Approximate planning and learning for partially observed systems

Aditya Mahajan
McGill University

Joint work with Jayakumar Subramanian
Thanks to Amit Sinha and Raihan Seraj for simulation results

Mila RL Reading Group
14 February 2020

Many successes of RL in recent years

- Algorithms based on comprehensive theory

Many successes of RL in recent years

- Algorithms based on comprehensive theory

Alpha Go

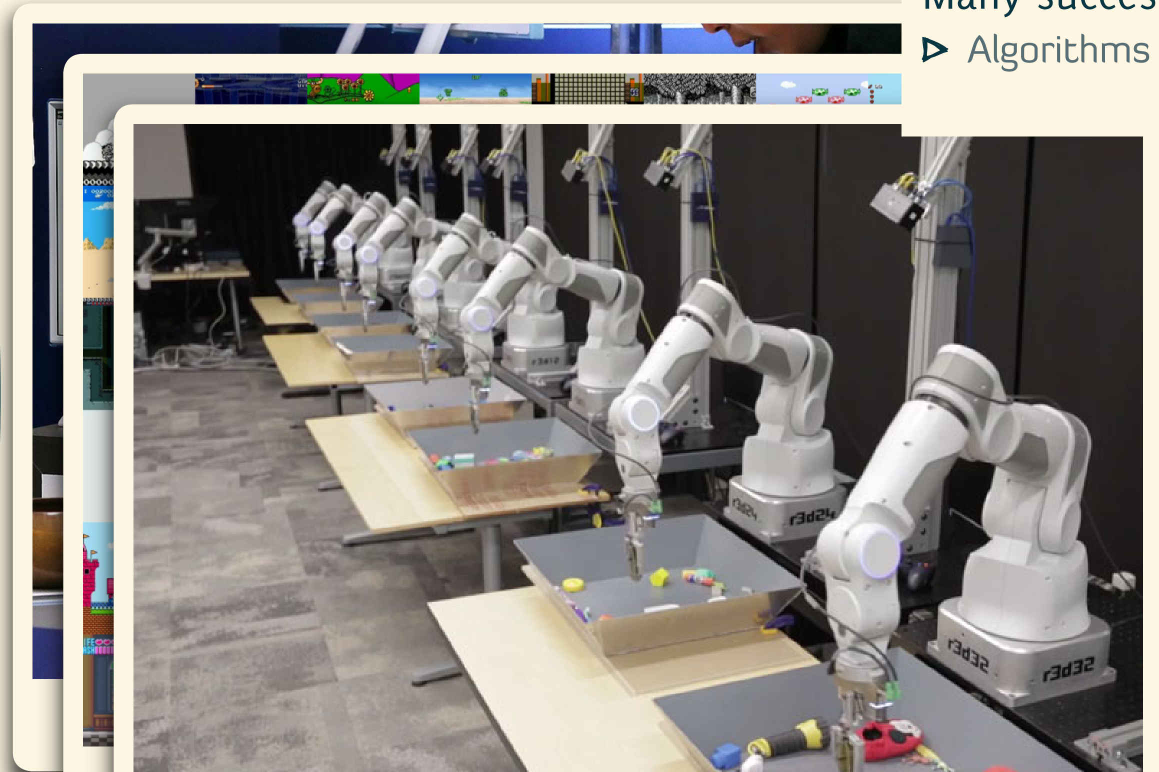
Many successes of RL in recent years

- Algorithms based on comprehensive theory

Arcade games

Many successes of RL in recent years

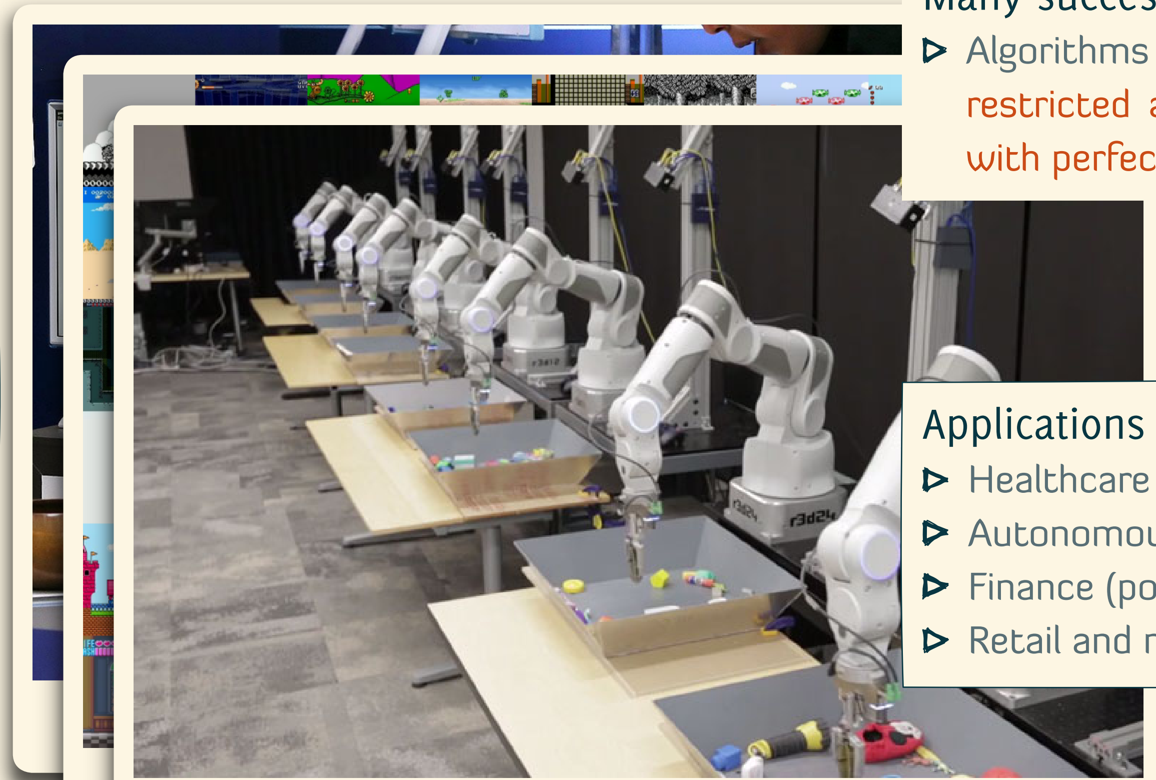
- Algorithms based on comprehensive theory



Robotics

Many successes of RL in recent years

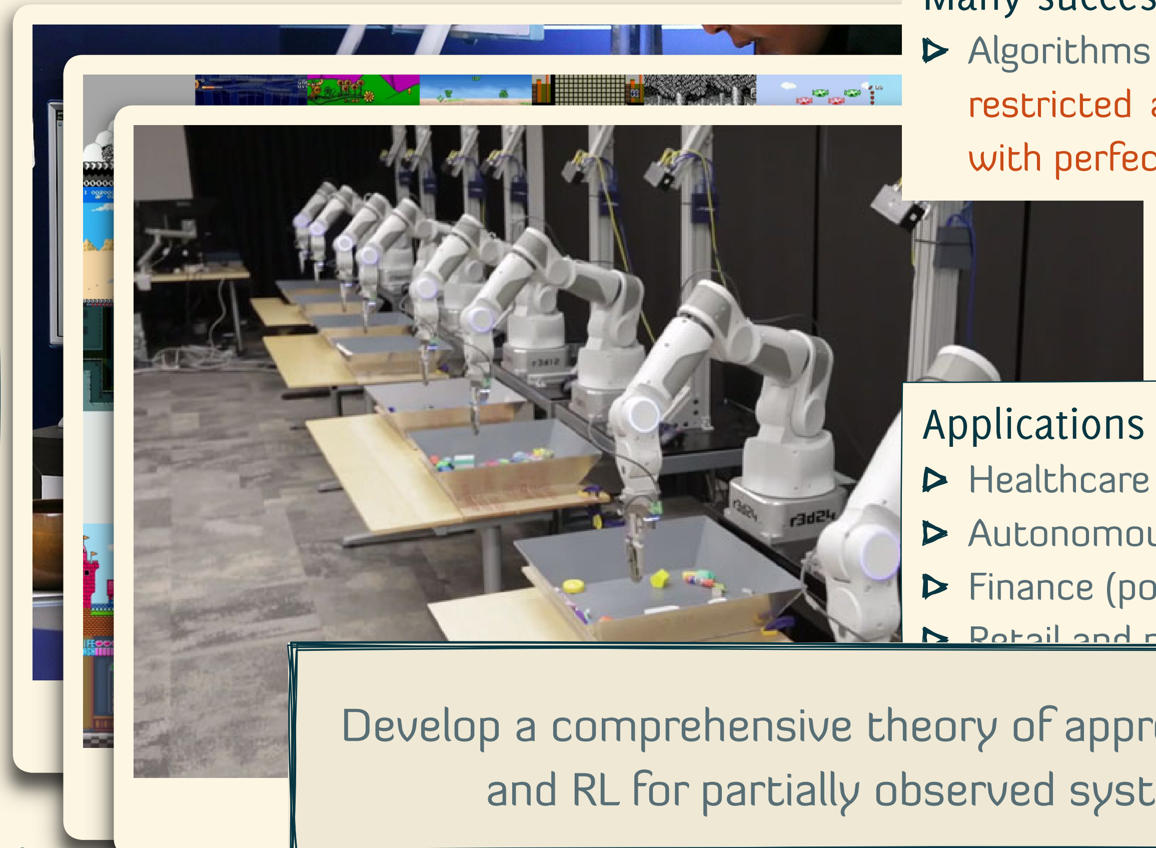
- Algorithms based on comprehensive theory restricted almost exclusively to systems with perfect state observations.



Robotics

Applications with partially observed state

- Healthcare
- Autonomous driving
- Finance (portfolio management)
- Retail and marketing



Many successes of RL in recent years

- Algorithms based on comprehensive theory restricted almost exclusively to systems with perfect state observations.

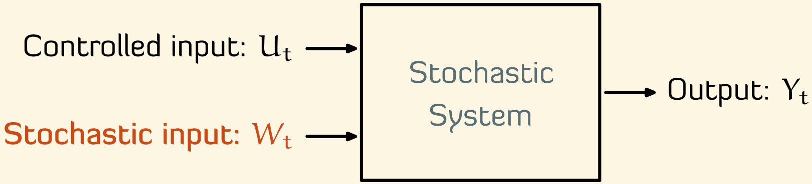
Applications with partially observed state

- Healthcare
- Autonomous driving
- Finance (portfolio management)
- Retail and marketing

Develop a comprehensive theory of approximate DP and RL for partially observed systems

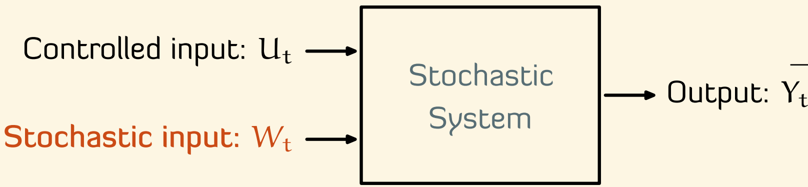
Belief state for partially observed systems

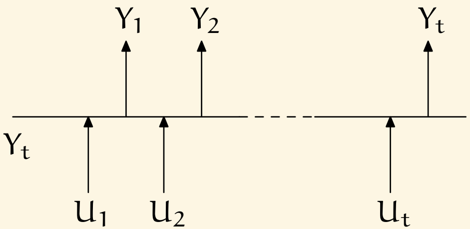
Belief state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

Belief state in partially observed stochastic dynamical systems

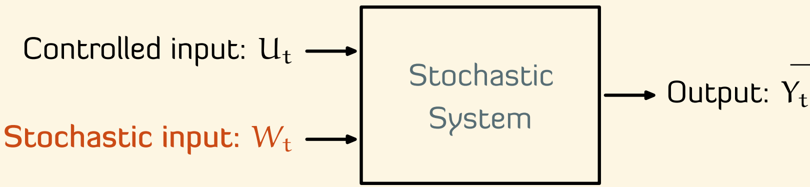




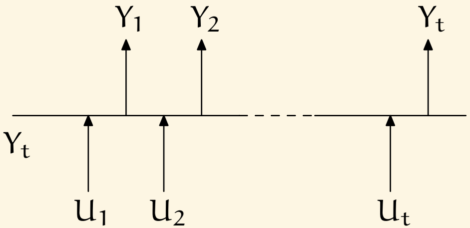
Timing diagram

$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

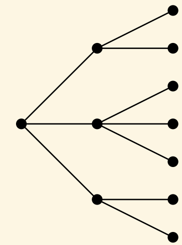
Belief state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

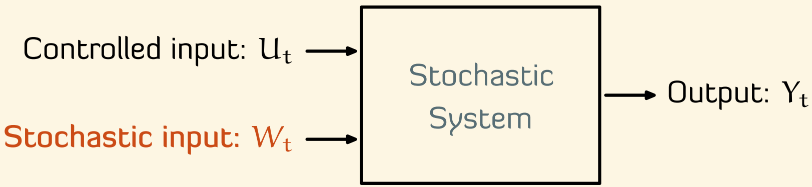


Timing diagram



Decision tree

Belief state in partially observed stochastic dynamical systems

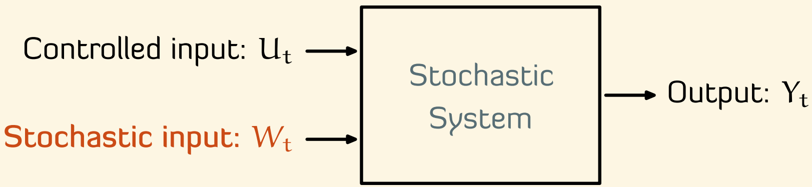


$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

STOCHASTIC INPUT IS NOT OBSERVED

Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

Belief state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

STOCHASTIC INPUT IS NOT OBSERVED

Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

TRADITIONAL SOLUTION: BELIEF STATES

Step 1 Identify a state $\{S_t\}_{t \geq 0}$ for predicting output assuming that the stochastic inputs are observed.

