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A Parametric Approach to Extended Source Localization

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Abstract: Point source modeling is frequently used in array processing. Although this assumption is good for many applications, there are some situations where point source modeling is unrealistic. For instance, in a multi-beam echo sounder, a reflected signal from the sea floor appears as a spatially extended source. In this paper we investigate distributed sources. The approach is based on the assumption that the correlation kernel of the distributed source belongs to a family of parametric functions. We generalize the MUSIC algorithm to a distributed signal parameter estimator (DSPE). The DSPE localizer minimizes a scalar product between an estimated basis for the noise subspace and the array manifold. We study two cases corresponding to completely correlated and totally uncorrelated signal distributions. We also discuss limitations of the application of ordinary beamformer techniques to spatially distributed signals by computing the array gain. It is shown that the array gain is upper bounded by a value which depends on the extension width of the source. Thus increasing the number of the sensors in a beamformer does not necessarily increase the resolution.

1. Introduction

A common assumption in array processing is that the signals are generated by sources that are highly localized in space (point sources). This modeling of a source is not a valid assumption in some applications. In a multi-beam echo sounder the penetration of the signal into the sea floor and reflection from different layers creates a spatially distributed source [1]. In radar, if the target is spread in range, the reflection of the signal from the target is observed as a spatially extended signal [2]. When a signal is distributed in space, it is usually modeled as a spatially colored noise [3]. The spatial extension of signal might occur in other applications such as reflection of sound in a reverberant room and communications through the reflection from the ionosphere and the troposphere.

In past, extended sources were treated as a cluster of point sources [1]. For such a model the dimensionality of the signal subspace grows with the number of point signals. To obtain a unqiue solution for this model, the number of point signals (the dimension of the signal subspace) should be smaller than the number of sensors (the dimension of the observation space). Thus, if the number of sensors is not large enough, the clustered point sources approximation will not provide a precise model of a distributed source.

Recently, we have presented a new method for the localization of spatially distributed sources [4]. This method is based on a parametric model for the spatial correlation function of the signal and has been applied to localization of coherently distributed (CD) and incoherently distributed (ID) source. The present paper is a continuation of the previous work. In this paper we provide additional details about the derivation of the method. In particular, we show that the new method is a generalization of the MUSIC algorithm. Furthermore, we study the performance of a conventional beamformer for distributed sources by finding the array gain for a uniform linear array.

2. Distributed source modeling

Assume a scenario with q spatially distributed narrowband sources arriving at an array of p sensors. The array output vector in the frequency domain can be formulated by

$$\mathbf{x} = \sum_{i=1}^{q} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta) s_i(\theta;\psi_i) d\theta + \mathbf{n}, \qquad (1)$$

where $\mathbf{a}(\theta)$ is the $p \times 1$ location vector, $\mathbf{n}$ is the $p \times 1$ noise vector, and $s_i(\theta;\psi_i)$ is the angular signal density of the i-th source which is also a function of the angle-of-arrival θ and the parameter vector ψ_i. The two limits of the directions-of-arrival (DOAs) for a uniform source distribution, and the angle of maximum power and the -3 dB extension width for a bell-shaped distribution are examples of the parameters.

The time samples of the noise vector are modeled as zero-mean, independent, circular, complex Gaussian random variables and uncorrelated from the signals. It is assumed that the noise is spatially white. The white noise assumption can be relaxed if the correlation matrix of the noise is known but for a scalar. With these assumptions the correlation matrix of the array output is given by

$$\mathbf{R}_x = \sum_{i=1}^{q} \sum_{j=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta) p_{ij}(\theta,\theta';\psi_i,\psi_j) \times \mathbf{a}^H(\theta')\, d\theta\, d\theta' + \sigma_n^2 \mathbf{I} \qquad (2)$$

where σ_n^2 is the noise power and

$$p_{ij}(\theta,\theta';\psi_i,\psi_j) = E[s_i(\theta;\psi_i)s_j^*(\theta';\psi_j)] \qquad (3)$$

is the angular cross-correlation kernel parameterized in terms of the unknown parameter vectors ψ_i and ψ_j. The superscripts H and $*$ represent the Hermitian transposition and the complex conjugation, respectively. In this paper we assume that the sources are uncorrelated from each other which results in a simplification of $p_{ij}(\theta,\theta';\psi_i,\psi_j)$ to

$$p_{ij}(\theta,\theta';\psi_i,\psi_j) = p(\theta,\theta';\psi_i)\delta_{ij} \qquad (4)$$

where δ_{ij} is the Kronecker delta and the same parametric model is used for all sources. This constraint can be relaxed if we model correlated sources as a single source with a new parametric correlation kernel which is the addition of the angular cross-correlation kernel of the sources.

If the signal components corresponding to different directions-of-arrival (DOAs) are uncorrelated, the angular cross-correlation kernel can be further simplified to

$$p(\theta,\theta';\psi_i) = p(\theta;\psi_i)\delta(\theta-\theta') \qquad (5)$$

where $\delta(\theta)$ is the Dirac delta function. We refer to such a source as incoherently distributed (ID). This model can be applied to scattering media.

