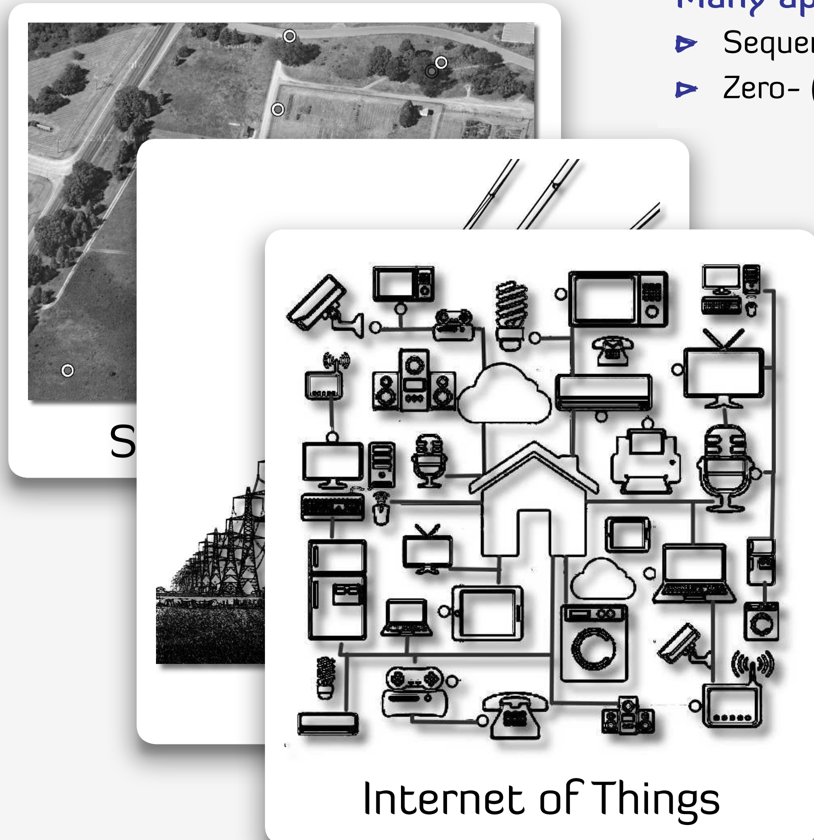

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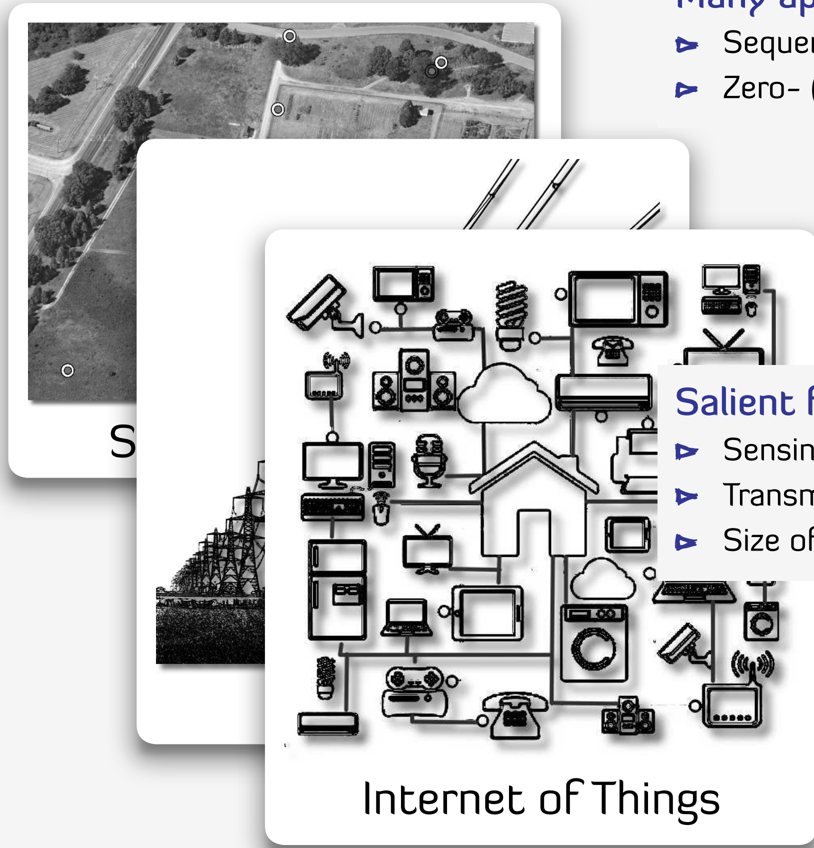

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Salient features

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical

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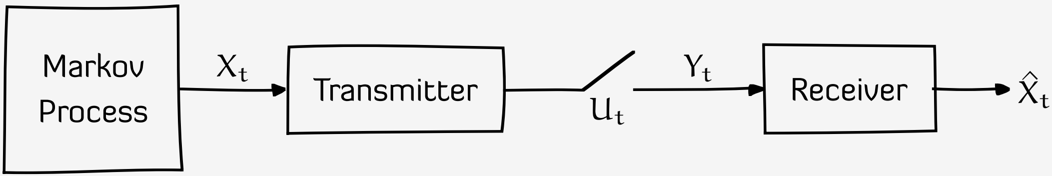
Analyze a stylized model and evaluate fundamental trade-offs

A completely solved example of a “simple” decentralized system with non-classical information structure

The system model



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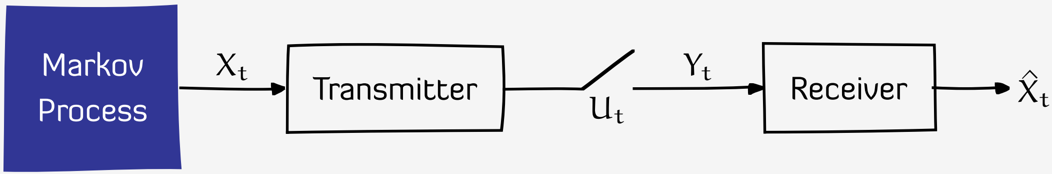
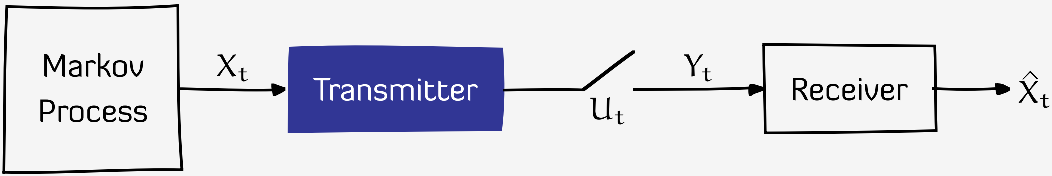
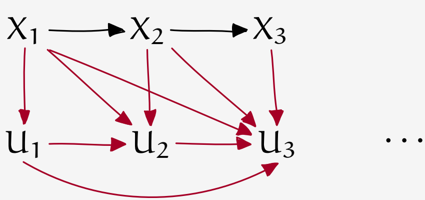


The system model

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \varepsilon, & \text{if } U_t = 0 \end{cases}$$



$$U_t = f_t(X_{1:t}, U_{1:t-1})$$



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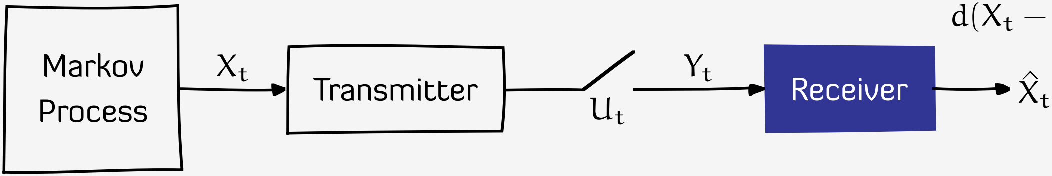
Distortion

$d(X_t - \hat{X}_t)$



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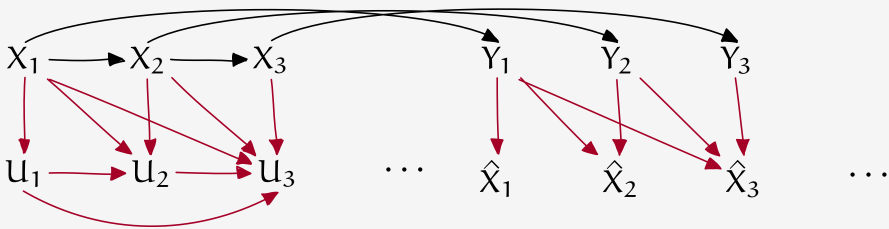

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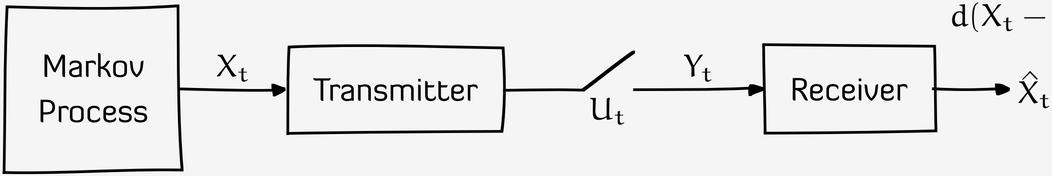

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Communication Strategies

- Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$.
- Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$.

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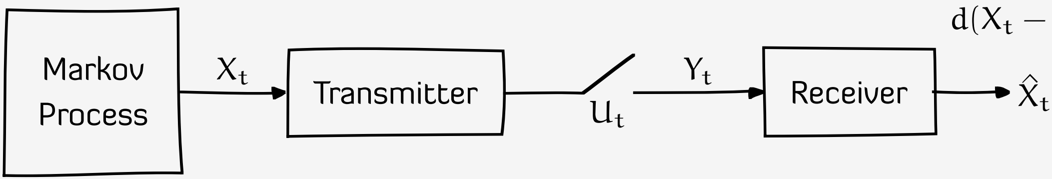

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1. Discounted setup, $\beta \in (0, 1)$

$$D_\beta(f, g) = (1-\beta)\, \mathbb{E}_0^{(f,g)} \Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \Big]; \qquad N_\beta(f, g) = (1-\beta)\, \mathbb{E}_0^{(f,g)} \Big[\sum_{t=0}^{\infty} \beta^t U_t \Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f, g) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_0^{(f,g)} \Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \Big]; \qquad N_1(f, g) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_0^{(f,g)} \Big[\sum_{t=0}^{T-1} U_t \Big]$$

Optimization problems

Costly communication

$$\text{For } \lambda \in \mathbb{R}_{>0}, \quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) + \lambda N_\beta(f,g) \}$$

Constrained communication

$$\text{For } \alpha \in (0,1), \quad D^*_\beta(\alpha) \coloneqq \inf_{(f,g)} \{ D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha \}$$

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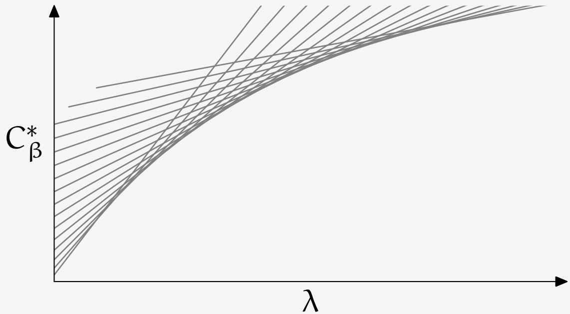

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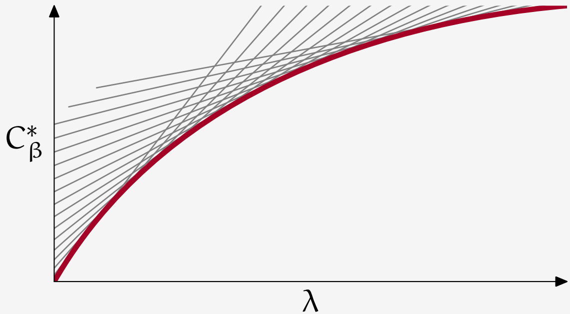

C^*_β is cts, inc, and concave

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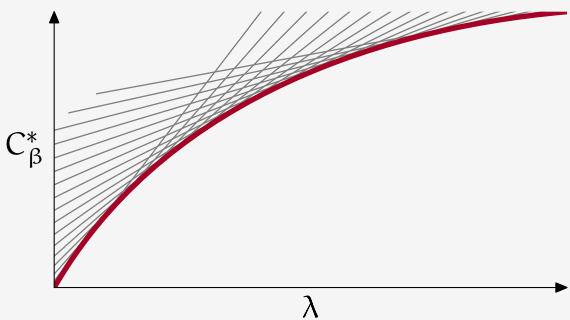

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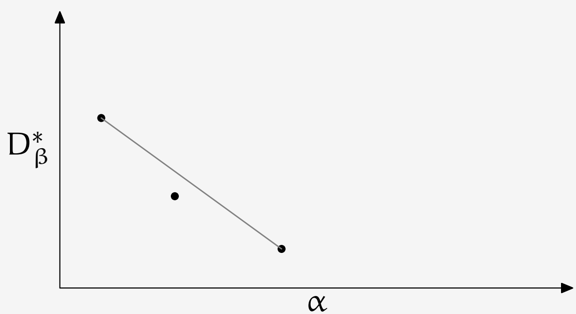