Step 2 Define a BELIEF STATE $B_t \in \Delta(\mathcal{S})$:

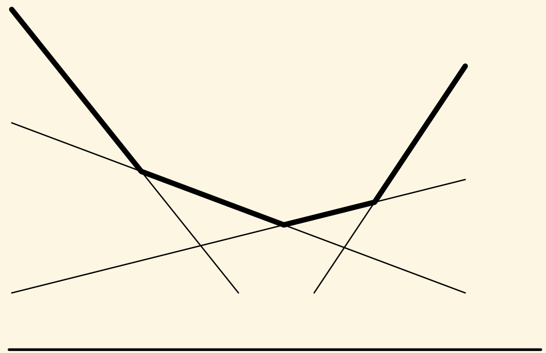
$$B_t(s) = \mathbb{P}(S_t = s \mid H_t = h_t), \quad s \in \mathcal{S}.$$

▷ Astrom, "Optimal control of Markov decision processes with incomplete state information," 1965. ▷ Striebel, "Sufficient statistics in the optimal control of stochastic systems," 1965.

▷ Stratonovich, "Conditional Markov processes," 1960. ▷ Baum and Petrie, "Statistical inference for probabilistic functions of finite state Markov chains," 1966.

Partially observed Markov decision processes (POMDPs): Pros and Cons of belief state representation

Value function is piecewise linear and convex.

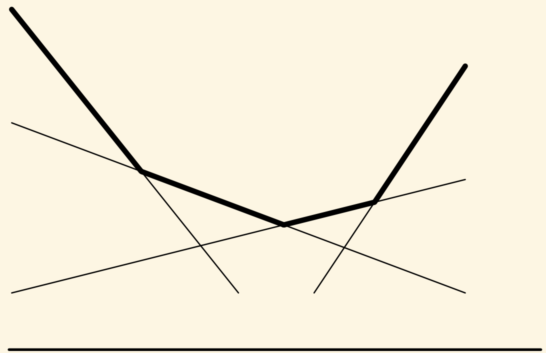


Is exploited by various efficient algorithms.

- Smallwood and Sondik, "The optimal control of partially observable Markov process over a finite horizon," 1973.
- Chen, "Algorithms for partially observable Markov decision processes," 1988.
- Kaelbling, Littmam, Cassandra, "Planning and acting in partially observable stochastic domains," 1998.
- Pineau, Gordon, Thrun, "Point-based value iteration: an anytime algorithm for POMDPs," 2003.

Partially observed Markov decision processes (POMDPs): Pros and Cons of belief state representation

Value function is piecewise linear and convex.



Is exploited by various efficient algorithms.

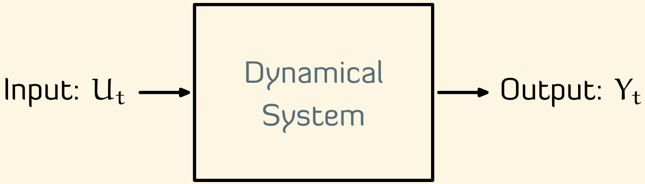
When the state space model is not known analytically (as is the case for black-box models and simulators as well as some real world application such as healthcare), belief states are difficult to construct and difficult to approximate from data.

- Smallwood and Sondik, "The optimal control of partially observable Markov process over a finite horizon," 1973.
- Chen, "Algorithms for partially observable Markov decision processes," 1988.
- Kaelbling, Littmam, Cassandra, "Planning and acting in partially observable stochastic domains," 1998.
- Pineau, Gordon, Thrun, "Point-based value iteration: an anytime algorithm for POMDPs," 2003.

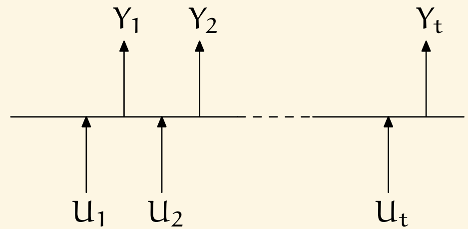
Is there another ways to model partially observed systems which is more amenable to approximations?

Let's go back to first principles.

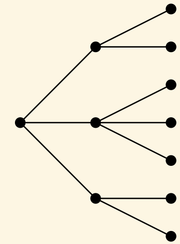
Notion of state in deterministic dynamical systems



$$Y_t = f_t(U_{1:t}).$$

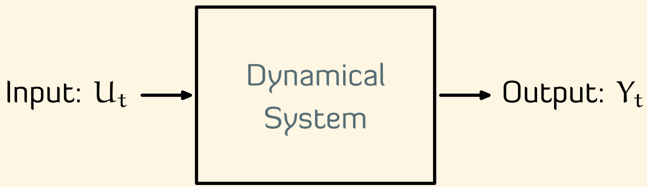


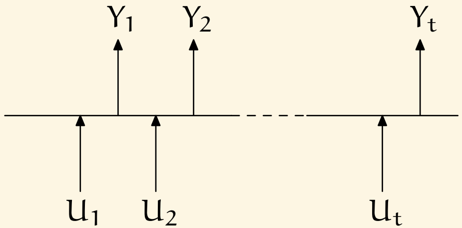
Timing diagram



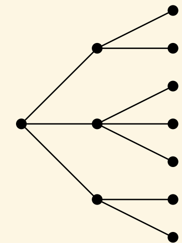
Decision tree

Notion of state in deterministic dynamical systems





Timing diagram



Decision tree

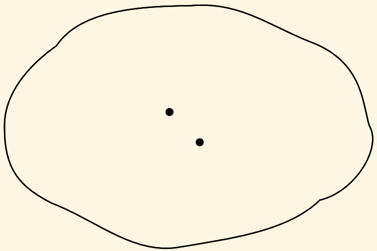
$$Y_t = f_t(U_{1:t}).$$

EQUIVALENCE RELATIONSHIP

Let $H_t = U_{1:t-1}$ denote the history of inputs until time t.

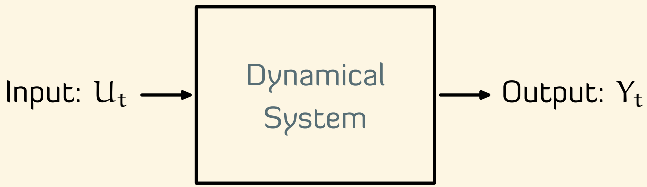
$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $U_{t:T}$, the future outputs $Y_{t:T}^{(1)}$ and $Y_{t:T}^{(2)}$ are the same:

$$f_{t:T}(H_t^{(1)}, U_{t:T}) = f_{t:T}(H_t^{(2)}, U_{t:T})$$



▷ Nerode, "Linear Automaton Transformation", 1958. ▷ Minsky, "Computation: Finite and Infinite Machines", 1967.
▷ Witsenhausen, "Some remarks on the concept of state", 1976.

Notion of state in deterministic dynamical systems



$$Y_t = f_t(U_{1:t}).$$

EQUIVALENCE RELATIONSHIP

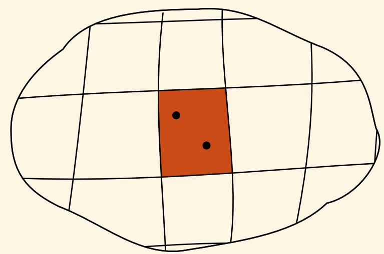
Let $H_t = U_{1:t-1}$ denote the history of inputs until time t.

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $U_{t:T}$, the future outputs $Y_{t:T}^{(1)}$ and $Y_{t:T}^{(2)}$ are the same:

$$f_{t:T}(H_t^{(1)}, U_{t:T}) = f_{t:T}(H_t^{(2)}, U_{t:T})$$

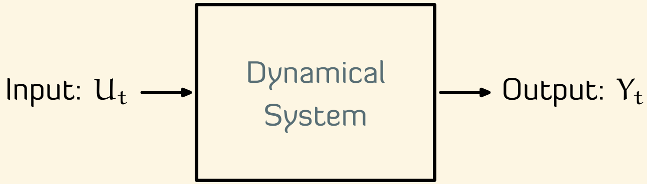
STATE SUFFICIENT FOR I/O MAPPING

Let $\mathcal{H}_t$ denote the space of all histories at time t. Then, the state space at time t is the quotient space $\mathcal{H}_t/\sim$.



▷ Nerode, "Linear Automaton Transformation", 1958. ▷ Minsky, "Computation: Finite and Infinite Machines", 1967.
▷ Witsenhausen, "Some remarks on the concept of state", 1976.

Notion of state in deterministic dynamical systems



$$Y_t = f_t(U_{1:t}).$$

EQUIVALENCE RELATIONSHIP

Let $H_t = U_{1:t-1}$ denote the history of inputs until time t.

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $U_{t:T}$, the future outputs $Y_{t:T}^{(1)}$ and $Y_{t:T}^{(2)}$ are the same:

$$f_{t:T}(H_t^{(1)}, U_{t:T}) = f_{t:T}(H_t^{(2)}, U_{t:T})$$

STATE SUFFICIENT FOR I/O MAPPING

Let $\mathcal{H}_t$ denote the space of all histories at time t. Then, the state space at time t is the quotient space $\mathcal{H}_t/\sim$.

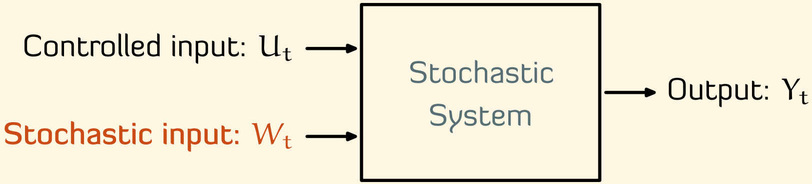
PROPERTIES OF STATE

The state X_t at time t is a “compression” of past inputs that satisfies the following:

- UPDATES IN A RECURSIVE MANNER:
  $$X_{t+1} = \text{function}(X_t, U_t).$$
- SUFFICIENT TO PREDICT OUTPUT:
  $$Y_t = \text{function}(X_t, U_t).$$

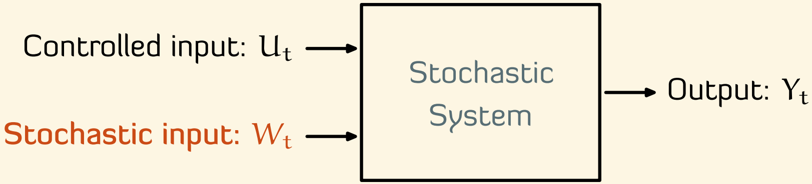
(Ignore: measurability and minimality)

Notion of state in stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

Notion of state in stochastic dynamical systems



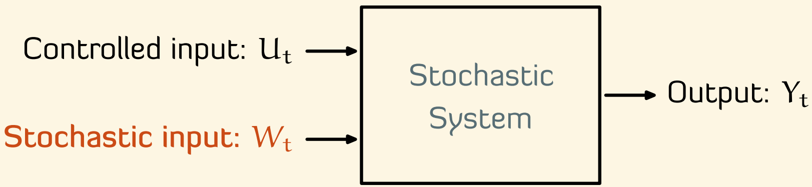
$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

STOCHASTIC INPUT IS OBSERVED

Let $H_t = (U_{1:t-1}, W_{1:t-1})$ denote the history of inputs until time t.