In some situations the signal components corresponding to different DOAs might be completely correlated with each other. Such a signal is called coherently distributed (CD) and has a separable kernel given by

$$p(\theta, \theta'; \psi_i) = \eta_i g_i(\theta; \psi_i) g_i^*(\theta'; \psi_i) \tag{6}$$

where η_i is a scalar representing the power of the i-th source observed at the reference point of the array, the superscript $*$ shows the complex conjugation, and $g_i(\theta; \psi_i)$ is a complex deterministic angular signal density defined in the interval $[\frac{-\pi}{2}\ \frac{\pi}{2}]$ and normalized according to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\theta; \psi_i) d\theta = 1. \tag{7}$$

In practice the signal components at different angles might be partially correlated. Partially correlated signals can be localized using the same method as for ID signals.

3. The array gain

Beamformers improve the array output SNR by steering a beam in the the direction of a signal. Because of the ease of implementation, these methods are practically important. However, they have relatively low resolution. A large number of sensors must be used in a conventional beamformer to achieve a high resolution. For point sources the array gain can be improved by increasing the number of sensors. Here, we show that for distributed sources the spatial correlation function of the signal is upper bounded by an exponentially decreasing function. Then, we derive the array gain and show that it is bounded and thus cannot increase linearly with the number of sensors. For the specific case of CD sources, we show that the array gain attains a maximum and then decreases exponentially as the number of sensors becomes very large.

The gain of an array of sensors is defined as the ratio of the SNR at the array output to the SNR at a single sensor [6]. Assuming that the noise is spatially white and that a conventional beamformer is used, the array gain is given by

$$G_a = \frac{\mathbf{a}^H \mathbf{R}_s \mathbf{a}}{\mathbf{a}^H \mathbf{a}} \tag{8}$$

where $\mathbf{a}$ is the location vector of the array steered towards the direction of interest and $\mathbf{R}_s$ is the correlation matrix of the array output in a noise-free environment.

3.1. The CD source case

Assume that the array output can be observed along a continuous linear aperture and denote the observation at point z by $x(z)$. For a single CD source in a noise-free environment we have

$$x(z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{j\frac{2\pi z}{\lambda}\sin\theta} \gamma g(\theta; \psi) d\theta \tag{9}$$

where γ is a zero-mean complex Gaussian random variable and $g(\theta; \psi)$ is the normalized deterministic angular signal density. Assuming that the source is uniformly distributed around θ_0, i.e.,

$$g(\theta; \psi) = \begin{cases} \frac{1}{2\Delta}, & |\theta - \theta_0| \leq \Delta, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

the observation vector can be written as

$$x(z) = \gamma \frac{1}{2\Delta} \int_{\theta_0 - \Delta}^{\theta_0 + \Delta} e^{j\frac{2\pi z}{\lambda}\sin\theta} d\theta. \tag{11}$$

For a small Δ, it is straightforward to show that

$$x(z) \approx \gamma e^{j\frac{2\pi z}{\lambda}\sin\theta_0} \mathrm{sinc}(\frac{2z}{\lambda}\Delta\cos\theta_0). \tag{12}$$

From (12) we arrive at the following result.

Property 1. *For a uniform CD source with a small extension, the spatial correlation function at the points z_1 and z_2 in a noise-free environment is bounded by*

$$|E[x(z_1)x^*(z_2)]| \leq K z_1^{-1} z_2^{-1} \tag{13}$$

where K is a positive scalar.

An example of the correlation between two points on a linear array for a uniform CD source is depicted in Fig. 1. It is assumed that the first point is the phase reference of the array. The second point varies along the array. The envelope of the correlation function exponentially decreases with the separation between the two points. Thus, as the aperture length of the array increases, the correlation between widely separated sensors decreases and the corresponding signals cannot be coherently added to increase the SNR. This suggests that the array gain does not increase linearly with the number of sensors.

For a uniform linear array with half the wavelength spacing between sensors, the component of the observation (12) at the position of the l-th sensor is

$$x_l = \gamma e^{j\pi l \sin\theta_0} \mathrm{sinc}(l\Delta\cos\theta_0). \tag{14}$$

Assuming that the power of the source is unity and $\theta_0 = 0$, the array gain is given by

$$G_a = \frac{1}{p}\Big[\sum_{l=0}^{p-1} \mathrm{sinc}(l\Delta)\Big]^2. \tag{15}$$

Note that for $\Delta = 0$ the array gain is equal to p which is the gain of a point source scenario. For $\Delta \neq 0$ and large p, the sum in (15) is approximated by $\pi/2$ which reveals that the array gain decreases with a rate of $1/p$. The array gain for a CD source as a function of the number of sensors p is illustrated in Fig. 2. The array gain has a maximum which depends on the extension width. Increasing the number of the sensors beyond the maximum point decreases the array gain. We have found that at the maximum point, the array length p_{MAX} can be approximated as

$$p_{MAX}\Delta^\circ \approx 40 \tag{16}$$

where Δ° is the extension width measured in degrees.