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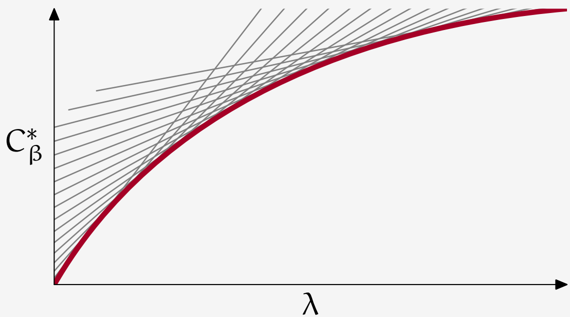

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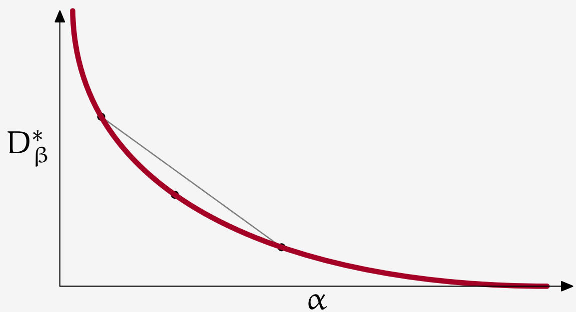

D^*_β is cts, dec, and convex

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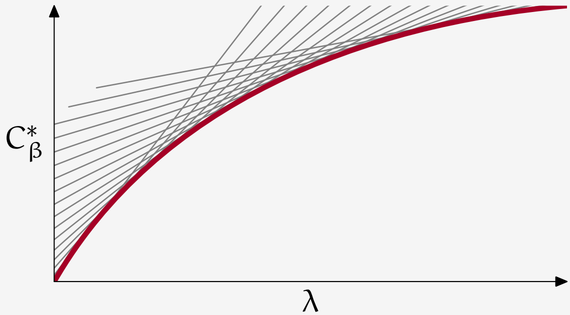

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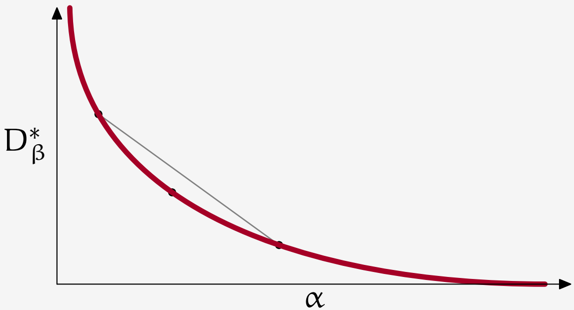
Distortion-transmission function



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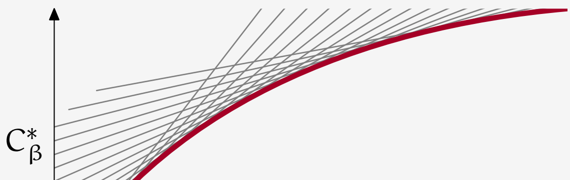

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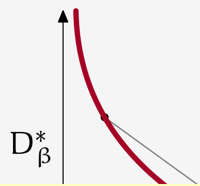

We provide explicit computable expressions for both curves

$$X_{t+1} = X_t + W_t,\; W_t \sim \mathcal{N}(0, 1)$$

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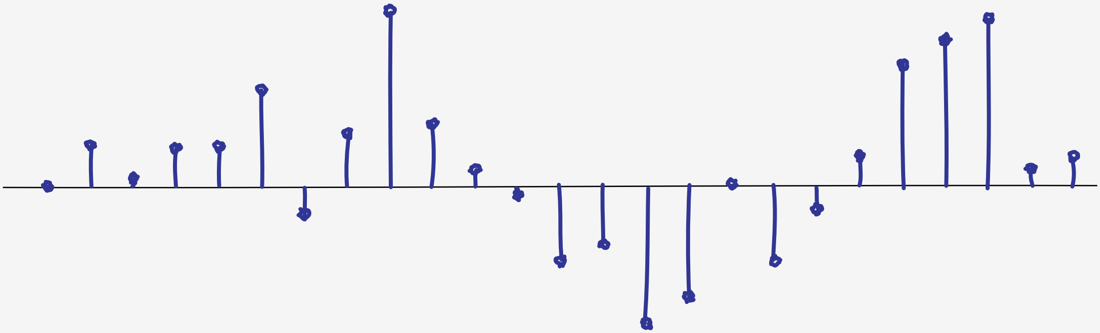
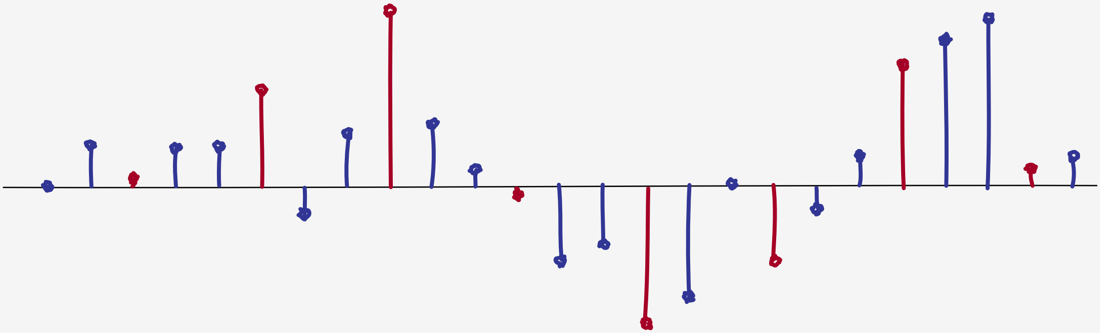


Periodic transmission strategy



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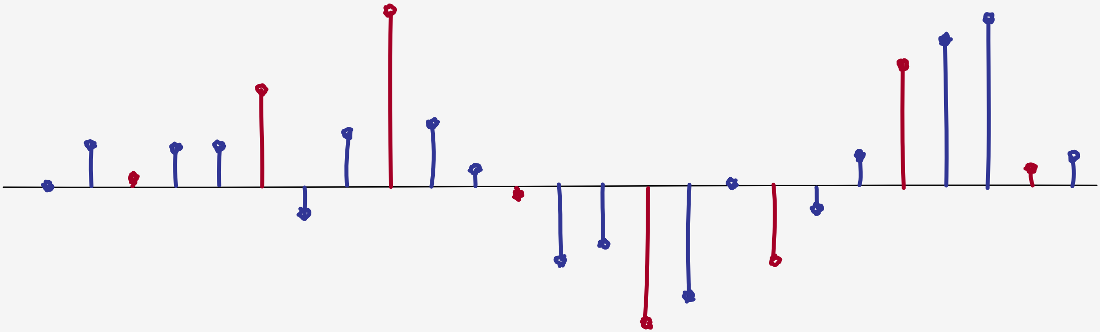
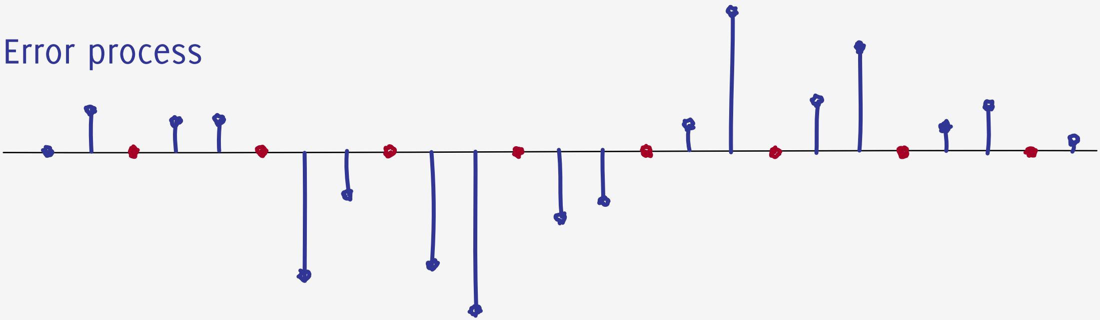


Error process



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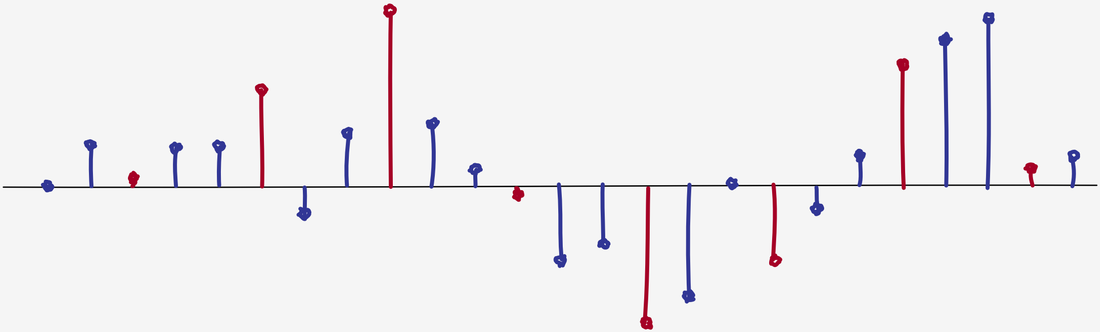

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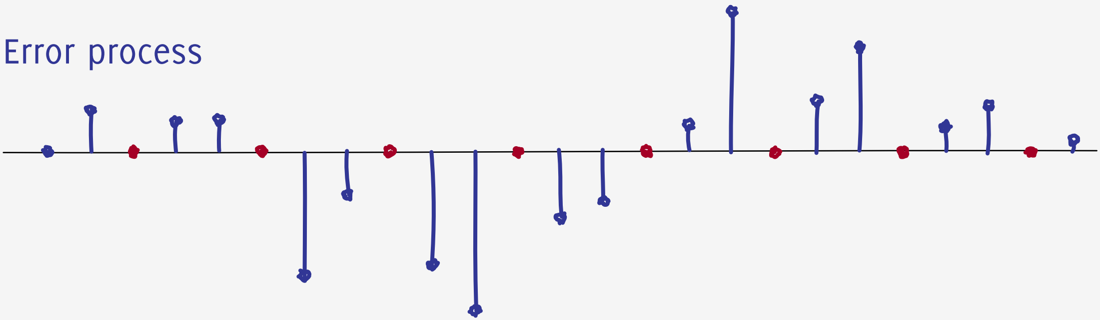
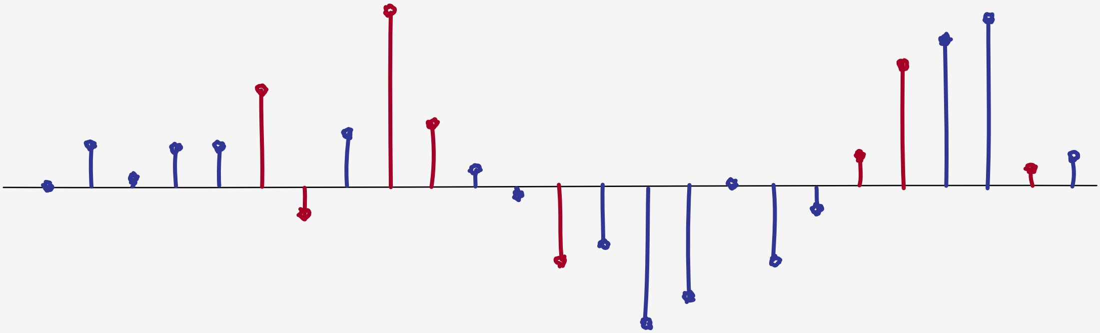


$D = 0.69 \quad N \approx 1/3$

An alternative strategy



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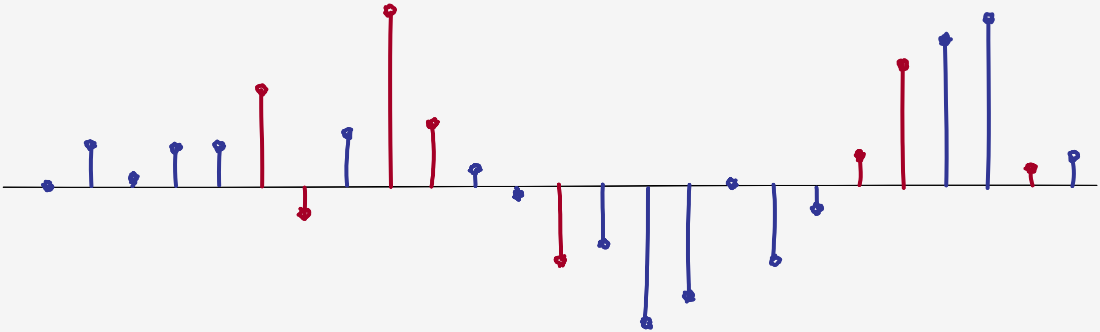