There are two ways to define state:

Notion of state in stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

PREDICTING OUTPUTS ALMOST SURELY

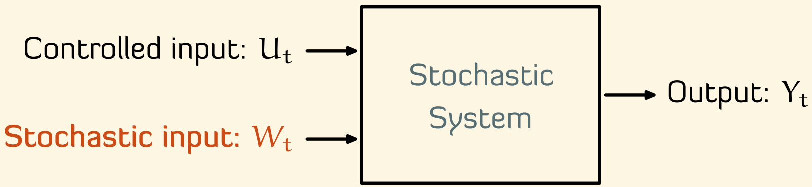
$$H_t^{(1)} \sim H_t^{(2)} \text{ if for all future inputs } (U_{t:T}, W_{t:T}),$$
$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

STOCHASTIC INPUT IS OBSERVED

Let $H_t = (U_{1:t-1}, W_{1:t-1})$ denote the history of inputs until time t.

There are two ways to define state:

Notion of state in stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

STOCHASTIC INPUT IS OBSERVED

Let $H_t = (U_{1:t-1}, W_{1:t-1})$ denote the history of inputs until time t.

There are two ways to define state:

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Now let's construct the state space

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

Now let's construct the state space

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

PROPERTIES OF STATE

The state X_t at time t is a "compression" of past inputs that satisfies the following:

- UPDATES IN A RECURSIVE MANNER:

$$X_{t+1} = \text{function}(X_t, U_t, W_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

$$Y_t = \text{function}(X_t, U_t, W_t).$$

- Kalman, "Mathematical description of linear dynamical systems", 1963.
- Balakrishnan, "Foundations of state-space theory of cts systems", 1967.
- Willems, "The generation of Lyapunov functions for I/O stable systems", 1971.

Now let's construct the state space

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

PROPERTIES OF STATE

The state X_t at time t is a "compression" of past inputs that satisfies the following:

- UPDATES IN A RECURSIVE MANNER:

$$X_{t+1} = \text{function}(X_t, U_t, W_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

$$Y_t = \text{function}(X_t, U_t, W_t).$$

Now let's construct the state space

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

PROPERTIES OF STATE

The state X_t at time t is a "compression" of past inputs that satisfies the following:

- UPDATES IN A RECURSIVE MANNER:

$$X_{t+1} = \text{function}(X_t, U_t, W_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

$$Y_t = \text{function}(X_t, U_t, W_t).$$

PROPERTIES OF STATE

The state X_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:

$$\mathbb{P}(X_{t+1} \mid H_t, U_t) = \mathbb{P}(X_{t+1} \mid X_t, U_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

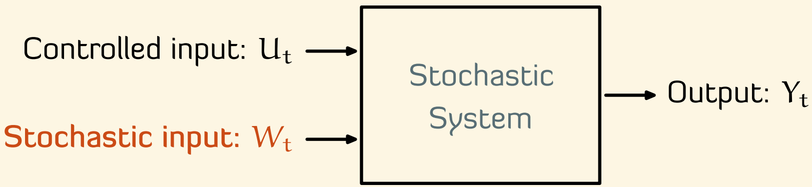
$$\mathbb{P}(Y_t \mid H_t, U_t) = \mathbb{P}(Y_t \mid X_t, U_t).$$

We recover the two basic models of Markov decision processes!

We recover the two basic models
of Markov decision processes!

What happens when the
stochastic input is **not** observered?

Notion of state in partially observed stochastic dynamical systems

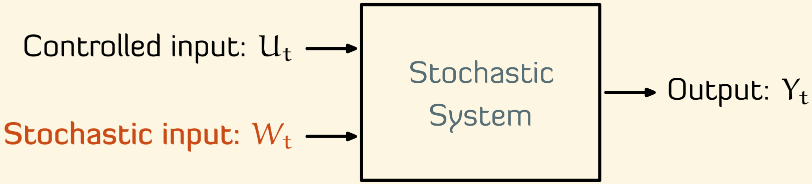


$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

WHEN THE STOCHASTIC INPUT IS NOT OBSERVED

Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

Notion of state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

WHEN THE STOCHASTIC INPUT IS NOT OBSERVED

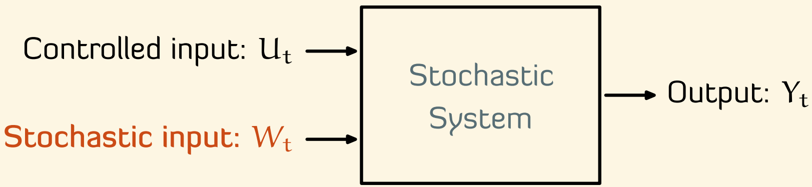
Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

Notion of state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

WHEN THE STOCHASTIC INPUT IS NOT OBSERVED

Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

PREDICTING OUTPUTS ALMOST SURELY

$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,

$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$

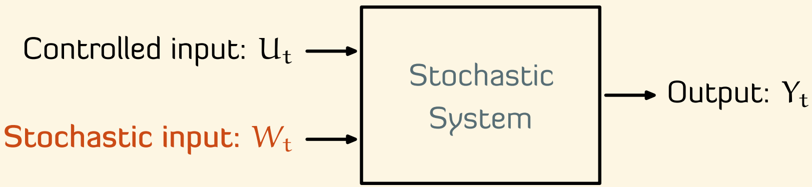
FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

- Grassberger, "Complexity and forecasting in dynamical systems," 1988.
- Cruthfield and Young, "Inferring statistical complexity," 1989.

Notion of state in partially observed stochastic dynamical systems



$$Y_t = f_t(U_{1:t}, W_{1:t}).$$

WHEN THE STOCHASTIC INPUT IS NOT OBSERVED

Let $H_t = (Y_{1:t-1}, U_{1:t-1})$ denote the history of inputs and OUTPUTS until time t.

~~**PREDICTING OUTPUTS ALMOST SURELY**~~

~~$H_t^{(1)} \sim H_t^{(2)}$ if for all future inputs $(U_{t:T}, W_{t:T})$,~~

~~$$Y_{t:T}^{(1)} = Y_{t:T}^{(2)}, \quad a.s.$$~~

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Too restrictive . . .

- Grassberger, "Complexity and forecasting in dynamical systems," 1988.
- Cruthfield and Young, "Inferring statistical complexity," 1989.

Now let's construct the state space

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Now let's construct the state space

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

PROPERTIES OF INFORMATION STATE

The info state Z_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:

$$\mathbb{P}(Z_{t+1} \mid H_t, U_t) = \mathbb{P}(Z_{t+1} \mid Z_t, U_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

$$\mathbb{P}(Y_t \mid H_t, U_t) = \mathbb{P}(Y_t \mid Z_t, U_t).$$

Now let's construct the state space

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Same complexity as identifying the state sufficient for forecasting outputs for the case of perfect observations (which was Step 1 for belief state formulations)

PROPERTIES OF INFORMATION STATE

The info state Z_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:

$$\mathbb{P}(Z_{t+1} \mid H_t, U_t) = \mathbb{P}(Z_{t+1} \mid Z_t, U_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

$$\mathbb{P}(Y_t \mid H_t, U_t) = \mathbb{P}(Y_t \mid Z_t, U_t).$$

Now let's construct the state space

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Same complexity as identifying the state sufficient for forecasting outputs for the case of perfect observations (which was Step 1 for belief state formulations)

PROPERTIES OF INFORMATION STATE

The info state Z_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:

$$\mathbb{P}(Z_{t+1} \mid H_t, U_t) = \mathbb{P}(Z_{t+1} \mid Z_t, U_t).$$

- SUFFICIENT TO PREDICT OUTPUT:

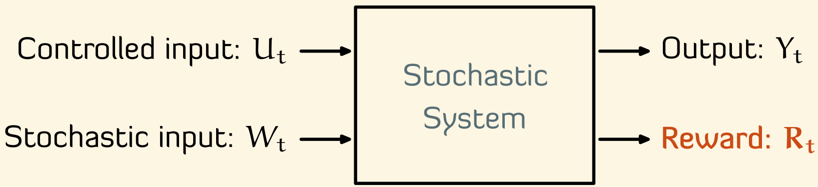
$$\mathbb{P}(Y_t \mid H_t, U_t) = \mathbb{P}(Y_t \mid Z_t, U_t).$$

KEY QUESTIONS

- Can this be used for dynamic programming?
- What is the right notion of approximations in this framework?

An information state for dynamic programming

Predicting output vs optimizing expected rewards over time



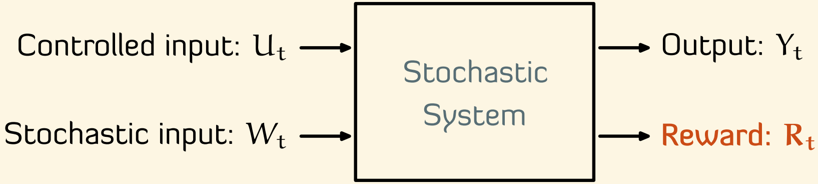
$$Y_t = f_t(U_{1:t}, W_{1:t}),$$

$$R_t = r_t(U_{1:t}, W_{1:t}).$$

Choose $U_t = g_t(Y_{1:t-1}, U_{1:t-1})$ to

$$\max \mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

Predicting output vs optimizing expected rewards over time



$$Y_t = f_t(U_{1:t}, W_{1:t}),$$

$$R_t = r_t(U_{1:t}, W_{1:t}).$$

Choose $U_t = g_t(Y_{1:t-1}, U_{1:t-1})$ to

$$\max \mathbb{E}\left[\sum_{t=1}^{T} R_t\right]$$

PROPERTIES OF INFORMATION STATE (SUFFICIENT FOR DYNAMIC PROGRAMMING)

The info state Z_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:

$$\mathbb{P}(Z_{t+1} \mid H_t, U_t) = \mathbb{P}(Z_{t+1} \mid Z_t, U_t).$$

- SUFFICIENT TO ESTIMATE EXPECTED REWARD:

$$\mathbb{E}[R_t \mid H_t, U_t] = \mathbb{E}[R_t \mid Z_t, U_t].$$

Dynamic programming using information state

PRELIMINARY THEOREM

If $\{Z_t\}_{t \geqslant 1}$ is any information state process. Then:

- There is no loss of optimality in restricting attention to policies of the form

$$U_t = \tilde{g}_t(Z_t).$$

- Bohlin (1970) - David and Varaiya (1972) - Kumar and Varaiya (1984).

Dynamic programming using information state

PRELIMINARY THEOREM

If $\{Z_t\}_{t \geqslant 1}$ is any information state process. Then:

- There is no loss of optimality in restricting attention to policies of the form

$$U_t = \tilde{g}_t(Z_t).$$

- Let $\{V_t\}_{t=1}^{T+1}$ denote the solution to the following dynamic program: $V_{T+1}(z_{T+1}) = 0$ and for $t \in \{T, \dots, 1\}$,

$$Q_t(z_t, u_t) = \mathbb{E}[R_t + V_{t+1}(Z_{t+1}) \mid Z_t = z_t, U_t = u_t],$$

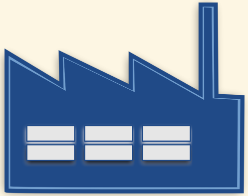
$$V_t(z_t) = \max_{u_t \in \mathcal{U}} Q_t(z_t, u_t).$$

A policy $\{\tilde{g}_t\}_{t=1}^{T}$, $\tilde{g}_t \colon \mathcal{Z}_t \to \mathcal{U}$, is optimal if it satisfies

$$\tilde{g}_t(z_t) \in \arg\max_{u_t \in \mathcal{U}} Q_t(z_t, u_t).$$

- Bohlin (1970)
- David and Varaiya (1972)
- Kumar and Varaiya (1984).