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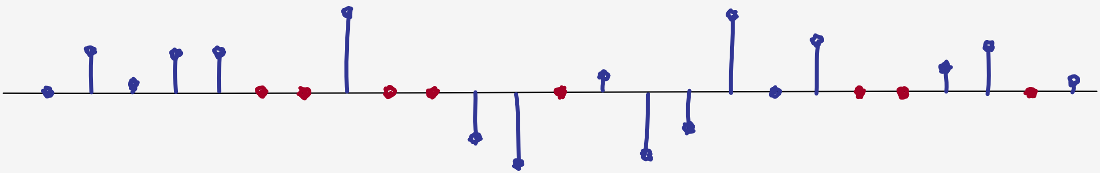

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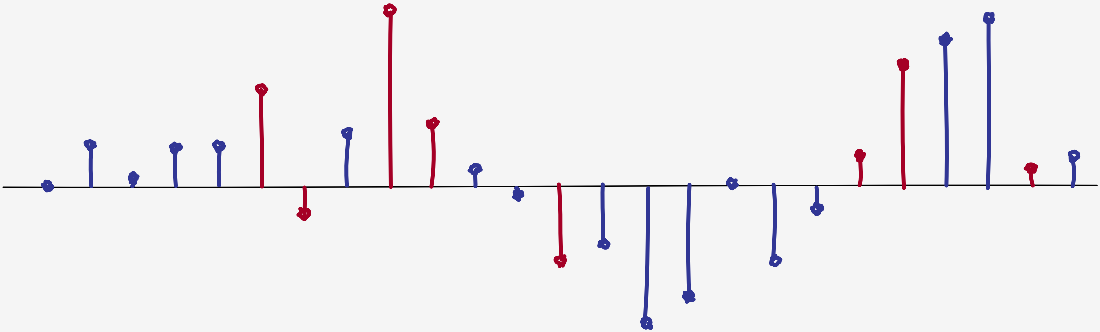

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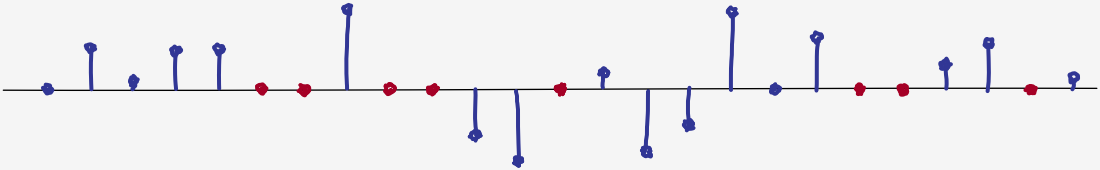
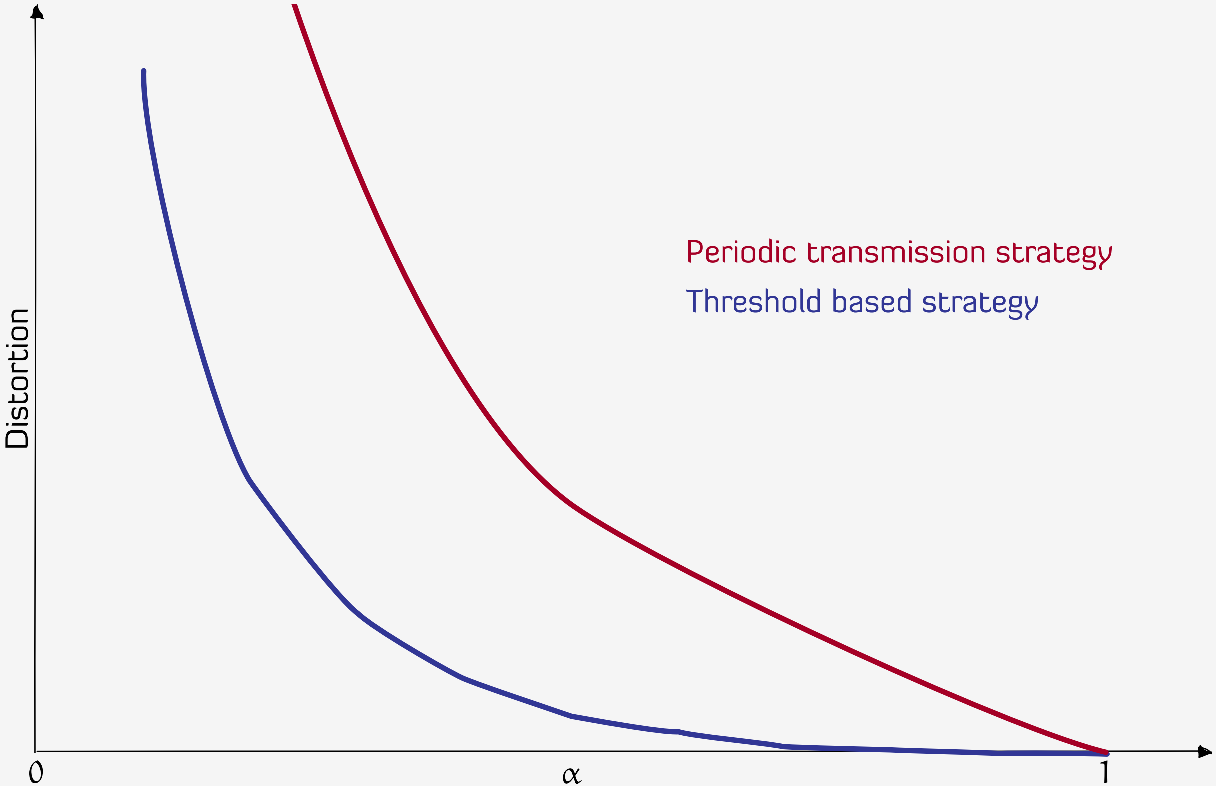


$D = 0.24 \quad N \approx 1/3$

Distortion-transmission function



Identify strategies that achieve the optimal trade-off

Simple and intuitive threshold based strategies are optimal.

Provide computable expressions for distortion-transmission function

Based on simple matrix calculations for discrete Markov processes

Based on solving Fredholm integral equations for Gaussian processes

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Beautiful example of stochastics and optimization

Decentralized stochastic control and POMDPs

Stochastic orders and majorization

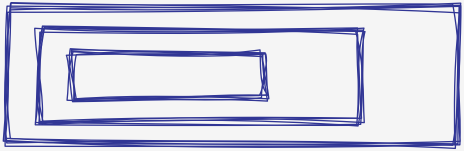
Markov chain analysis, stopping times, and renewal theory

Constrained MDPs and Lagrangian relaxations

So how do we start?

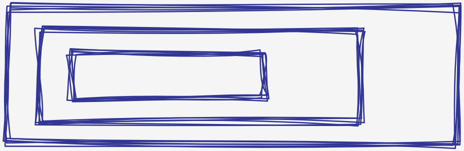
Decentralized stochastic control

Dealing with non-classical information structure



Classical info. struct.

Dealing with non-classical information structure



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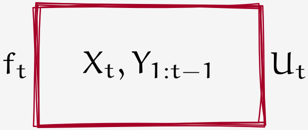

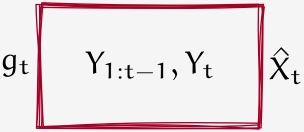

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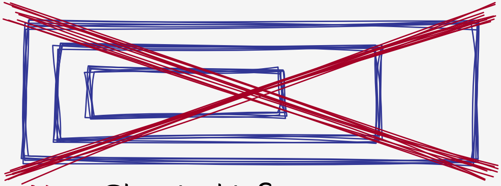

Non-Classical info. struct.

f_t $\boxed{X_t, Y_{1:t-1}}$ U_t

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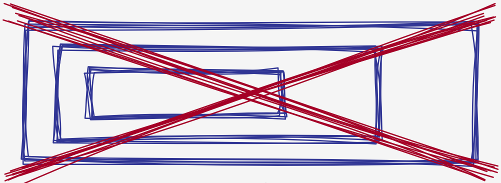

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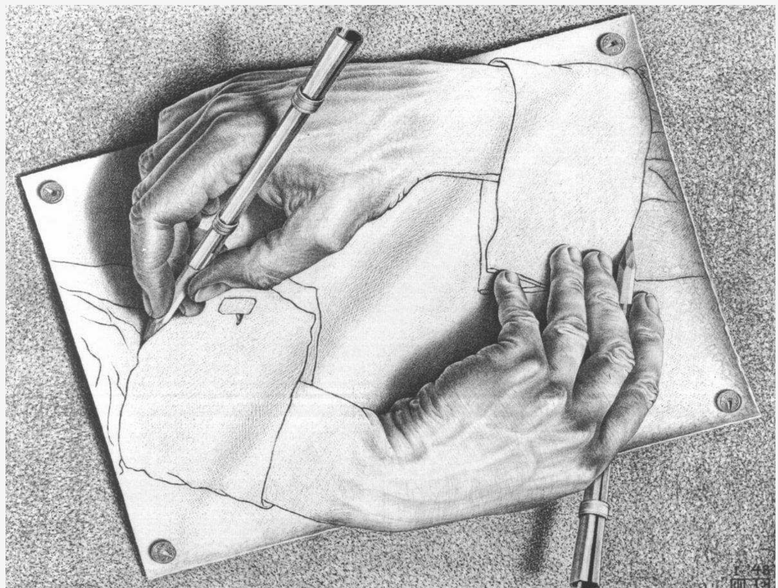

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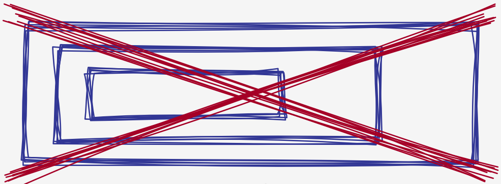

Non-Classical info. struct.

Common info $C_t := \bigcap_{s \geqslant t} \bigcap_{i=1}^{n} I_s^i$

Local info $L_t^i := I_t^i \setminus C_t$

$$g(C, L) = \psi(C)(L)$$

Belongs to the class of **tractable non-classical information structures** (called partial-history sharing) identified in [Mahajan-Nayyar-Teneketzis 2013]

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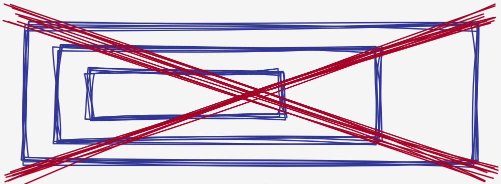
Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Dealing with non-classical information structure



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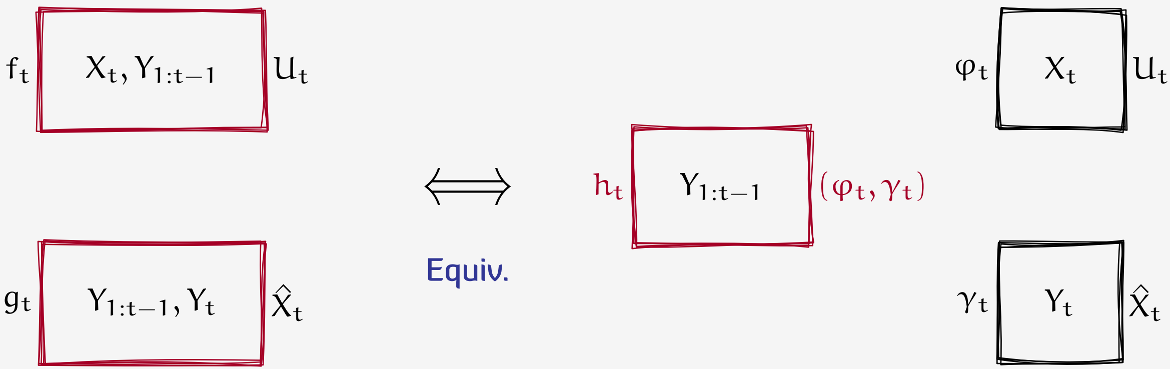

▶ Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Dealing with non-classical information structure

Non-Classica

The coordinated system is a centralized
(i.e., single-agent) partially observed system

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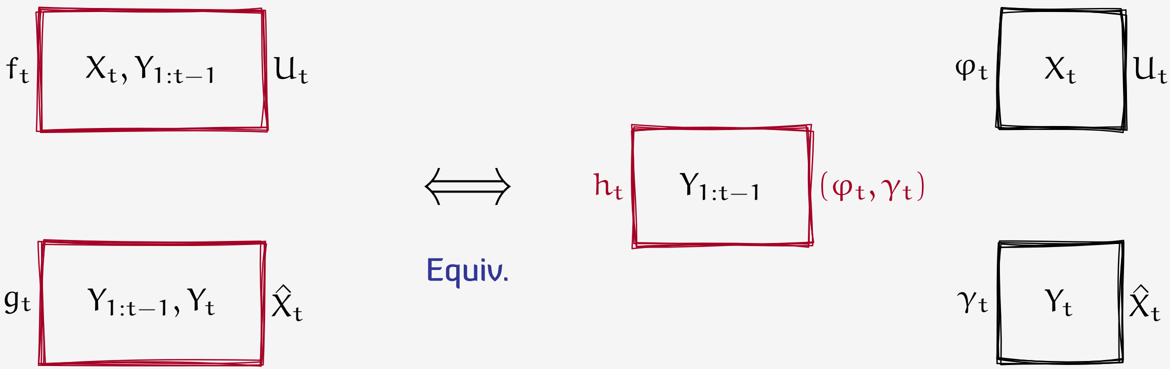

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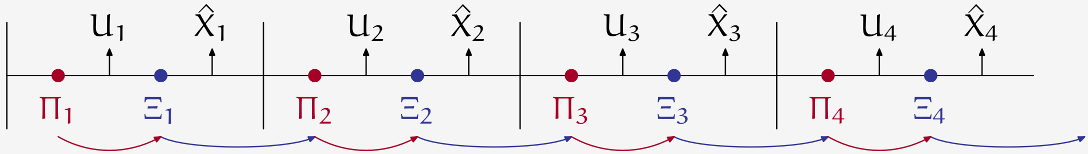
Information states and dynamic program

Information states

Pre-transmission belief : $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$.