An example: Machine repair

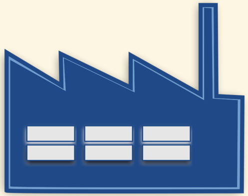


$$P(\text{RUN}) = P_0, \qquad P(\text{INSPECT}) = I$$

$$P(\text{REPLACE}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- State $\in \{1, 2, \dots, n\}$
- Action $\in \{\text{RUN}, \text{INSPECT + REPAIR}\}$.
- cost(state,action) = running cost(state) + inspection cost + repair cost

An example: Machine repair



$$P(\text{RUN}) = P_0, \qquad P(\text{INSPECT}) = I$$

$$P(\text{REPLACE}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- State $\in \{1, 2, \dots, n\}$
- Action $\in \{\text{RUN}, \text{INSPECT + REPAIR}\}$.
- cost(state,action) = running cost(state) + inspection cost + repair cost

Belief state: $\mathbb{P}(S_t | Y_{1:t-1}) \in \mathbb{R}^n$

Alternative information state $(S_\tau, t - \tau) \in \{1, \dots, n\} \times \mathbb{N}$

What about approximations?

Preliminary: A family of pseudometrics on probability distribution

INTEGRAL PROBABILITY METRIC (IPM)

Let $\mathcal{P}$ denote the set of probability measures on a measurable space $(\mathcal{X}, \mathfrak{G})$.

Given a class $\mathfrak{F}$ of real-valued bounded measurable functions on $(\mathcal{X}, \mathfrak{G})$, the integral probability metric (IPM) between two probability distributions $\mu, \nu \in \mathcal{P}$ is given by:

$$d_{\mathfrak{F}}(\mu, \nu) = \sup_{f \in \mathfrak{F}} \left| \int_{\mathcal{X}} f d\mu - \int_{\mathcal{X}} f d\nu \right|.$$

Müller, "Integral probability metrics and their generating classes of functions," 1997.

Preliminary: A family of pseudometrics on probability distribution

INTEGRAL PROBABILITY METRIC (IPM)

Let $\mathcal{P}$ denote the set of probability measures on a measurable space $(\mathcal{X}, \mathfrak{G})$.

Given a class $\mathfrak{F}$ of real-valued bounded measurable functions on $(\mathcal{X}, \mathfrak{G})$, the integral probability metric (IPM) between two probability distributions $\mu, \nu \in \mathcal{P}$ is given by:

$$d_{\mathfrak{F}}(\mu, \nu) = \sup_{f \in \mathfrak{F}} \left| \int_{\mathcal{X}} f d\mu - \int_{\mathcal{X}} f d\nu \right|.$$

EXAMPLES

- If $\mathfrak{F} = \{f : \|f\|_\infty \leqslant 1\}$,
  $d_{\mathfrak{F}} =$ Total variation distance.
- If $\mathfrak{F} = \{f : |f|_L \leqslant 1\}$,
  $d_{\mathfrak{F}} =$ Wasserstein distance.
- If $\mathfrak{F} = \{f : \|df/dx\|_{1-1/p} \leqslant 1\}$,
  $d_{\mathfrak{F}} =$ Cramér p distance
- . . .

We say a function f has a $\mathfrak{F}$-constant K if $f/K \in \mathfrak{F}$.

Müller, "Integral probability metrics and their generating classes of functions," 1997.

Approximate information state

(ε, δ)-APPROXIMATE INFORMATION STATE (AIS)

Given a function class $\mathfrak{F}$, a compression $\{Z_t\}_{t \geqslant 1}$ of history (i.e., $Z_t = \varphi_t(H_t)$) is called an $\{(\varepsilon_t, \delta_t)\}_{t \geqslant 1}$ AIS if there exist:

- a function $\tilde{R}_t(Z_t, U_t)$, and
- a stochastic kernel $\nu_t(Z_{t+1} | Z_t, U_t)$

such that

- $\left| \mathbb{E}[R_t | H_t = h_t, U_t = u_t] - \tilde{R}_t(\varphi_t(h_t), u_t) \right| \leqslant \varepsilon_t$
- For any Borel set A of $\mathcal{Z}_{t+1}$, define

  $$\mu_t(A) = \mathbb{P}(Z_{t+1} \in A \mid H_t = h_t, U_t = u_t)$$

  Then,

  $$d_{\mathfrak{F}}(\mu_t, \nu_t(\cdot | \varphi_t(h_t), u_t)) \leqslant \delta_t.$$

Approximate dynamic programming using AIS

MAIN THEOREM

Given a function class $\mathfrak{F}$, let $\{Z_t\}_{t \geqslant 1}$, where $Z_t = \varphi_t(H_t)$, be an $\{(\varepsilon_t, \delta_t)\}_{t \geqslant 1}$ AIS.

Recursively define the following functions:

$$\hat{V}_{T+1}(z_{T+1}) = 0$$

and for $t \in \{T, \dots, 1\}$:

$$\hat{V}_t(z_t) = \max_{u_t \in \mathcal{U}} \Big\{ \tilde{R}_t(z_t, u_t) + \int \hat{V}_{t+1}(z_{t+1}) \nu_t(dz_{t+1} \mid z_t, u_t) \Big\}.$$

Let $\pi = (\pi_1, \dots, \pi_T)$ denote the corresponding policy.

Approximate dynamic programming using AIS

MAIN THEOREM

Given a function class $\mathfrak{F}$, let $\{Z_t\}_{t \geqslant 1}$, where $Z_t = \varphi_t(H_t)$, be an $\{(\varepsilon_t, \delta_t)\}_{t \geqslant 1}$ AIS.

Recursively define the following functions:

$$\hat{V}_{T+1}(z_{T+1}) = 0$$

and for $t \in \{T, \dots, 1\}$:

$$\hat{V}_t(z_t) = \max_{u_t \in \mathcal{U}} \Big\{ \tilde{R}_t(z_t, u_t) + \int \hat{V}_{t+1}(z_{t+1}) \nu_t(dz_{t+1} \mid z_t, u_t) \Big\}.$$

Let $\pi = (\pi_1, \dots, \pi_T)$ denote the corresponding policy.

Then, if the value function $\hat{V}_t$ has $\mathfrak{F}$-constant K_t, then

- for any history h_t,

$$\Big| V_t(h_t) - \hat{V}_t(\varphi_t(h_t)) \Big| \leqslant \varepsilon_T + \sum_{s=t}^{T} (\varepsilon_s + K_s \delta_s).$$

- for any history h_t,

$$\Big| V_t(h_t) - V_t^{\pi}(h_t) \Big| \leqslant 2\Big[\varepsilon_T + \sum_{s=t}^{T} (\varepsilon_s + K_s \delta_s)\Big].$$

AIS: Some remarks

In the definition of AIS, we can replace

$$d_{\mathfrak{F}}(\mathbb{P}(\mu_t, \nu_t(\cdot | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$$

by

- $Z_{t+1} = \text{function}(Z_t, Y_t, U_t)$
- $d_{\mathfrak{F}}(\mathbb{P}(Y_t | H_t = h_t, U_t = u_t), \mathbb{P}(Y_t | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t.$

AIS: Some remarks

In the definition of AIS, we can replace

$$d_{\mathfrak{F}}(\mathbb{P}(\mu_t, \nu_t(\cdot | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$$

by

- $Z_{t+1} = \text{function}(Z_t, Y_t, U_t)$
- $d_{\mathfrak{F}}(\mathbb{P}(Y_t | H_t = h_t, U_t = u_t), \mathbb{P}(Y_t | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t.$

The AIS process $\{Z_t\}_{t \geqslant 1}$ need not be Markov !!

AIS: Some remarks

In the definition of AIS, we can replace

$$d_{\mathfrak{F}}(\mathbb{P}(\mu_t, \nu_t(\cdot|Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$$

by

- $Z_{t+1} = \text{function}(Z_t, Y_t, U_t)$
- $d_{\mathfrak{F}}(\mathbb{P}(Y_t|H_t = h_t, U_t = u_t), \mathbb{P}(Y_t|Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$.

The AIS process $\{Z_t\}_{t \geqslant 1}$ need not be Markov !!

Two ways to interpret the results:

- Given the information state space $\mathcal{Z}$, find the best compression $\varphi_t : \mathcal{H}_t \to \mathcal{Z}$
- Given any compression function $\varphi_t : \mathcal{H}_t \to \mathcal{Z}_t$, find the approximation error.

AIS: Some remarks

In the definition of AIS, we can replace

$$d_{\mathfrak{F}}(\mathbb{P}(\mu_t, \nu_t(\cdot | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$$

by

- $Z_{t+1} = \text{function}(Z_t, Y_t, U_t)$
- $d_{\mathfrak{F}}(\mathbb{P}(Y_t | H_t = h_t, U_t = u_t), \mathbb{P}(Y_t | Z_t = \varphi_t(h_t), U_t = u_t)) \leqslant \delta_t$.

The AIS process $\{Z_t\}_{t \geqslant 1}$ need not be Markov !!

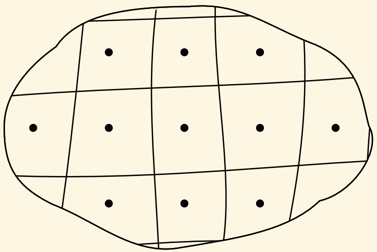
Two ways to interpret the results:

- Given the information state space $\mathcal{Z}$, find the best compression $\varphi_t : \mathcal{H}_t \to \mathcal{Z}$
- Given any compression function $\varphi_t : \mathcal{H}_t \to \mathcal{Z}_t$, find the approximation error.

Results naturally extend to infinite horizon

Some examples

Example 1: Error bounds on state aggregation

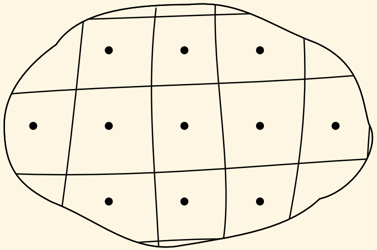


Consider an MDP with state space $\mathcal{X}$ and per-step reward $R_t = r(X_t, U_t)$.

Suppose $\mathcal{X}$ is quantized to a discrete set $\mathcal{Z}$ using $\varphi \colon \mathcal{X} \to \mathcal{Z}$.

- Let $z = \varphi(x)$ denote the label for x.
- Then $\varphi^{-1}(z)$ denote all states which have label z.

Example 1: Error bounds on state aggregation



Consider an MDP with state space $\mathcal{X}$ and per-step reward $R_t = r(X_t, U_t)$.

Suppose $\mathcal{X}$ is quantized to a discrete set $\mathcal{Z}$ using $\varphi\colon \mathcal{X} \to \mathcal{Z}$.

- Let $z = \varphi(x)$ denote the label for x.
- Then $\varphi^{-1}(z)$ denote all states which have label z.

$\{Z_t\}_{t\geqslant 1}$ IS AN (ε, δ) AIS

$$\varepsilon = \sup_{(x,u)\in\mathcal{X}\times\mathcal{U}} \Big| r(x,u) - r(\varphi(x),u) \Big|$$

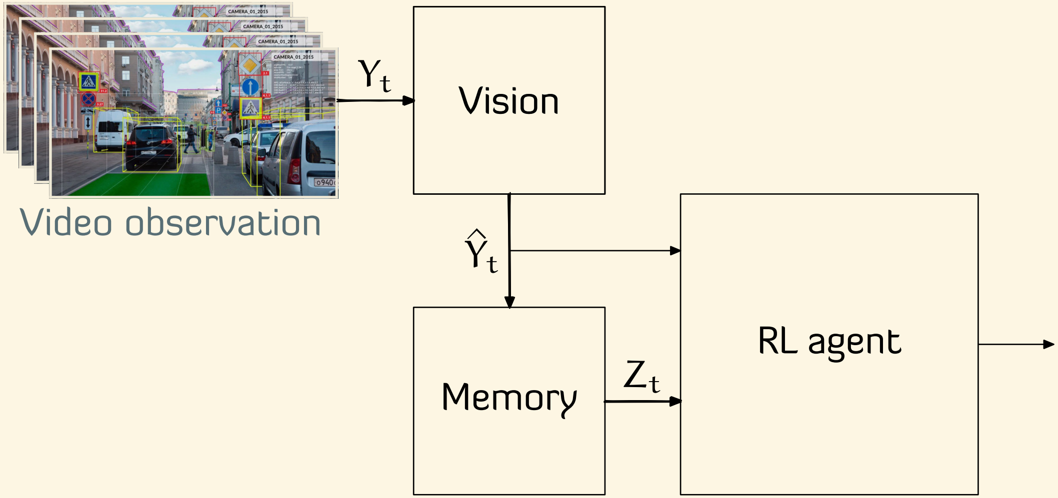
$$\delta = \sup_{(x,u)\in\mathcal{X}\times\mathcal{U}} d_{\mathfrak{F}}(\mathbb{P}(X_+ \mid X = x, U = u), \mathbb{P}(X_+ \mid X \in \varphi^{-1}(\varphi(x)), U = u)).$$

or equivalently, $r(\cdot, u)$ has a $\mathfrak{F}$-constant K_r, $\mathbb{P}(X_+ | X = \cdot, U = u)$ has a $\mathfrak{F}$-constant K_p, then

$$\varepsilon = K_r D, \quad \delta = K_p D, \quad \text{where } D = \max\{\|x - y\| : \varphi(x) = \varphi(y)\}.$$

- Bertsekas, "Convergence of discretization procedures in dynamic programming," 1975.