Post-transmission belief : $\Xi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.



Information states and dynamic program

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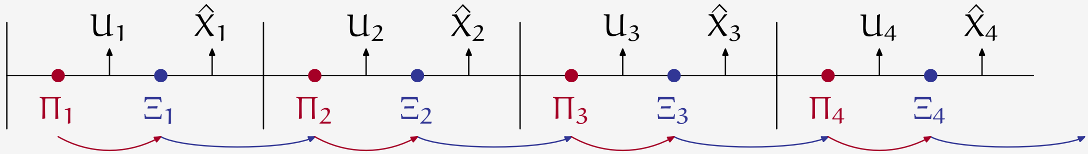

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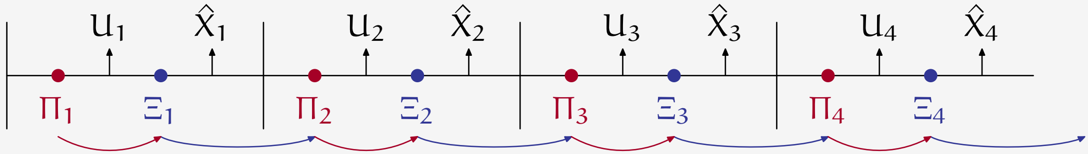

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Information states and dynamic program

Informa… $x \mid Y_{1:t-1})$.

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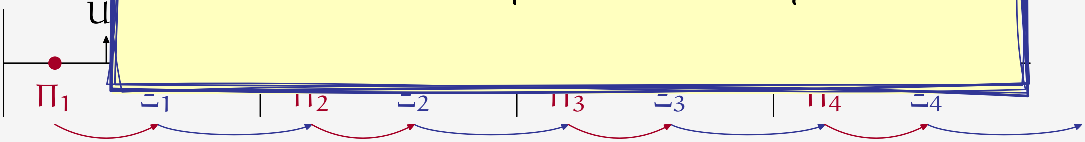
"Standard" POMDP. Optimal strategies can be computed numerically.



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Can we use the DP to say something more about the optimal strategy?

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Markov process $$X_{t+1} = X_t + W_t$$

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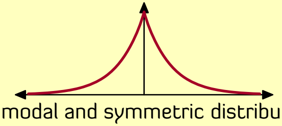

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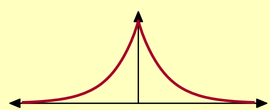
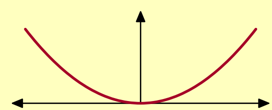
Distortion	Even and increasing $d(e) = d(-e) \leqslant d(e+1)$	Mean-squared $d(e) = \lvert e \rvert^2$



Unimodal and symmetric distribution



Even and increasing distortion

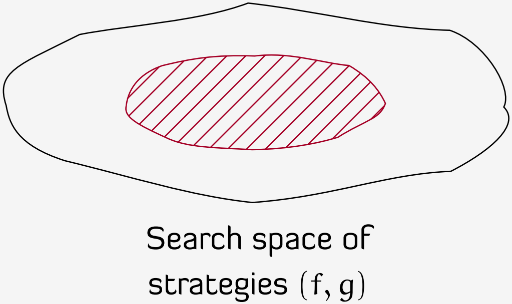
Step 1 Structure of optimal strategies

Step 2 Performance of arbitrary threshold strategies $f^{(k)}$

Step 3 Optimal costly comm.

Step 4 Distortion-transmission trade-off

Step 1 Structure of optimal strategies



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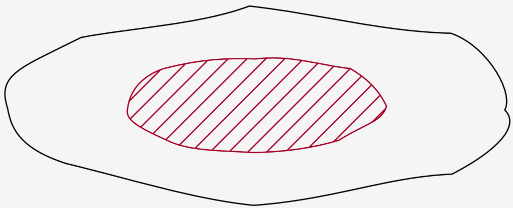

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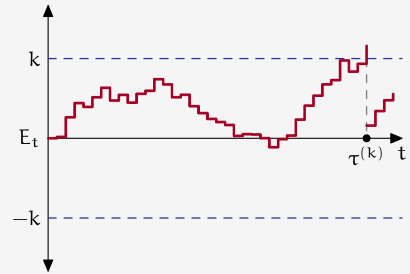

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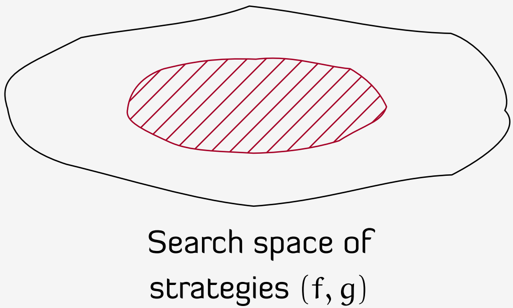

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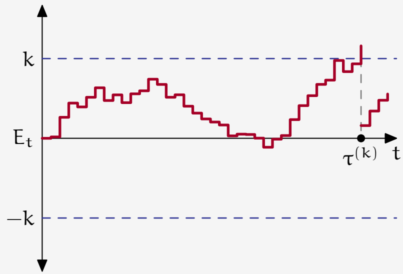

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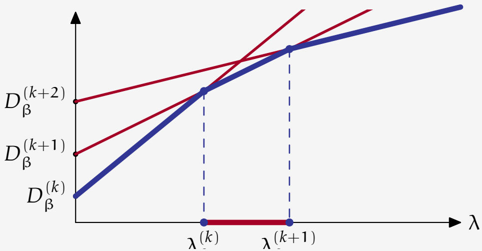

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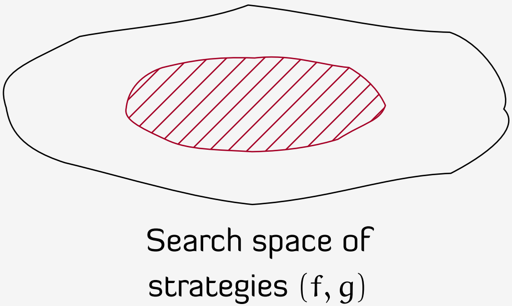

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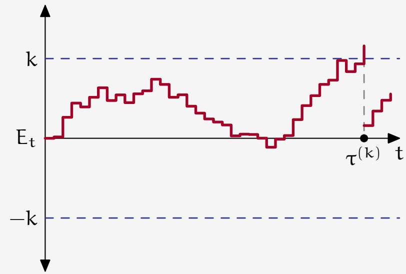

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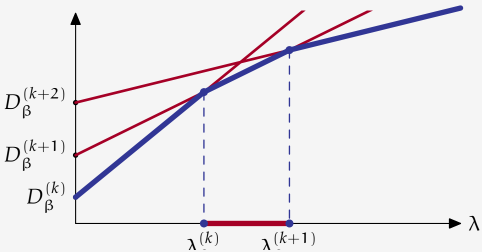

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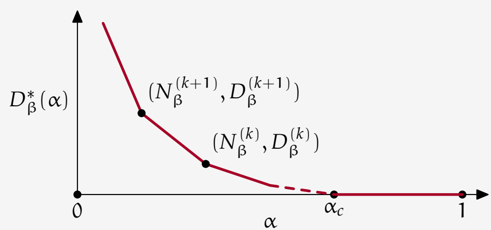

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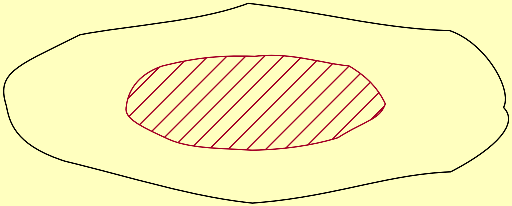
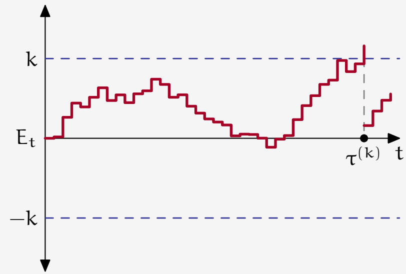


Search space of strategies (f, g)

Step 2 Performance of arbitrary threshold strategies $f^{(k)}$



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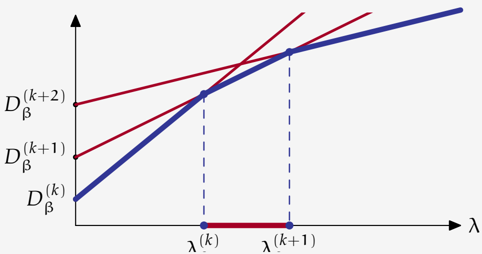

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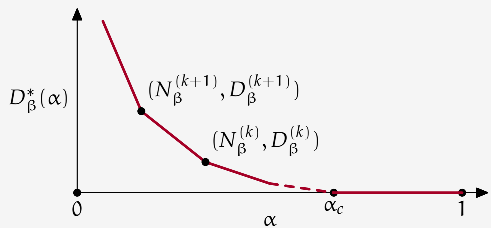
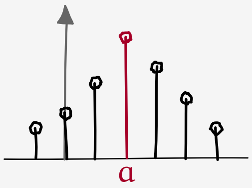


Preliminaries

[LM11, NBTV13]

Almost uniform and unimodal (ASU) distribution about a



$\pi_a \geqslant \pi_{a+1} \geqslant \pi_{a-1} \geqslant \pi_{a+2} \geqslant \cdots$

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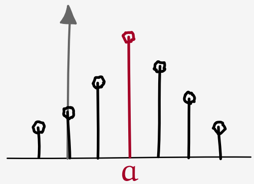

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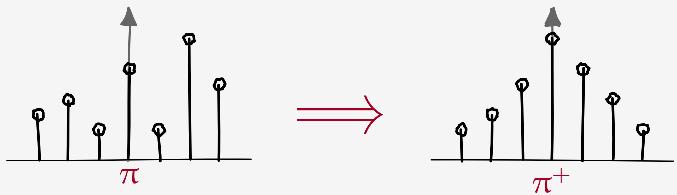
ASU Rearrangement



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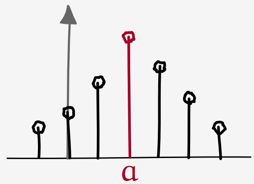

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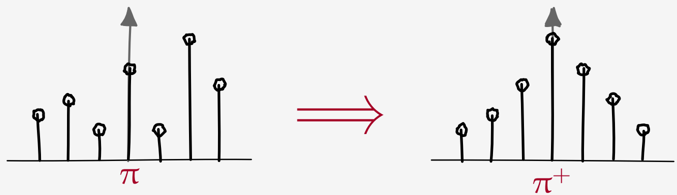
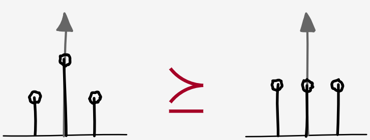


Majorization

$\pi \succeq \xi$ iff

$$\sum_{i=-n}^{n} \pi_i^+ \geqslant \sum_{i=-n}^{n} \xi_i^+ \quad \text{and} \quad \sum_{i=-n}^{n+1} \pi_i^+ \geqslant \sum_{i=-n}^{n+1} \xi_i^+$$

Invariant to permutations.



Step 1 Properties of the value functions [LM11, NBTV13]

Definition $\xi \triangleright \tilde{\xi}$ if $\xi \succeq \tilde{\xi}$ and $\tilde{\xi}$ is ASU about some point a

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Similar to Schur-concavity

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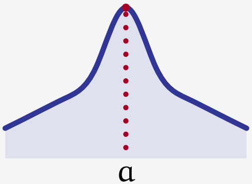

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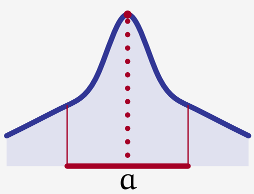
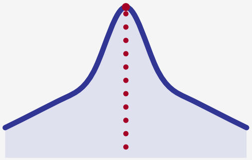


is of the form

$$\varphi(x) = \begin{cases} 1, & \text{if } |x-a| > k(\pi) \\ 0, & \text{if } |x-a| < k(\pi) \\ q_+, & \text{if } x-a = k(\pi) \\ q_-, & \text{if } x-a = -k(\pi) \end{cases}$$

Step 1 Structure of optimal strategies

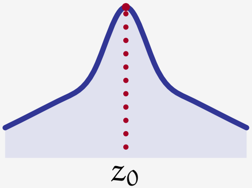
[LM11, NBTV13]



π_1 is ASU about z_0

Step 1 Structure of optimal strategies

[LM11, NBTV13]



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Is $|x_1 - z_0| > k_1$?

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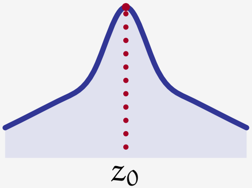
[LM11, NBTV13]



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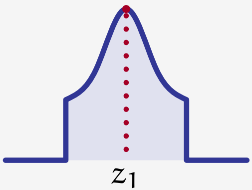

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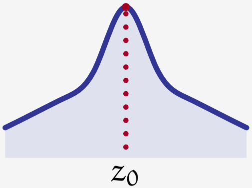

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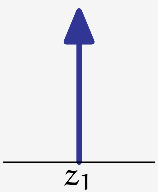

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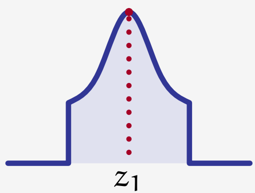

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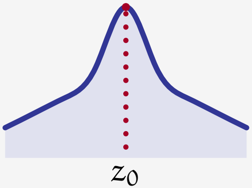
Step 1 Structure of optimal strategies

[LM11, NBTV13]



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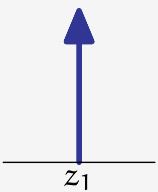

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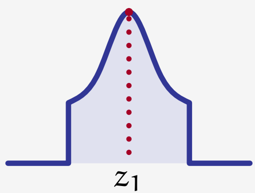

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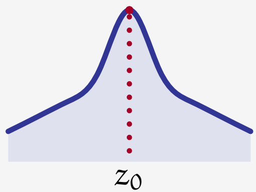
In both cases: $\hat{x}_1 = z_1$

Step 1 Structure of optimal strategies

[LM11, NBTV13]



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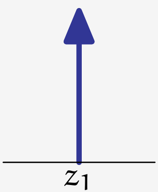

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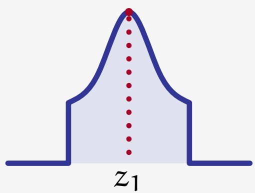

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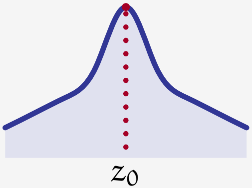
$t = 2$

Structure of optimal strategies

[LM11, NBTV13]



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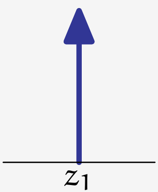

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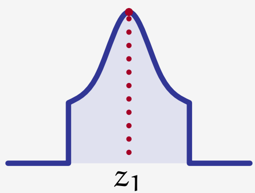

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$t = 2$

$$X_2 = X_1 + W_1 \implies \pi_1 = \xi_1 * p$$

π_1 is ASU about z_1

etc. ...

Step 1 Structure of optimal estimator [LM11, NBTV13]

Transmitted Process

Let Z_t denote the most recently transmitted value of the Markov process.

Step 1 Structure of optimal estimator

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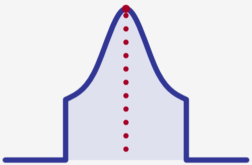
Transmitted Process

Let Z_t denote the most recently transmitted value of the Markov process.



Lemma

Ξ_t is ASU about Z_t

Step 1 Structure of optimal estimator [LM11, NBTV13]

Transmitted Process

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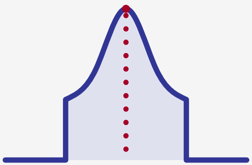


Lemma

Ξ_t is ASU about Z_t

Theorem

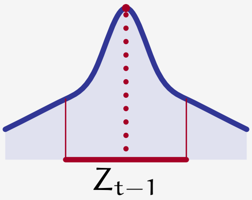
$\hat{X}_t = g_t^*(\Xi_t) = Z_t$

Remark

The optimal estimation strategy is time-homogeneous and can be specified in closed form.

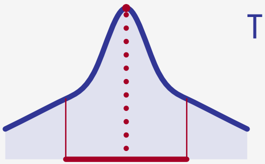
Step 1 Structure of optimal transmitter [LM11, NBTV13]

Lemma Π_t is ASU about Z_{t-1}



Step 1 Structure of optimal transmitter [LM11, NBTV13]

Lemma Π_t is ASU about Z_{t-1}



Theorem
$$U_t = f_t(X_t, \Pi_t) = \begin{cases} 1, & \text{if } |X_t - E_t| \geq k_t \\ 0, & \text{if } |X_t - E_t| < k_t \end{cases}$$

Step 1 Structure of optimal transmitter [LM11, NBTV13]

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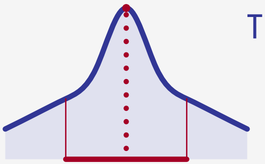

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Error process Let $E_t = X_t - Z_{t-1}$ denote the error process. $\{E_t\}_{t=0}^{\infty}$ is a controlled Markov process where

$$E_0 = 0 \quad \text{and} \quad \mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} p_{|e-n|}, & \text{if } u = 0; \\ p_n, & \text{if } u = 1. \end{cases}$$

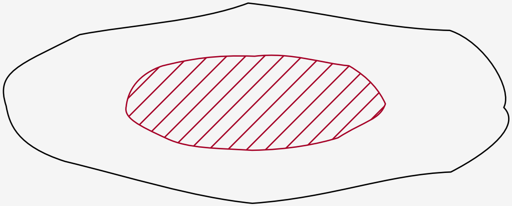
Remark The optimal transmission strategy is a function of the error process.

The results extend to infinite horizon setup under appropriate regularity conditions.

Time-homogeneous threshold-based strategies are optimal.

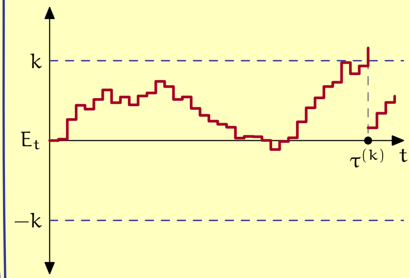
How do we find the optimal threshold-based strategy?

Step 1 Structure of optimal strategies

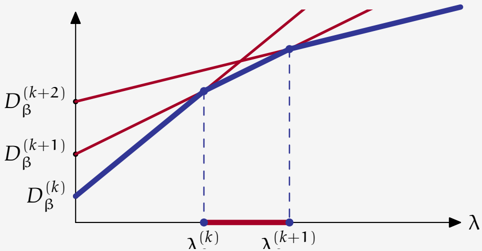


Search space of strategies (f, g)

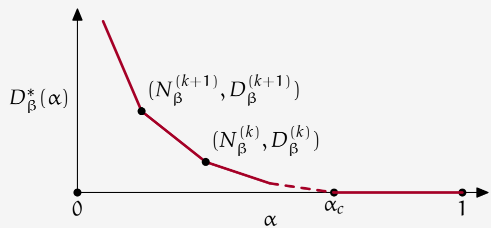
Step 2 Performance of arbitrary threshold strategies $f^{(k)}$



Step 3 Optimal costly comm.



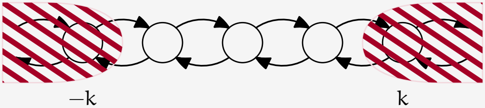
Step 4 Distortion-transmission trade-off



Step 2 Performance of threshold strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } |e| \geq k \\ 0 & \text{otherwise} \end{cases}$$



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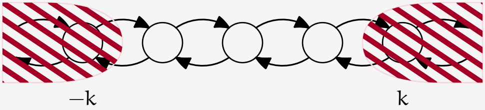
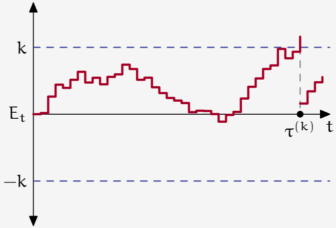


Let $\tau^{(k)}$ denote the stopping time of first transmission (starting at $E_0 = 0$).



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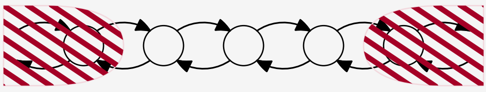

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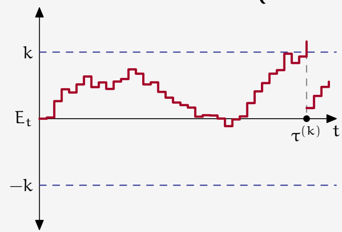

Define

$$L_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\Bigg[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Bigg| E_0 = e\Bigg].$$

$$M_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\Bigg[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Bigg| E_0 = e\Bigg].$$

Step 2 Performance of threshold strategies

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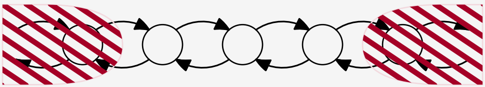

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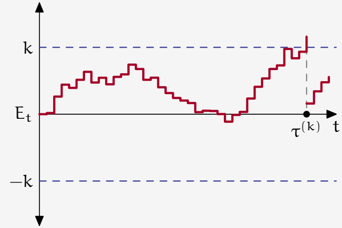

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Proposition $\{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal theory,

$$D_\beta^{(k)} \coloneqq D_\beta(f^{(k)}, g^*) = \frac{L_\beta^{(k)}(0)}{M_\beta^{(k)}(0)} \quad \text{and} \quad N_\beta^{(k)} \coloneqq N_\beta(f^{(k)}, g^*) = \frac{1}{M_\beta^{(k)}(0)} - (1-\beta).$$

Step 2 Performance of threshold strategies

Consid… of …0)).

$f^{(k)}$

Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$ is sufficient to compute the performance of $f^{(k)}$ (i.e., to compute $D_\beta^{(k)}$ and $N_\beta^{(k)}$).

Define

$$L_\beta^{(k)}(e) = (1-\beta)\,\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big].$$

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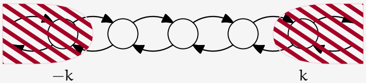
Step 2 Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$

Markov chain setup

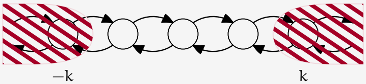


$$L_\beta^{(k)}(e) = d(e) + \beta \sum_{n=-k}^{k} p_{n-e} L_\beta^{(k)}(n)$$

$$M_\beta^{(k)}(e) = 1 + \beta \sum_{n=-k}^{k} p_{n-e} M_\beta^{(k)}(n)$$

Step 2 Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$

Markov chain setup



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Proposition

$$L_\beta^{(k)} = \big[[I - \beta P^{(k)}]^{-1} d^{(k)}\big].$$

$P^{(k)}$ is substochastic.

$$M_\beta^{(k)} = \big[[I - \beta P^{(k)}]^{-1} \mathbf{1}^{(k)}\big].$$

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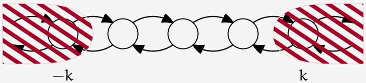
Markov chain setup



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Gauss-Markov setup

$$L_\beta^{(k)}(e) = d(e) + \beta \int_{-k}^{k} \varphi(n-e) L_\beta^{(k)}(n)\, dn$$

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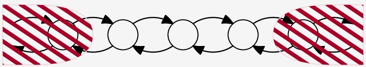
Step 2 Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$

Markov chain setup



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Fredholm Integral Equations of the 2nd kind.

Solutions exist and are unique.

Step 2 Computing $L_\beta^{(k)}$ and $M_\beta^{(k)}$

$D_\beta^{(k)}$ and $N_\beta^{(k)}$ can be computed using these expressions.

Proposition

$$L_\beta^{(k)} = \big[[I - \beta P^{(k)}]^{-1} d^{(k)}\big].$$

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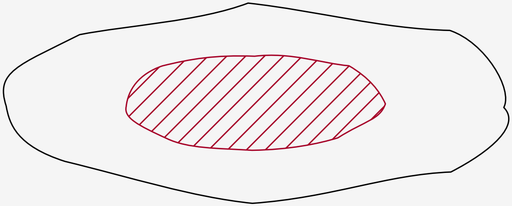
Fredholm Integral Equations of the 2nd kind.

Solutions exist and are unique.

We found the performance of a generic threshold-based strategy

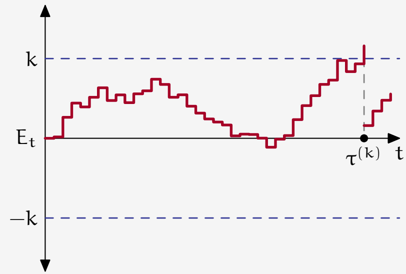
How does this lead to identifying an optimal strategy?

Step 1 Structure of optimal strategies

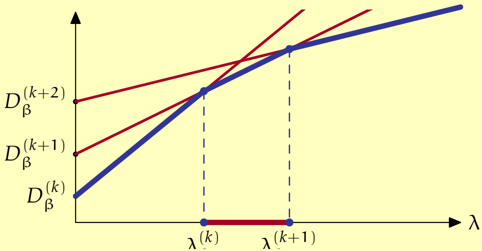


Search space of strategies (f, g)

Step 2 Performance of arbitrary threshold strategies $f^{(k)}$



Step 3 Optimal costly comm.