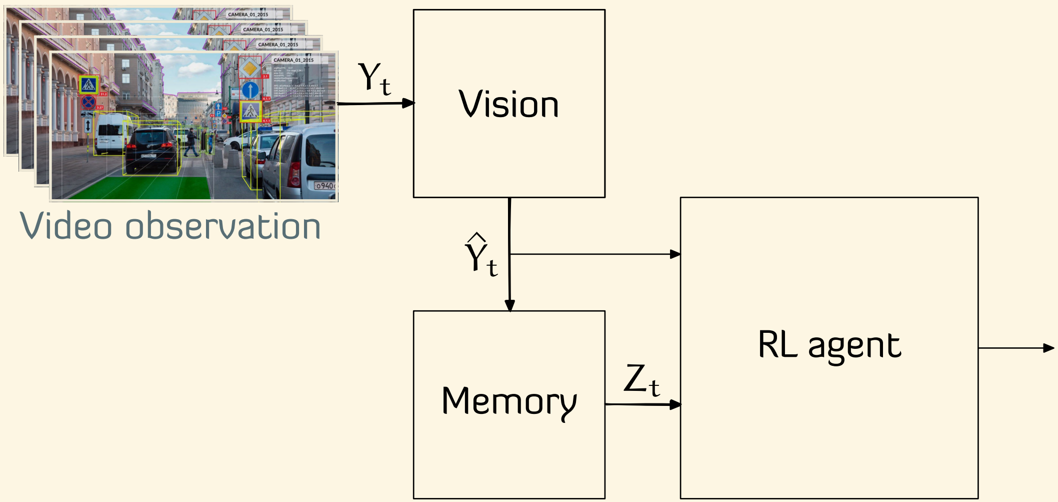
Example 2: Approximation bounds for using quantized obs.



▷ Ha, Schmidhuber, "World Models", 2018.

Example 2: Approximation bounds for using quantized obs.

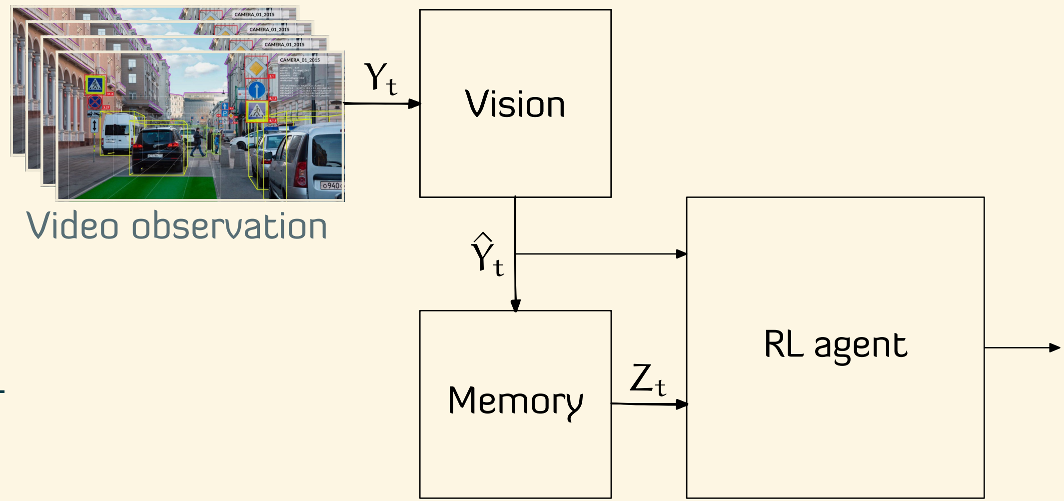
- Proposed as a heuristic algorithms
- No performance bounds



▷ Ha, Schmidhuber, “World Models”, 2018.

Example 2: Approximation bounds for using quantized obs.

- Proposed as a heuristic algorithms
- No performance bounds



$\{Z_t\}_{t\geqslant 1}$ IS AN (ε, δ) AIS

$$\varepsilon_t = \sup_{h_t, u_t} \left| \mathbb{E}[R_t | h_t, u_t] - \tilde{R}_t(\varphi_t(h_t), u_t) \right|$$

$$\delta_t = \sup_{h_t, u_t} d_{\mathfrak{F}}(\mathbb{P}(\hat{Y}_{t+1} | h_t, u_t), \mathbb{P}(\hat{Y}_{t+1} | \varphi_t(h_t), u_t))$$

Ha, Schmidhuber, “World Models”, 2018.

Example 3: Approximation bounds for mean-field teams

n agents: state X_t^i, control U_t^i.

- Empirical mean-field:
$$M_t(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^i}(x).$$
- Statistical mean-field:
$$\bar{m}_t(x) = \mathbb{P}(X_t^i = x).$$

Example 3: Approximation bounds for mean-field teams

n agents: state X_t^i, control U_t^i.

- Dynamics

$$\mathbb{P}(\mathbf{X}_{t+1} | \mathbf{X}_t, \mathbf{U}_t) = \prod_{i=1}^{n} P(X_{t+1}^i | X_t^i, U_t^i, M_t)$$

- Per-step reward

$$R(\mathbf{X}_t, \mathbf{U}_t) = \frac{1}{n} \sum_{i=1}^{n} r(X_t^i, U_t^i, M_t)$$

- Empirical mean-field:

$$M_t(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^i}(x).$$

- Statistical mean-field:

$$\bar{m}_t(x) = \mathbb{P}(X_t^i = x).$$

Example 3: Approximation bounds for mean-field teams

n agents: state X^i_t, control U^i_t.

- Dynamics

$$\mathbb{P}(\mathbf{X}_{t+1} | \mathbf{X}_t, \mathbf{U}_t) = \prod_{i=1}^{n} P(X^i_{t+1} | X^i_t, U^i_t, M_t)$$

- Per-step reward

$$R(\mathbf{X}_t, \mathbf{U}_t) = \frac{1}{n} \sum_{i=1}^{n} r(X^i_t, U^i_t, M_t)$$

- Empirical mean-field:

$$M_t(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^i_t}(x).$$

- Statistical mean-field:

$$\bar{m}_t(x) = \mathbb{P}(X^i_t = x).$$

- **Info structure**: $I^i_t = \{X^i_t, M_t\}$
- **Infinite population limit**: $\tilde{I}^i_t = \{X^i_t, \bar{m}_t\}$.

$$\bar{m}_{t+1} = \mathcal{P}_g m_t,$$

where

$$[\mathcal{P}_g m](y) = \sum_{x,u} m(x) g(u|x) P(y|x,u,m).$$

Example 3: Approximation bounds for mean-field teams

n agents: state X^i_t, control U^i_t.

- Dynamics

$$\mathbb{P}(\mathbf{X}_{t+1}|\mathbf{X}_t, \mathbf{U}_t) = \prod_{i=1}^{n} P(X^i_{t+1}|X^i_t, U^i_t, M_t)$$

- Per-step reward

$$R(\mathbf{X}_t, \mathbf{U}_t) = \frac{1}{n}\sum_{i=1}^{n} r(X^i_t, U^i_t, M_t)$$

- Empirical mean-field:

$$M_t(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X^i_t}(x).$$

- Statistical mean-field:

$$\bar{m}_t(x) = \mathbb{P}(X^i_t = x).$$

- **Info structure**: $I^i_t = \{X^i_t, M_t\}$
- **Infinite population limit**: $\tilde{I}^i_t = \{X^i_t, \bar{m}_t\}$.

$$\bar{m}_{t+1} = \mathcal{P}_g m_t,$$

where

$$[\mathcal{P}_g m](y) = \sum_{x,u} m(x) g(u|x) P(y|x,u,m).$$

(A) $r(x,u,m)$ and $P(y|x,u,m)$ are Lipschitz in x, u, and m.

$\{\bar{m}_t\}_{t\geq 1}$ is an (ε, δ) AIS for expanded info structure, where $\varepsilon, \delta \in \mathcal{O}(1/\sqrt{n})$.

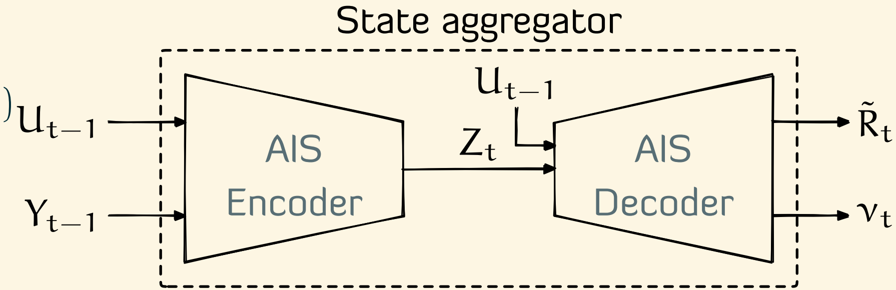
Now to reinforcement learning
for partially observed systems.

Reinforcement learning setup

- **State aggregator**:

  $\mathcal{L}_{AIS} = \alpha_t|\tilde{R}_t - R_t| + (1-\alpha_t)d_{\mathfrak{F}}(\nu_t, \mu_t)$

  ξ: Parameters of the aggregator
  Updated using SGD with LR a_k



Reinforcement learning setup

- **State aggregator**:

  $\mathcal{L}_{AIS} = \alpha_t |\tilde{R}_t - R_t| + (1 - \alpha_t) d_{\mathfrak{F}}(\nu_t, \mu_t)$

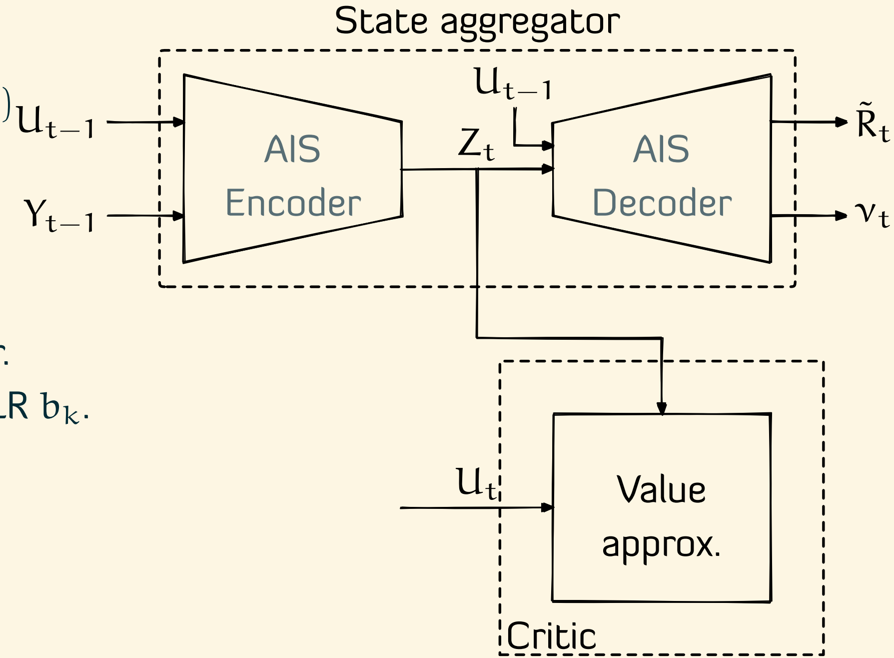
  ξ: Parameters of the aggregator

  Updated using SGD with LR a_k

- **Value approximator**:

  φ: parameters of $Q(z, u)$ approximator.

  Updated using TD(0) or TD(λ) with LR b_k.