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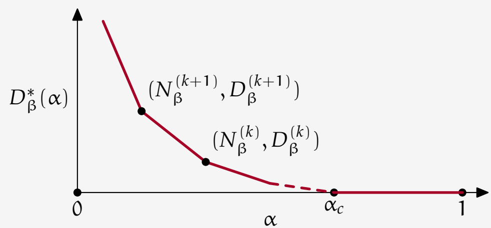

Step 3 Properties of optimal thresholds

Monotonicity $\quad L_\beta^{(k+1)} > L_\beta^{(k)}$ and $M_\beta^{(k+1)} > M_\beta^{(k)}$

Depends on unimodularity of noise

Step 3 Properties of optimal thresholds

Monotonicity

$$L_\beta^{(k+1)} > L_\beta^{(k)} \quad \text{and} \quad M_\beta^{(k+1)} > M_\beta^{(k)}$$

Use DP and monotonicity of Bellman operator

Implication:

$$D_\beta^{(k+1)} \geqslant D_\beta^{(k)} \quad \text{and} \quad N_\beta^{(k+1)} < N_\beta^{(k)}$$

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Submodularity $C_\beta^{(k)}(\lambda) \coloneqq D_\beta^{(k)} + \lambda N_\beta^{(k)}$ is submodular in (k, λ).

Step 3 Properties of optimal thresholds

Monotonicity

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Submodularity

$C_\beta^{(k)}(\lambda) \coloneqq D_\beta^{(k)} + \lambda N_\beta^{(k)}$ is submodular in (k, λ).

Proposition

$k_\beta^*(\lambda) \coloneqq \arg\min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}(\lambda)$ is increasing in λ.

Step 3 Properties of optimal thresholds

Monotonicity $L_\beta^{(k+1)} > L_\beta^{(k)}$ and $M_\beta^{(k+1)} > M_\beta^{(k)}$

Implication:

$D_\beta^{(k+1)} \geqslant D_\beta^{(k)}$ and $N_\beta^{(k+1)} < N_\beta^{(k)}$

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Proposition $k_\beta^*(\lambda) \coloneqq \arg\min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}(\lambda)$ is increasing in λ.

Thus, optimal threshold increases with increase in λ.

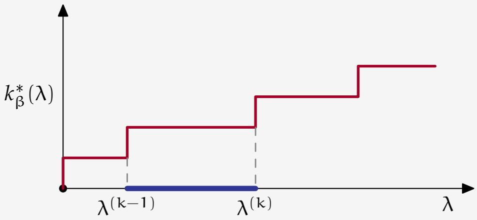
Characterizing the optimal threshold for a given communication cost **is tricky.**

Instead, we will characterize the optimal communication cost for a given threshold.

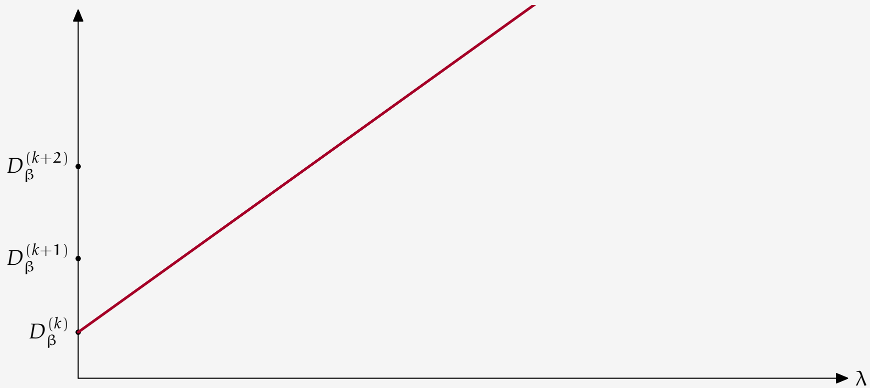
Optimal costly communication: Markov chain

Define $\Lambda_\beta^{(k)} \coloneqq \{\lambda \in \mathbb{R}_{\geqslant 0} : k_\beta^*(\lambda) = k\}$
$= [\lambda_\beta^{(k-1)}, \lambda_\beta^{(k)}]$.

$$C_\beta^{(k)}(\lambda_\beta^{(k)}) = C_\beta^{(k+1)}(\lambda_\beta^{(k)})$$



Step 3 Optimal costly communication: Markov chain



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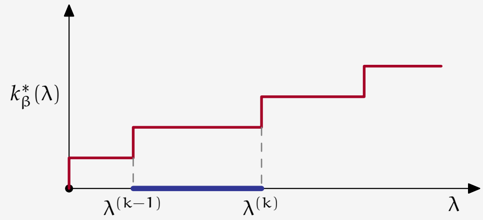

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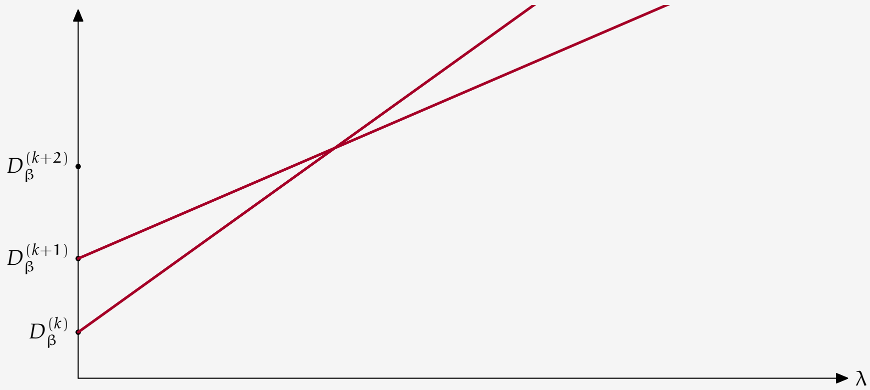

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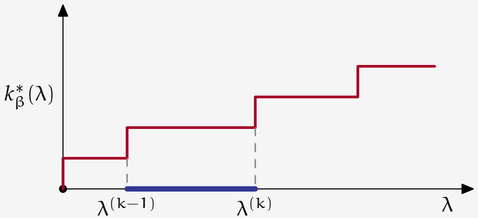
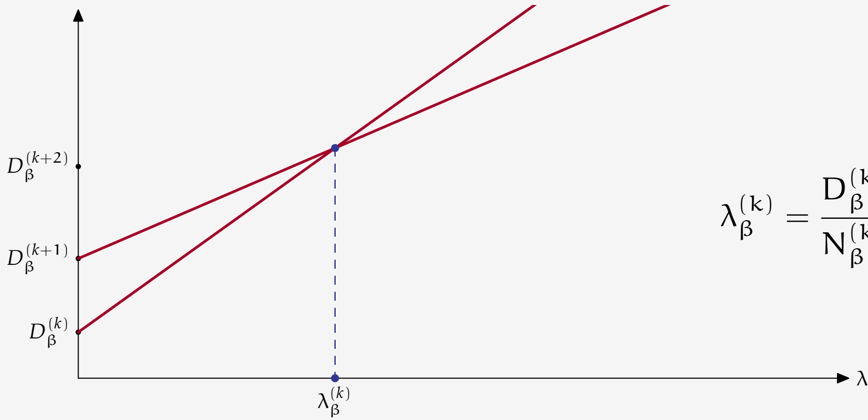


Step 3 Optimal costly communication: Markov chain



$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Define $\Lambda_\beta^{(k)} \coloneqq \{\lambda \in \mathbb{R}_{\geqslant 0} : k^*_\beta(\lambda) = k\}$
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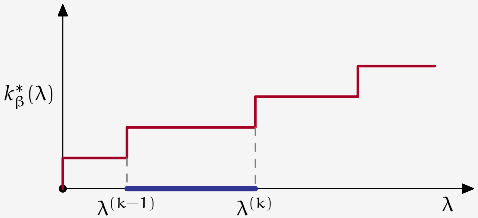
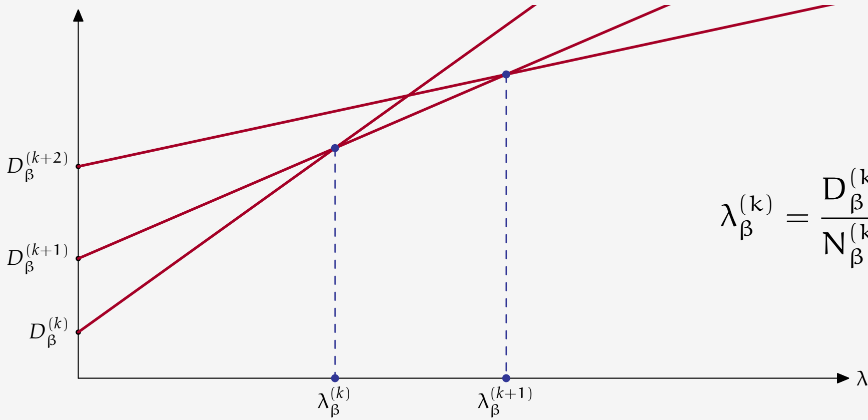


Optimal costly communication: Markov chain



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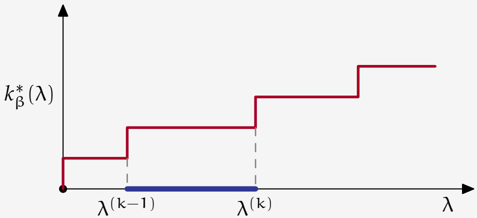
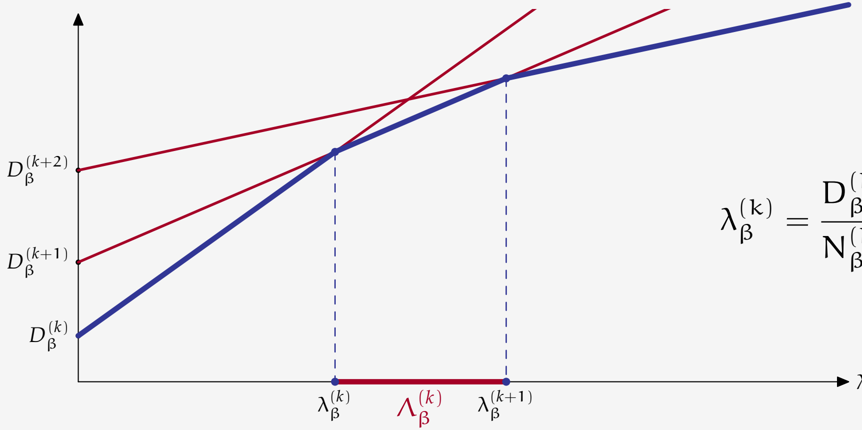


Step 3 Optimal costly communication: Markov chain



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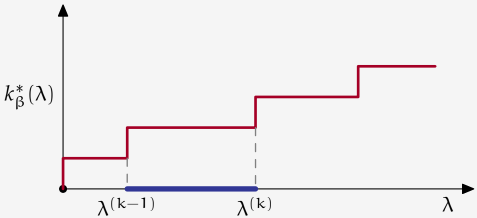
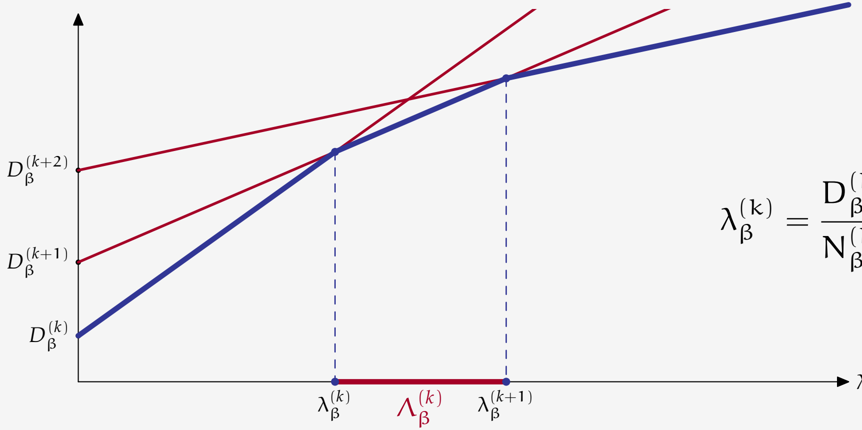


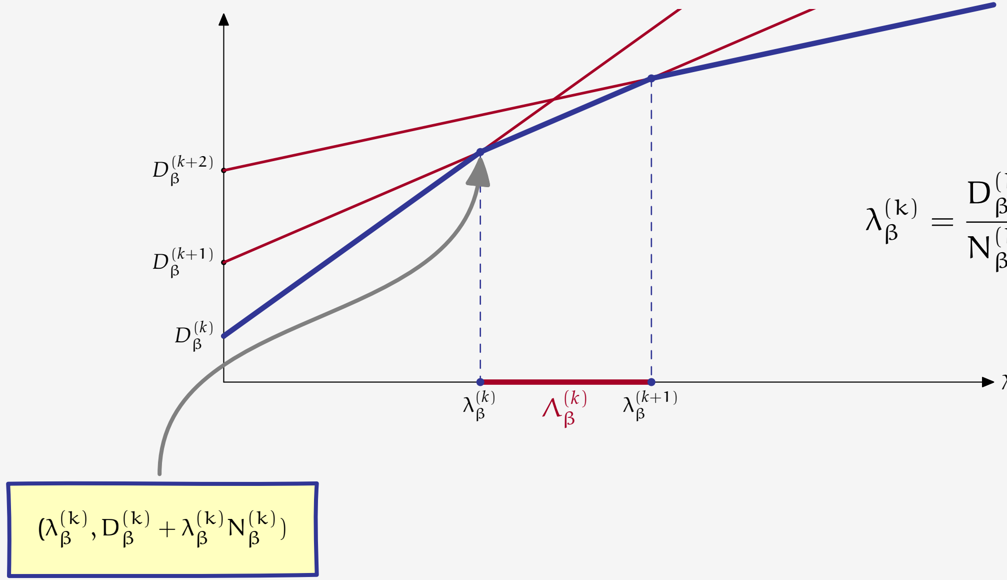
Step 3 Optimal costly communication: Markov chain



$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Theorem Strategy $f^{(k+1)}$ is optimal for $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$.