Reinforcement learning setup

- **State aggregator**:

  $\mathcal{L}_{AIS} = \alpha_t |\tilde{R}_t - R_t| + (1 - \alpha_t) d_{\mathfrak{F}}(\nu_t, \mu_t)$

  ξ: Parameters of the aggregator

  Updated using SGD with LR a_k

- **Value approximator**:

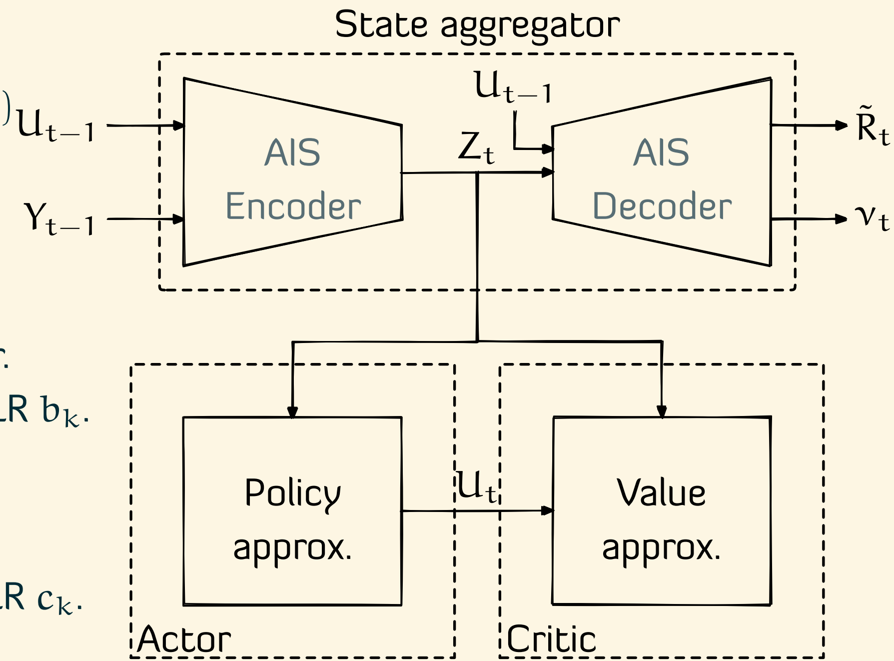
  φ: parameters of $Q(z, u)$ approximator.

  Updated using TD(0) or TD(λ) with LR b_k.

- **Policy approximator**:

  θ: parameters of $\pi(u \mid z)$

  Updated using policy gradient with LR c_k.



Reinforcement learning setup

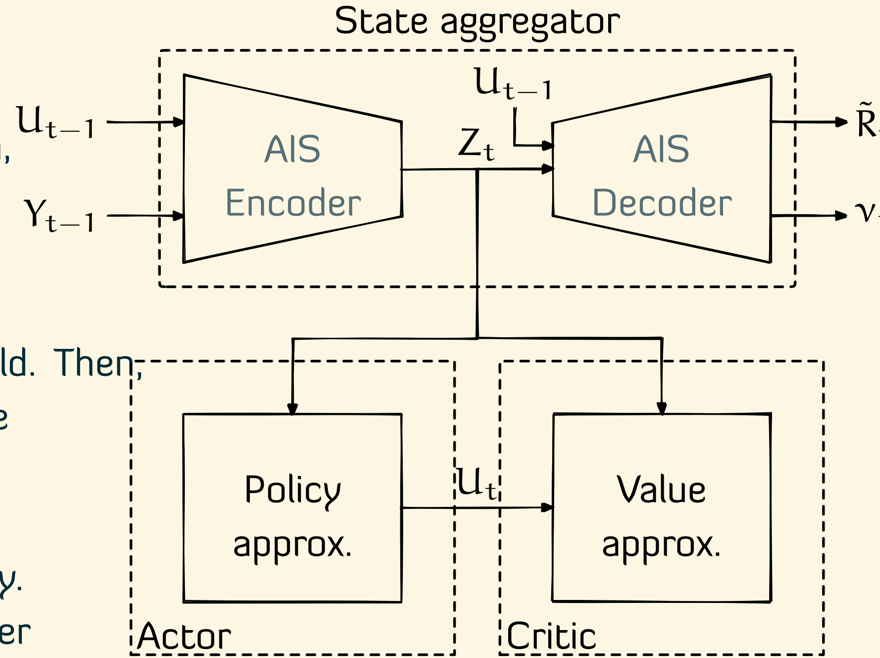
CONVERGENCE RESULT

If the learning rates satisfy conditions for three time-scale stochastic approximation, the compatibility condition

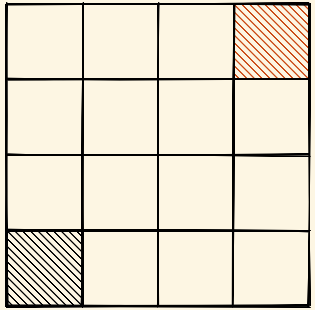
$$\frac{\partial Q(z, u)}{\partial \varphi} = \frac{1}{\pi(u|z)} \frac{\partial \pi(u|z)}{\partial \theta}$$

and additional mild technical conditions hold. Then,

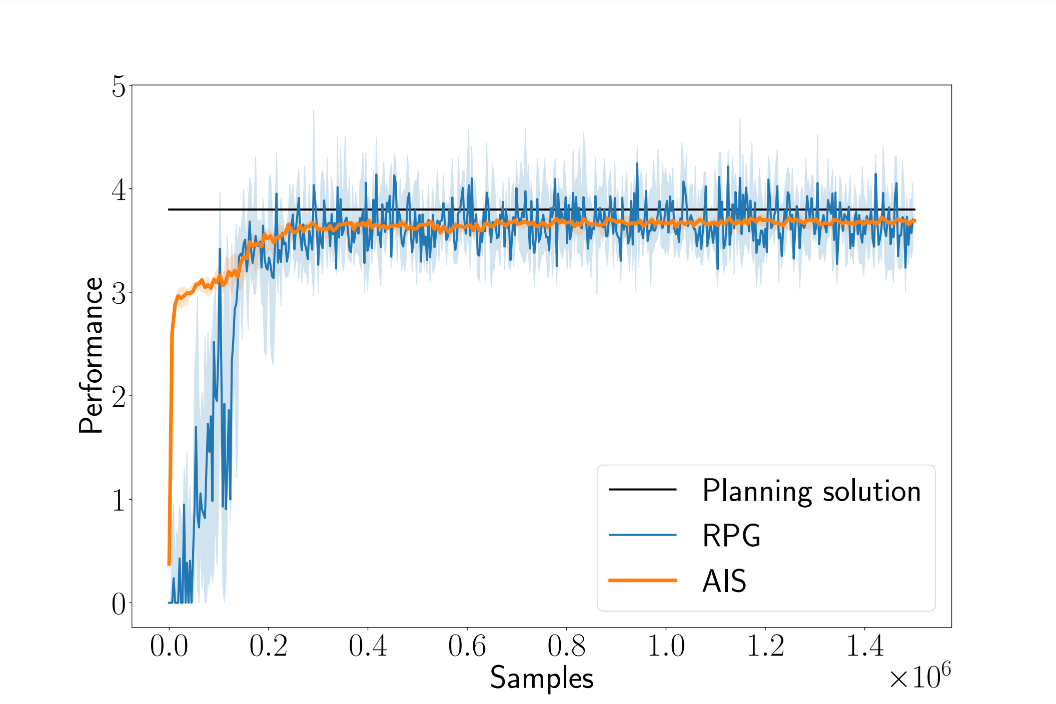
- State aggregator converges (with some approximation error)
- The critic converges to the best approximator within the specified family.
- The actor converges to a local maximizer within the family of policy approximators.

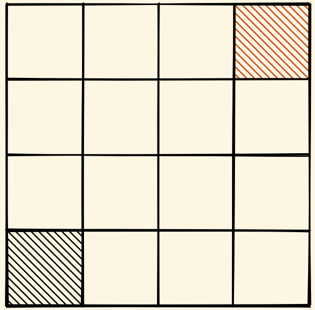


Numerical Results: 4×4 Grid Environment



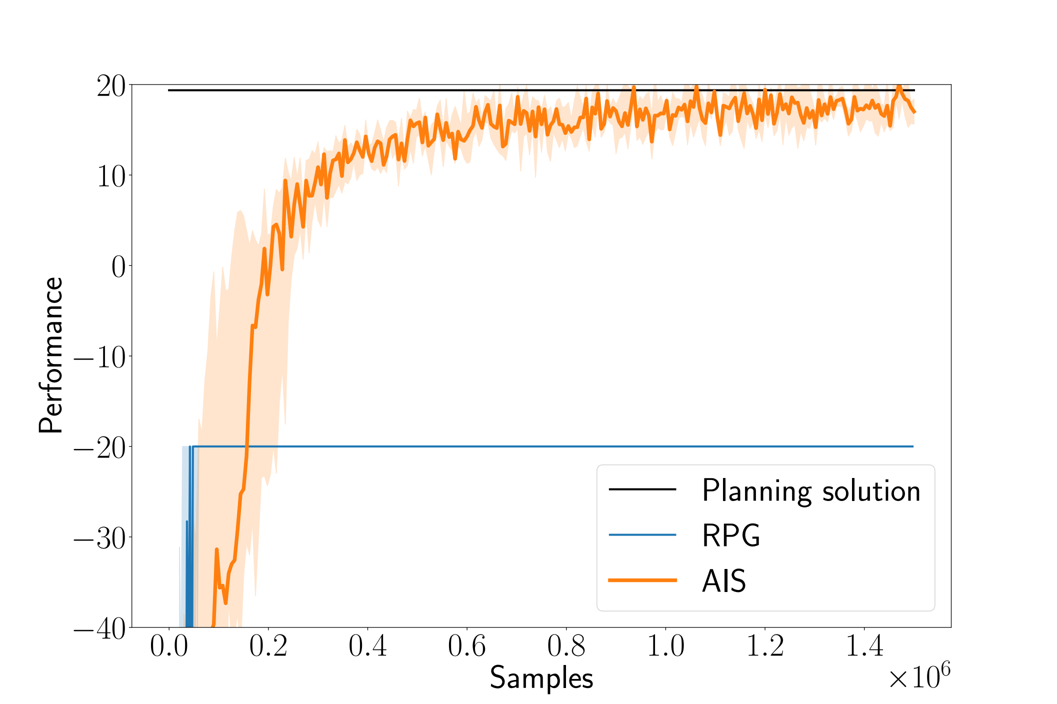
Numerical Results: 4×4 Grid Environment



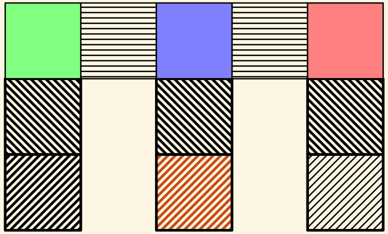


Numerical Results: Tiger Environment

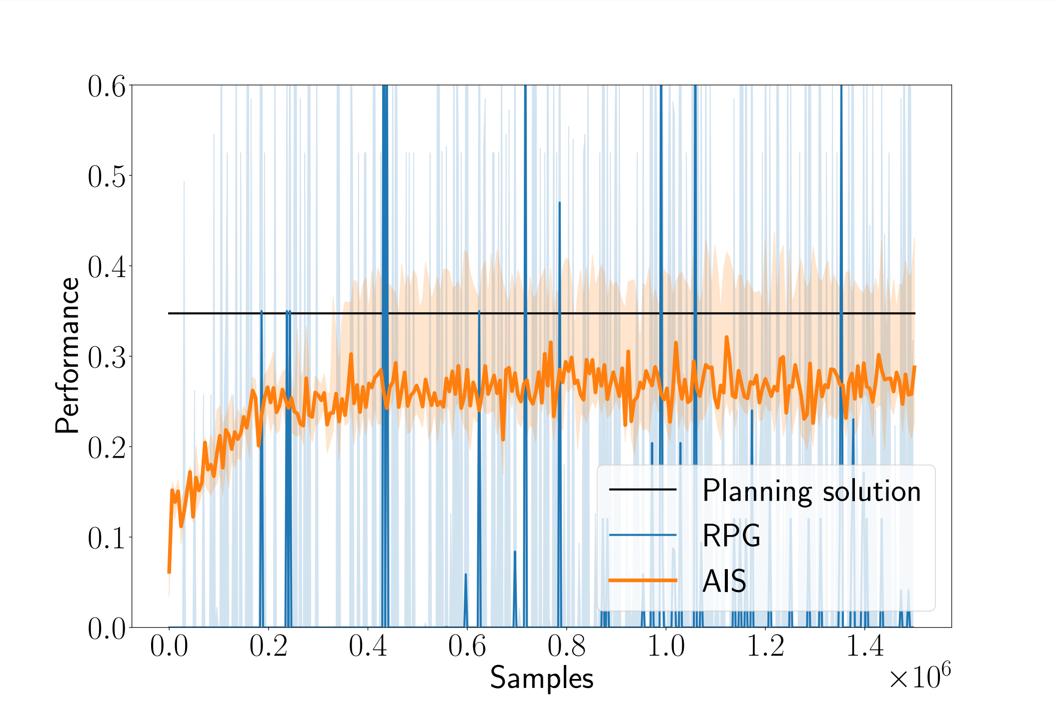
Numerical Results: Tiger Environment

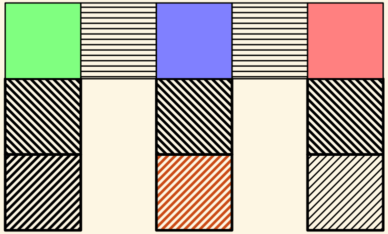


Numerical Results: Cheese Maze Environment

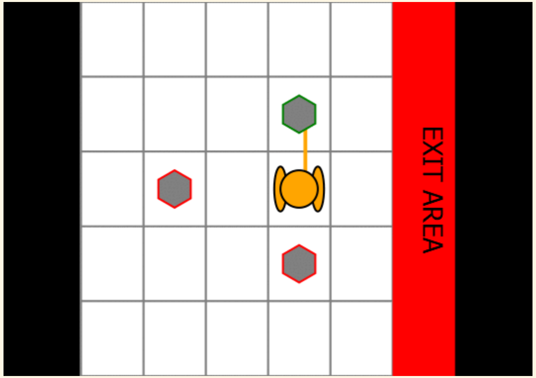


Numerical Results: Cheese Maze Environment

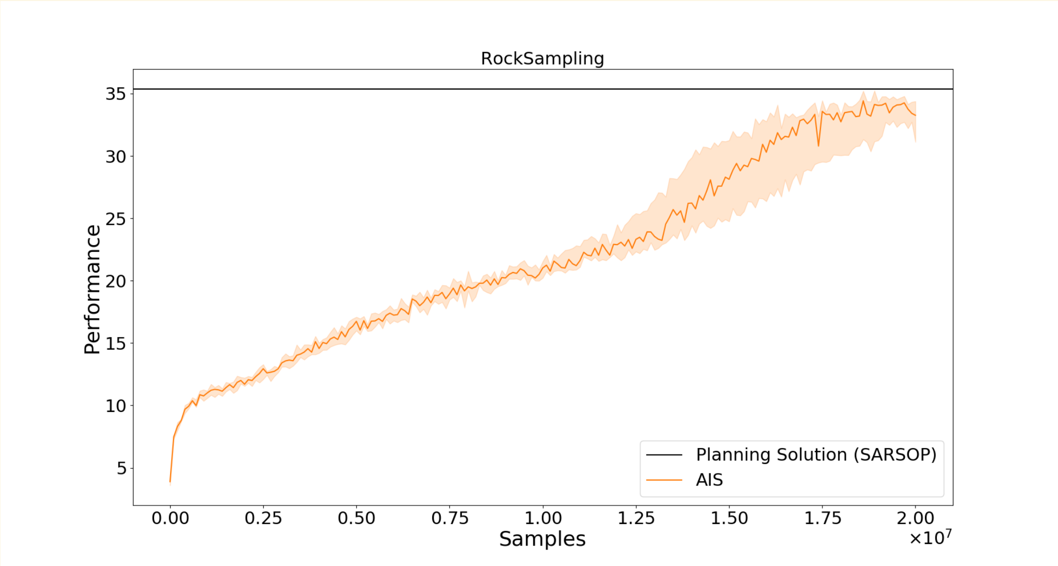


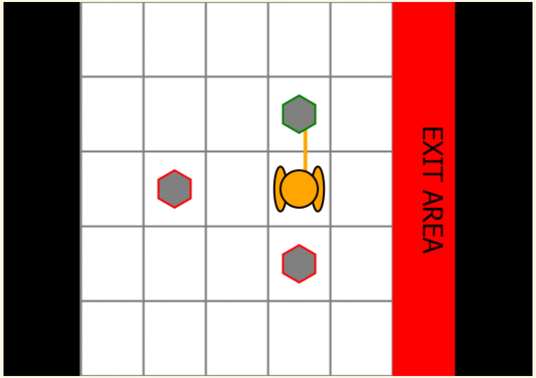


Numerical Results: Rock Sample

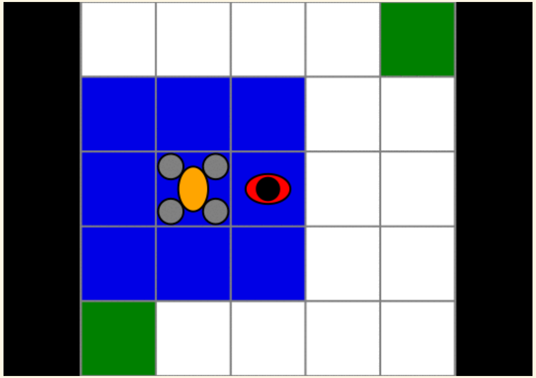


Numerical Results: Rock Sample

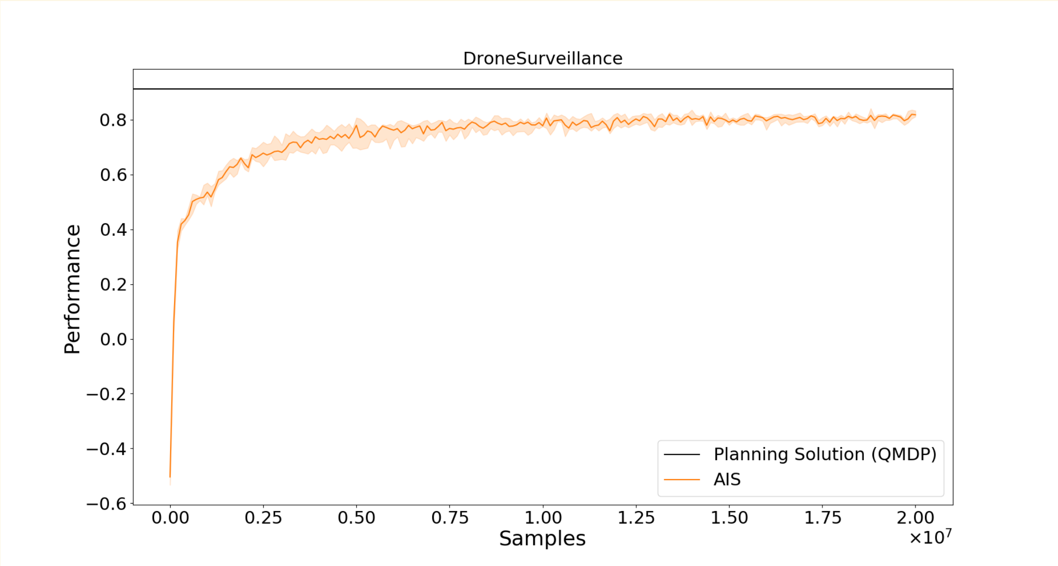


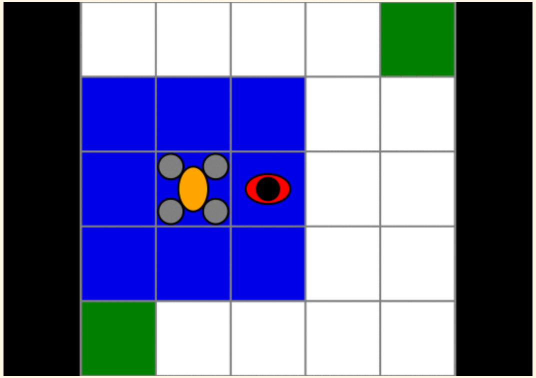


Numerical Results: Drone Surveillance



Numerical Results: Drone Surveillance





Summary

Summary

Now let's construct the state space

FORECASTING OUTPUTS IN DISTRIBUTION

$H_t^{(1)} \sim H_t^{(2)}$ if for all future CONTROL inputs $U_{t:T}$,

$$\mathbb{P}(Y_{t:T}^{(1)} \mid H_t^{(1)}, U_{t:T}) = \mathbb{P}(Y_{t:T}^{(2)} \mid H_t^{(2)}, U_{t:T})$$

Same complexity as identifying the state sufficient for forecasting outputs for the case of perfect observations (which was Step 1 for belief state formulations)

PROPERTIES OF INFORMATION STATE

The info state Z_t at time t is a "compression" of past inputs that satisfies the following:

- SUFFICIENT TO PREDICT ITSELF:
$$\mathbb{P}(Z_{t+1} \mid H_t, U_t) = \mathbb{P}(Z_{t+1} \mid Z_t, U_t).$$
- SUFFICIENT TO PREDICT OUTPUT:
$$\mathbb{P}(Y_t \mid H_t, U_t) = \mathbb{P}(Y_t \mid Z_t, U_t).$$

KEY QUESTIONS

- Can this be used for dynamic programming?
- What is the right notion of approx-imations in this framework?

Summary

Now let's construct the state space

Approximate information state

(ε, δ)-APPROXIMATE INFORMATION STATE (AIS)

Given a function class $\mathfrak{F}$, a compression $\{Z_t\}_{t \geqslant 1}$ of history (i.e., $Z_t = \varphi_t(H_t)$) is called an $\{(\varepsilon_t, \delta_t)\}_{t \geqslant 1}$ AIS if there exist:

- a function $\tilde{R}_t(Z_t, U_t)$, and
- a stochastic kernel $\nu_t(Z_{t+1} | Z_t, U_t)$

such that

- $\left| \mathbb{E}[R_t | H_t = h_t, U_t = u_t] - \tilde{R}_t(\varphi_t(h_t), u_t) \right| \leqslant \varepsilon_t$
- For any Borel set A of $\mathcal{Z}_{t+1}$, define

  $$\mu_t(A) = \mathbb{P}(Z_{t+1} \in A \mid H_t = h_t, U_t = u_t)$$

  Then,

  $$d_{\mathfrak{F}}(\mu_t, \nu_t(\cdot | \varphi_t(h_t), u_t)) \leqslant \delta_t.$$

Summary

Now let's construct the state space

Approximate dynamic programming using AIS

MAIN THEOREM

Given a function class $\mathfrak{F}$, let $\{Z_t\}_{t\geqslant 1}$, where $Z_t = \varphi_t(H_t)$, be an $\{(\varepsilon_t, \delta_t)\}_{t\geqslant 1}$ AIS.

Recursively define the following functions:

$$\hat{V}_{T+1}(z_{T+1}) = 0$$

and for $t \in \{T, \dots, 1\}$:

$$\hat{V}_t(z_t) = \max_{u_t \in \mathcal{U}} \Big\{ \tilde{R}_t(z_t, u_t) + \int \hat{V}_{t+1}(z_{t+1}) \nu_t(dz_{t+1} \mid z_t, u_t) \Big\}.$$

Let $\pi = (\pi_1, \dots, \pi_T)$ denote the corresponding policy.

Then, if the value function $\hat{V}_t$ has $\mathfrak{F}$-constant K_t, then

- for any history h_t,

$$\big|V_t(h_t) - \hat{V}_t(\varphi_t(h_t))\big| \leqslant \varepsilon_T + \sum_{s=t}^{T} (\varepsilon_s + K_s \delta_s).$$

- for any history h_t,

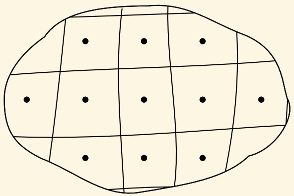
$$\big|V_t(h_t) - V_t^{\pi}(h_t)\big| \leqslant 2\Big[\varepsilon_T + \sum_{s=t}^{T} (\varepsilon_s + K_s \delta_s)\Big].$$

Summary

Now let's construct the state space

Approximate dynamic programming using AIS

Example 1: Error bounds on state aggregation



Consider an MDP with state space $\mathcal{X}$ and per-step reward $R_t = r(X_t, U_t)$.

Suppose $\mathcal{X}$ is quantized to a discrete set $\mathcal{Z}$ using $\varphi\colon \mathcal{X} \to \mathcal{Z}$.

- Let $z = \varphi(x)$ denote the label for x.
- Then $\varphi^{-1}(z)$ denote all states which have label z.

$\{Z_t\}_{t \geqslant 1}$ IS AN (ε, δ) AIS

$$\varepsilon = \sup_{(x,u) \in \mathcal{X} \times \mathcal{U}} \big| r(x,u) - r(\varphi(x), u) \big|$$

$$\delta = \sup_{(x,u) \in \mathcal{X} \times \mathcal{U}} d_{\mathfrak{F}}(\mathbb{P}(X_+ \mid X = x, U = u), \mathbb{P}(X_+ \mid X \in \varphi^{-1}(\varphi(x)), U = u)).$$

or equivalently, $r(\cdot, u)$ has a $\mathfrak{F}$-constant K_r, $\mathbb{P}(X_+ | X = \cdot, U = u)$ has a $\mathfrak{F}$-constant K_p, then

$$\varepsilon = K_r D, \quad \delta = K_p D, \quad \text{where } D = \max\{\|x - y\| : \varphi(x) = \varphi(y)\}.$$

Bertsekas, "Convergence of discretization procedures in dynamic programming," 1975.