$C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

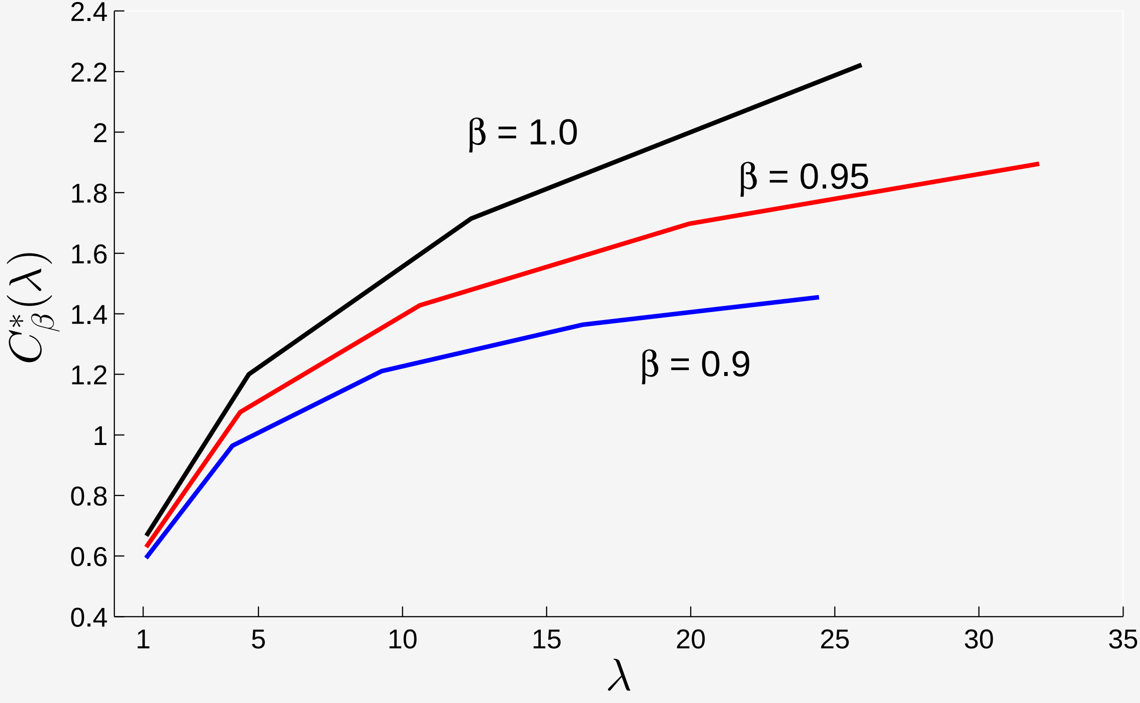


$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}$$

Theorem Strategy $f^{(k+1)}$ is optimal for $\lambda \in (\lambda_\beta^{(k)}, \lambda_\beta^{(k+1)}]$.

$C_\beta^*(\lambda) = \min_{k \in \mathbb{Z}_{\geqslant 0}} C_\beta^{(k)}$ is piecewise linear, continuous, concave, and increasing function of λ.

Example Symmetric birth-death Markov chain ($p = 0.3$)



Step 3 Optimal costly communication: Gauss-Markov

Lemma

$D_\beta^{(k)}$ is increasing in k and $N_\beta^{(k)}$ is decreasing in k.

$D_\beta^{(k)}$ and $N_\beta^{(k)}$ are differentiable in k.

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Theorem

Strategy $f^{(k)}$ is optimal for $\lambda = -\dfrac{\partial_k D^{(k)}_\beta}{\partial_k N^{(k)}_\beta}$

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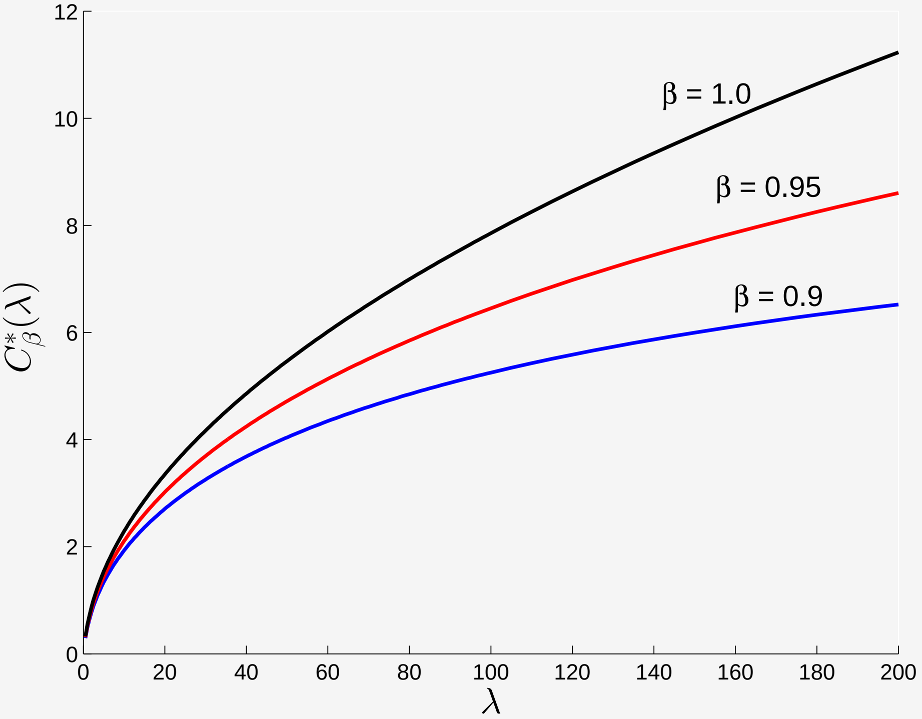
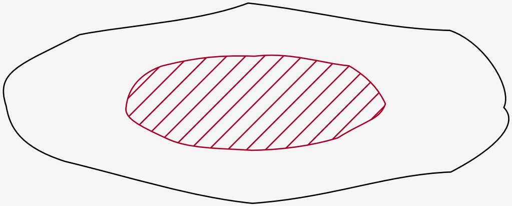
Example Gauss-Markov with $\sigma^2 = 1$

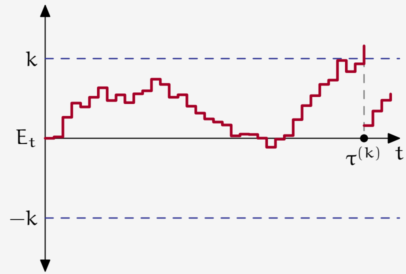


Step 1 Structure of optimal strategies

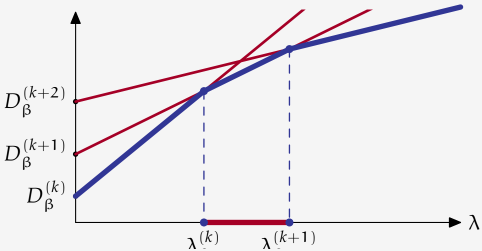


Search space of strategies (f, g)

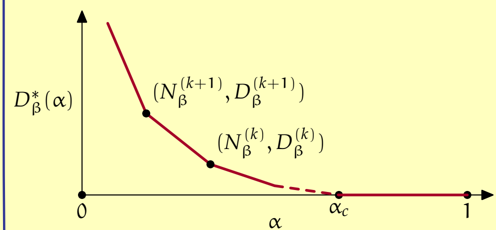
Step 2 Performance of arbitrary threshold strategies $f^{(k)}$



Step 3 Optimal costly comm.



Step 4 Distortion-transmission trade-off



Step 4 Distortion-transmission trade-off: Markov chain

Sufficient conditions for constrained optimality

A strategy (f°, g°) is optimal for the constrained communication problem if

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$

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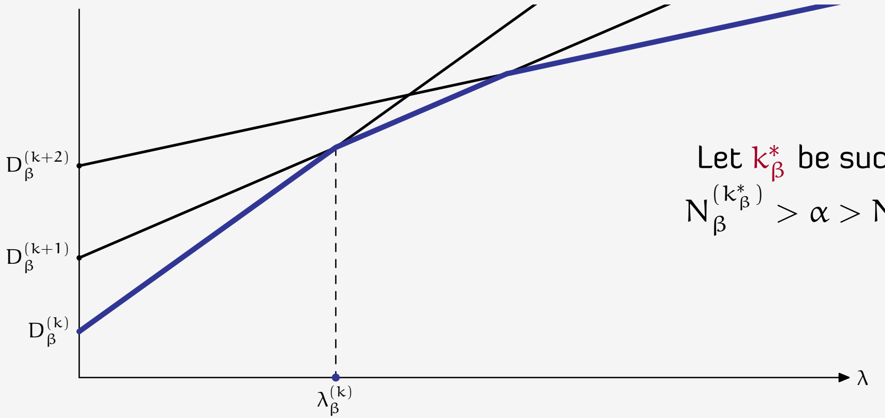

Let k^*_β be such that
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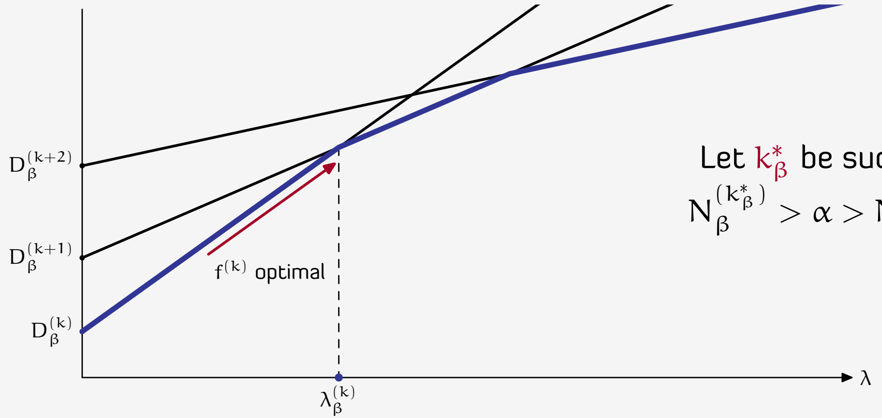

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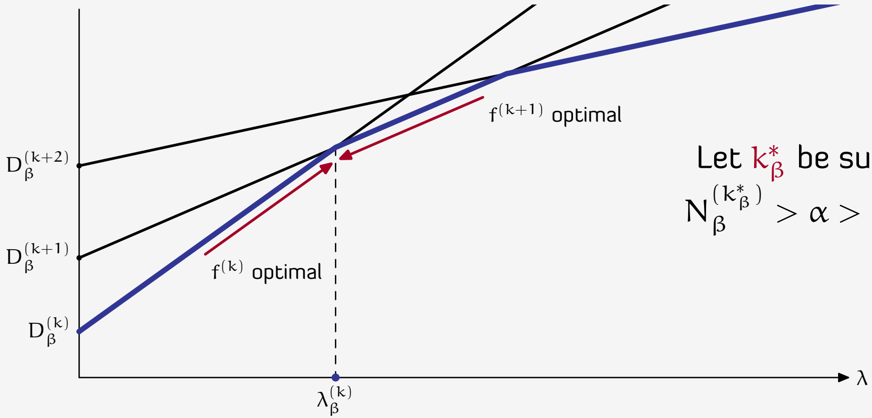

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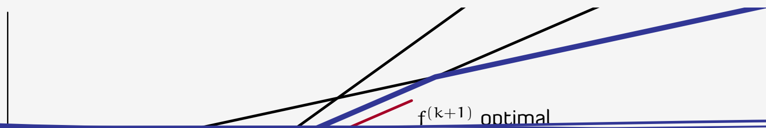

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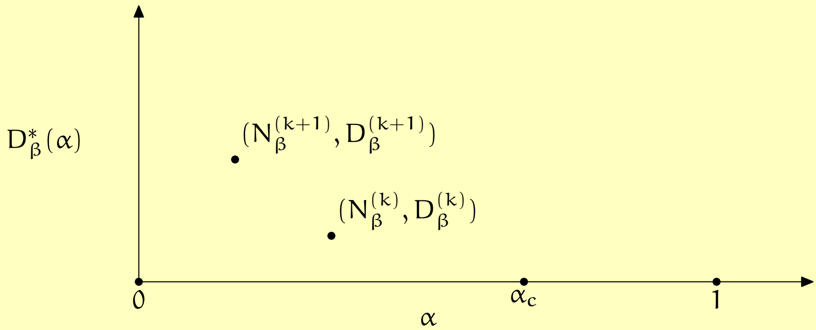