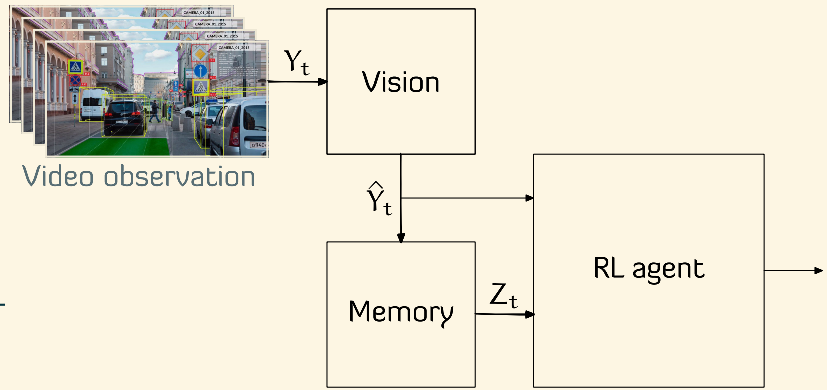
Summary

Now let's construct the state space

Approximate dynamic programming using AIS

Example 2: Approximation bounds for using quantized obs.

- Proposed as a heuristic algorithms
- No performance bounds



$\{Z_t\}_{t\geqslant 1}$ IS AN (ε, δ) AIS

$$\varepsilon_t = \sup_{h_t, u_t} \left| \mathbb{E}[R_t | h_t, u_t] - \tilde{R}_t(\varphi_t(h_t), u_t) \right|$$

$$\delta_t = \sup_{h_t, u_t} d_{\mathfrak{F}}(\mathbb{P}(\hat{Y}_{t+1} | h_t, u_t), \mathbb{P}(\hat{Y}_{t+1} | \varphi_t(h_t), u_t))$$

Ha, Schmidhuber, "World Models", 2018.

Summary

Now let's construct the state space

Approximate dynamic programming using AIS

Example 3: Approximation bounds for mean-field teams

n agents: state X^i_t, control U^i_t.

- Dynamics

$$\mathbb{P}(\mathbf{X}_{t+1} | \mathbf{X}_t, \mathbf{U}_t) = \prod_{i=1}^{n} P(X^i_{t+1} | X^i_t, U^i_t, M_t)$$

- Per-step reward

$$R(\mathbf{X}_t, \mathbf{U}_t) = \frac{1}{n} \sum_{i=1}^{n} r(X^i_t, U^i_t, M_t)$$

- Empirical mean-field:

$$M_t(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^i_t}(x).$$

- Statistical mean-field:

$$\bar{m}_t(x) = \mathbb{P}(X^i_t = x).$$

- **Info structure**: $I^i_t = \{X^i_t, M_t\}$
- **Infinite population limit**: $\tilde{I}^i_t = \{X^i_t, \bar{m}_t\}$.

$$\bar{m}_{t+1} = \mathcal{P}_g m_t,$$

where

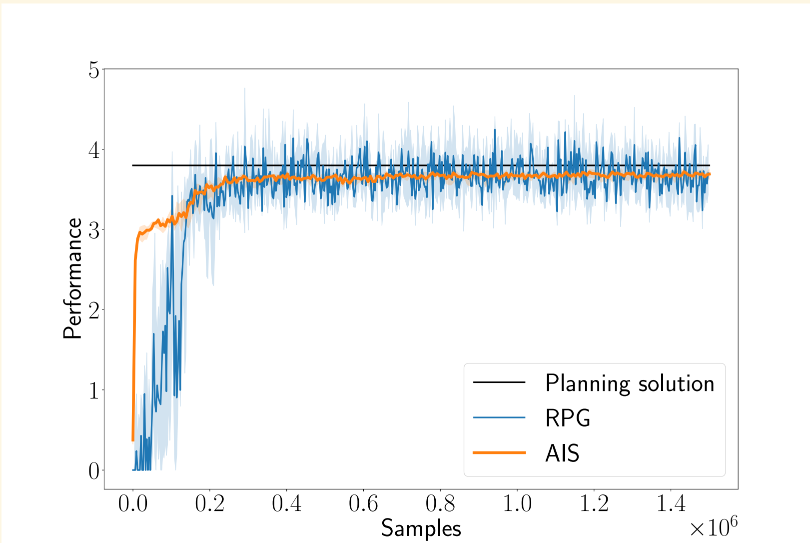
$$[\mathcal{P}_g m](y) = \sum_{x,u} m(x) g(u|x) P(y|x,u,m).$$

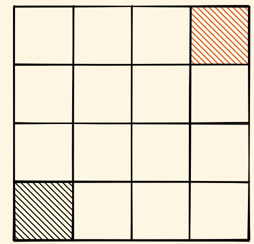
(A) $r(x,u,m)$ and $P(y|x,u,m)$ are Lipschitz in x, u, and m.

$\{\bar{m}_t\}_{t \geqslant 1}$ is an (ε, δ) AIS for expanded info structure, where $\varepsilon, \delta \in \mathcal{O}(1/\sqrt{n})$.

Summary

Numerical Results: 4×4 Grid Environment





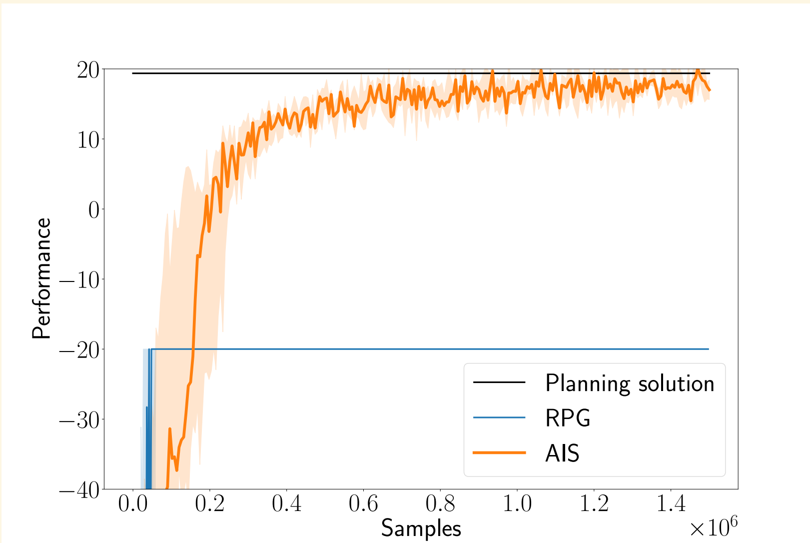
Summary

Now let's construct the state space

Approximate dynamic programming using AIS

Example 3: Approximation bounds for mean field teams

Numerical Results: Tiger Environment



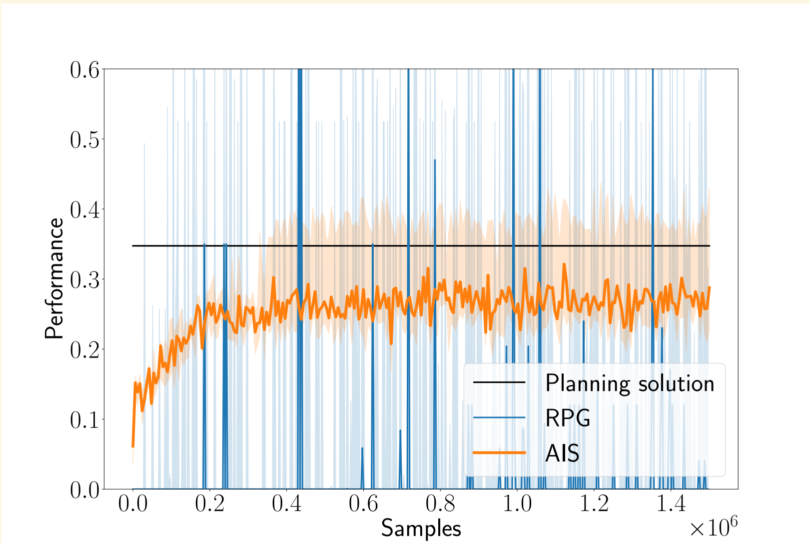
Summary

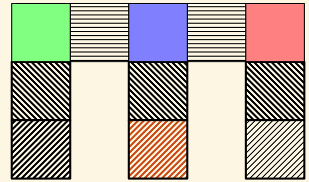
Now let's construct the state space

Approximate dynamic programming using AIS

Example 2: Approximation bounds for mean field teams

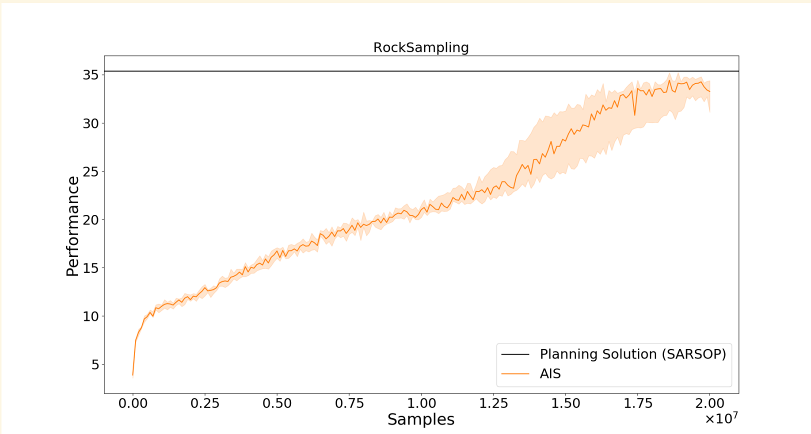
Numerical Results: Cheese Maze Environment

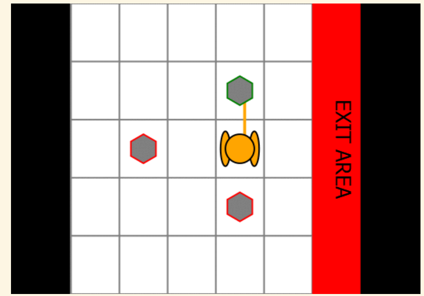




Summary

Numerical Results: Rock Sample





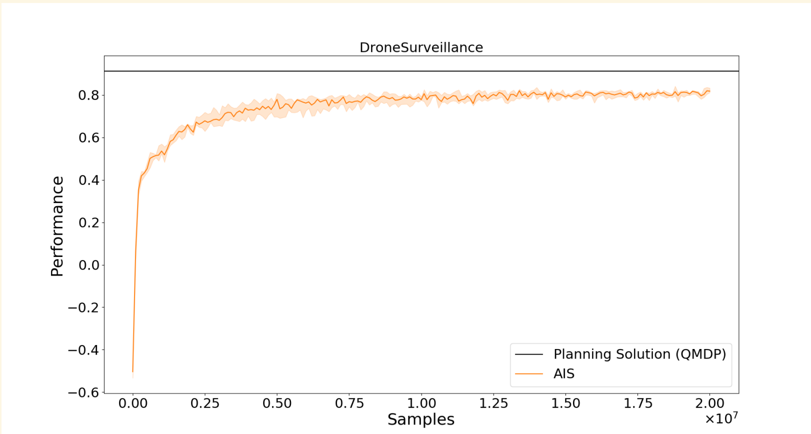
Summary

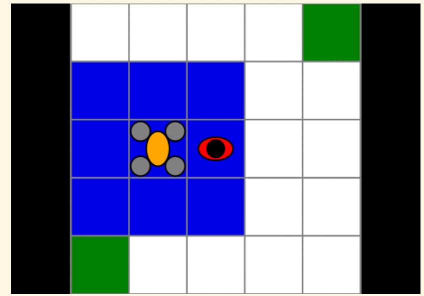
Now let's construct the state space

Approximate dynamic programming using AIS

Example 2: Approximation bounds for mean field teams

Numerical Results: Drone Surveillance





Summary

Now let's construct the state space

Approximate dynamic programming using AIS

Example 2: Approximation bounds for mean field teams

Numerical Results: Drone Surveillance

AIS provides a conceptually clean framework for approximate DP and online RL in partially observed systems