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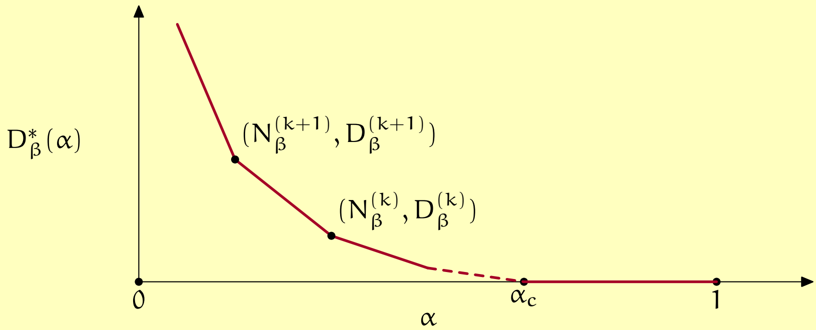

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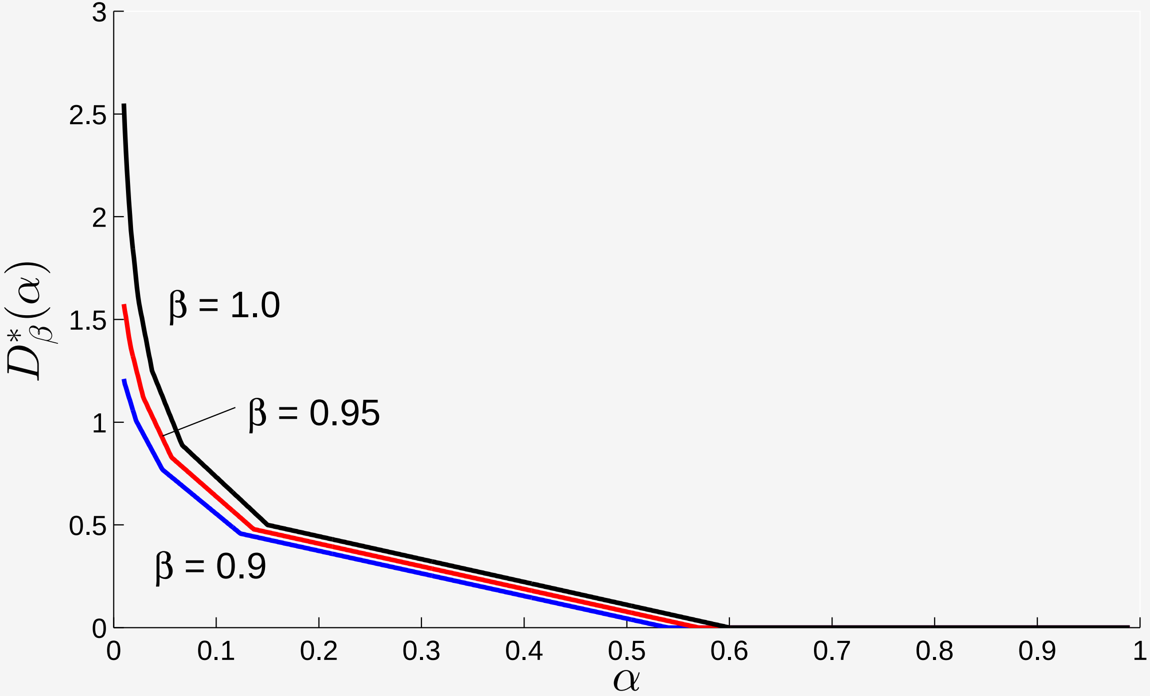

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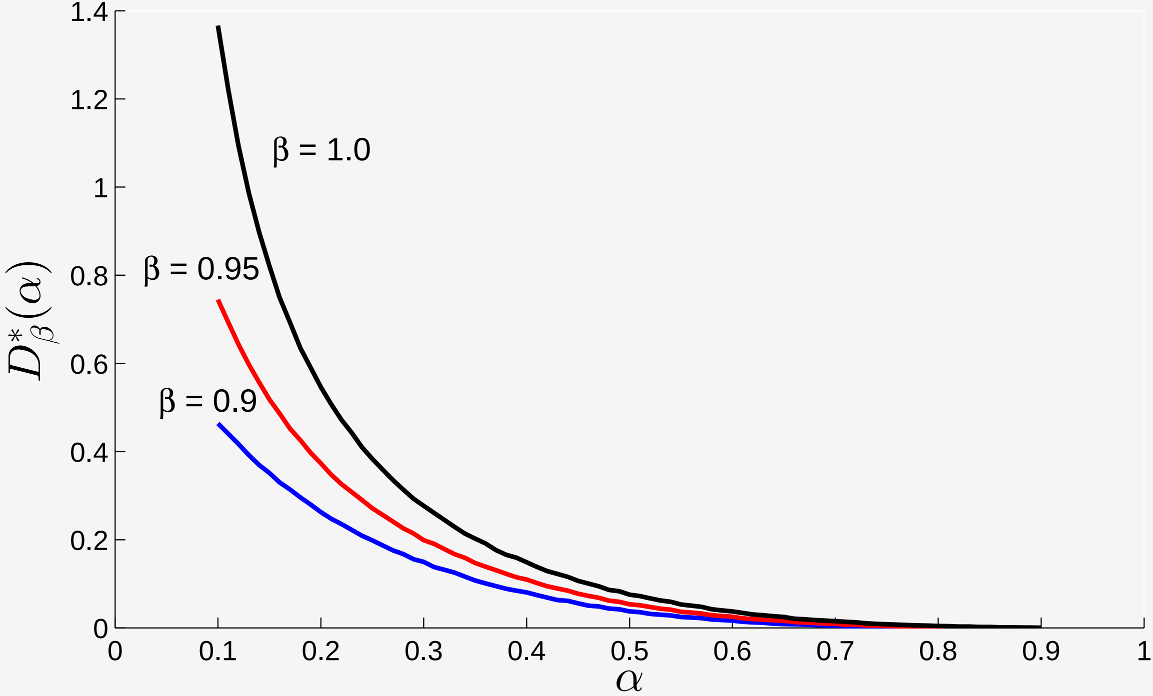
Example Gauss-Markov with $\sigma^2 = 1$



Conclusion

Analyze fundamental limits of estimation under communication constraints

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Possible generalizations to more realistic models

- Packet drops
- Rate constraints (effect of quantization)
- Network delays

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A simple non-trivial "toy-problem" for decentralized control

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- It is important to identify "easy" problems and positive results.

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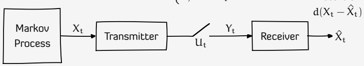
Full version available at arXiv:1505.04829.

The system model



$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \varepsilon, & \text{if } U_t = 0 \end{cases}$$

$$U_t = f_t(X_{1:t}, U_{1:t-1}) \qquad \hat{X}_t = g_t(Y_{1:t})$$

1. Discounted setup, $\beta \in (0,1)$

$$D_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t)\Big]; \qquad N_\beta(f,g) = (1-\beta)\,\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{\infty}\beta^t U_t\Big]$$

2. Average cost setup, $\beta = 1$

$$D_1(f,g) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t)\Big]; \qquad N_1(f,g) = \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_0^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t\Big]$$

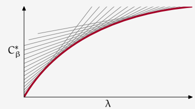
Optimization problems

Costly communication

For $\lambda \in \mathbb{R}_{>0}$, $\quad C^*_\beta(\lambda) = C_\beta(f^*, g^*; \lambda) := \inf_{(f,g)} \{D_\beta(f,g) + \lambda N_\beta(f,g)\}$

Constrained communication

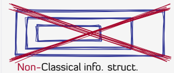
For $\alpha \in (0,1)$, $\quad D^*_\beta(\alpha) := \inf_{(f,g)} \{D_\beta(f,g) : N_\beta(f,g) \leqslant \alpha\}$



C^*_β is cts, inc, and concave

D^*_β is cts, dec, and convex

Dealing with non-classical information structure



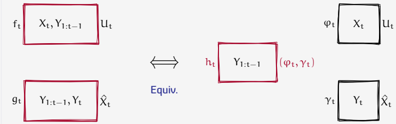
Non-Classical info. struct.

Common info $C_t := \bigcap_{s \geqslant t} \bigcap_{i=1}^{n} I^i_s$

Local info $L^i_t := I^i_t \setminus C_t$

$g(C, L) = \psi(C)(L)$

Belongs to the class of **tractable non-classical information structures** (called partial-history sharing) identified in [Mahajan-Nayyar-Teneketzis 2013]



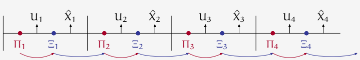
▶ Nayyar, Mahajan and Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," IEEE TAC 2013.

Information states and dynamic program

Information states

Pre-transmission belief : $\Pi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t-1})$.

Post-transmission belief : $\Xi_t(x) = \mathbb{P}(X_t = x \mid Y_{1:t})$.



Structural results

There is no loss of optimality in using

$$U_t = f_t(X_t, \Pi_t) \quad \text{and} \quad \hat{X}_t = g_t(\Xi_t).$$

Dynamic Program

$$W_{T+1}(\pi) = 0$$

and for $t = T, \dots, 0$

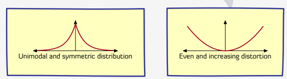
$$V_t(\xi) = \min_{\hat{x} \in \mathcal{X}} \mathbb{E}[d(X_t - \hat{x}) + W_{t+1}(\Pi_{t+1}) \mid \Xi_t = \xi],$$

$$W_t(\pi) = \min_{\varphi : \mathcal{X} \to \{0,1\}} \mathbb{E}[\lambda\varphi(X_t) + V_t(\Xi_t) \mid \Pi_t = \pi, \varphi_t = \varphi].$$

Simplifying modeling assumptions

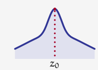
Markov process $\quad X_{t+1} = X_t + W_t$

	Markov chain setup	**Guass-Markov setup**
State spaces	$X_t, W_t \in \mathbb{Z}$	$X_t, W_t \in \mathbb{R}$
Noise distribution	Unimodal and symmetric $p_e = p_{-e} \geqslant p_{e+1}$	Zero-mean Gaussian $\varphi_\sigma(\cdot)$
Distortion	Even and increasing $d(e) = d(-e) \leqslant d(e+1)$	Mean-squared $d(e) = \lvert e\rvert^2$



Step 1 Structure of optimal strategies

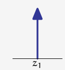
[LM11, NBTV13]



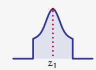
π_1 is ASU about z_0

Is $|x_1 - z_0| > k_1$?

YES. $u_1 = 1$, $z_1 = x_1$ NO. $u_1 = \varepsilon$, $z_1 = z_0$



ξ_1 is ASU about z_1



ξ_1 is ASU about z_1

In both cases: $\hat{x}_1 = z_1$

$t = 2$

$X_2 = X_1 + W_1 \implies \pi_1 = \xi_1 * p$

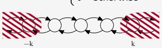
π_1 is ASU about z_1

etc. ...

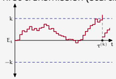
Step 2 Performance of threshold strategies

Consider a threshold-based strategy

$$f^{(k)}(e) = \begin{cases} 1 & \text{if } |e| \geqslant k \\ 0 & \text{otherwise} \end{cases}$$



Let $\tau^{(k)}$ denote the stopping time of first transmission (starting at $E_0 = 0$).



Define

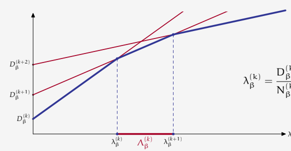
$$L^{(k)}_\beta(e) = (1-\beta)\,\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \Big| E_0 = e\Big].$$

$$M^{(k)}_\beta(e) = (1-\beta)\,\mathbb{E}\Big[\sum_{t=0}^{\tau^{(k)}-1} \beta^t \Big| E_0 = e\Big].$$

Proposition $\quad \{E_t\}_{t=0}^{\infty}$ is a regenerative process. By renewal theory,

$$D^{(k)}_\beta := D_\beta(f^{(k)}, g^*) = \frac{L^{(k)}_\beta(0)}{M^{(k)}_\beta(0)} \quad \text{and} \quad N^{(k)}_\beta := N_\beta(f^{(k)}, g^*) = \frac{1}{M^{(k)}_\beta(0)} - (1-\beta).$$

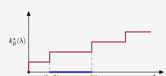
Step 3 Optimal costly communication: Markov chain



$$\lambda^{(k)}_\beta = \frac{D^{(k+1)}_\beta - D^{(k)}_\beta}{N^{(k)}_\beta - N^{(k+1)}_\beta}$$

Define $\Lambda^{(k)}_\beta := \{\lambda \in \mathbb{R}_{\geqslant 0} : k^*_\beta(\lambda) = k\}$
$= [\lambda^{(k-1)}_\beta, \lambda^{(k)}_\beta]$.

$$C^{(k)}_\beta(\lambda^{(k)}_\beta) = C^{(k+1)}_\beta(\lambda^{(k)}_\beta)$$



Step 4 Distortion-transmission trade-off: Markov chain

Sufficient conditions for constrained optimality

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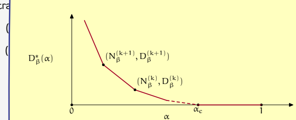

Randomized strategy $(\theta^*, f^{(k)}, f^{k+1})$ is optimal where

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