3.2. The ID source case

For an ID source the spatial cross-correlation function at the two observation points z and z' is given by [4]

$$E[x(z)x(z')] = \frac{1}{\Delta} e^{j\frac{2\pi}{\lambda}(z-z')\sin\theta_0} \mathrm{sinc}[\frac{2}{\lambda}(z-z')\Delta\cos\theta_0]. \tag{17}$$

From this we can easily arrive at the following result.

Property 2. *For a uniform ID source with a small extension width, the spatial correlation function decreases exponentially with the separation and is upper bounded by*

$$|E[x(z_1)x^*(z_2)]| \leq \frac{K}{z_1 - z_2} \tag{18}$$

where K is a positive scalar.

Since the spatial correlation function decreases exponentially with distance, the array gain cannot increase linearly with the number of sensors. For a uniform linear array with half the wavelength spacing between sensors, the spatial cross-correlation function between the l-th and the k-th sensors is given by

$$E[x_l x_k^*] = e^{j\pi(l-k)\sin\theta_0} \mathrm{sinc}[(l-k)\Delta\cos\theta_0]. \tag{19}$$

Assuming that $\theta_0 = 0$, the array gain is given by

$$G_a = \frac{1}{p}\Big[\sum_{l=0}^{p-1}\sum_{k=0}^{p-1} \mathrm{sinc}[(l-k)\Delta]\Big]. \tag{20}$$

Again, it is seen that for $\Delta = 0$, we obtain the same result as a point source case. With a change of variable the array gain can be represented as

$$G_a = \frac{1}{p}\left[p + 2\sum_{r=1}^{p-1}(p-r)\mathrm{sinc}(r\Delta)\right]. \qquad (21)$$

The array gain for an ID source is depicted in Fig. 3. For a fixed extension width, the maximum array gain for the ID source is higher than that for the CD source.

4. A generalization of MUSIC

Let us denote by $L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$ the Hilbert space of all complex valued square integrable functions defined over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The inner product and the norm in this space are defined by

$$\langle s_i, s_j \rangle_c = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s_i^*(\theta) s_j(\theta) d\theta \qquad (22)$$

$$\|s_i\|_c = \sqrt{\langle s_i, s_i \rangle_c} \qquad (23)$$

where the subscript c indicates the continuous nature of the waveform. According to (1), the observation vector $\mathbf{x}$ at the array output can be expressed as

$$\mathbf{x} = \sum_{i=1}^{q} \mathcal{L} s_i(.;\psi_i) + \mathbf{n} \qquad (24)$$

where $\mathcal{L}$ is a linear operator that maps $L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$ into a complex observation vector space $\mathbb{C}^p$ with dimensionality p such that

$$\mathcal{L} : L^2[-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \longrightarrow \mathbb{C}^p \qquad (25)$$

$$\mathcal{L}s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta) s(\theta) d\theta. \qquad (26)$$

The inner product and the norm in $\mathbb{C}^p$ are defined by

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle_d = \mathbf{x}_i^H \mathbf{x}_j \qquad (27)$$

$$\|\mathbf{x}_i\|_d = \sqrt{\langle \mathbf{x}_i, \mathbf{x}_i \rangle_d} \qquad (28)$$

where the subscript d indicates the discrete nature of the function.

The adjoint operator $\mathcal{L}^+ : \mathbb{C}^p \rightarrow L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies

$$\langle \mathcal{L}s, \mathbf{x} \rangle_d = \langle s, \mathcal{L}^+ \mathbf{x} \rangle_c. \qquad (29)$$

For the linear operator (26), we have

$$\langle \mathcal{L}s, \mathbf{x} \rangle_d = [\mathcal{L}s]^H \mathbf{x} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s^*(\theta) \mathbf{a}^H(\theta) d\theta \; \mathbf{x} = \langle s, \mathbf{a}^H \mathbf{x} \rangle_c. \qquad (30)$$

Thus the adjoint is given by

$$\mathcal{L}^+ \mathbf{x} = \mathbf{a}^H(\theta)\mathbf{x}. \qquad (31)$$

As a starting point, we extend the definition of the signal and noise subspaces to distributed sources. Note that the source signal $s_i(\theta;\psi_i)$ in (24) is a random signal which is also a function of the DOA θ and the parameter vector ψ_i. By the *source subspace* we mean the span of all realizations of the source signals $s_i(\theta;\psi_i)$, $i = 1, \ldots, q$ where the ψ_i's are fixed. This subspace is denoted by $\mathcal{S}$ and is defined as

$$\mathcal{S} = \mathrm{Span}\{s_i(\theta;\psi_i) : i = 1, \ldots, q, \text{ and all realizations}\}. \qquad (32)$$

The source subspace $\mathcal{S}$ is a subspace of $L^2[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of the linear operator $\mathcal{L}$ under $\mathcal{S}$ is the *signal subspace* and is represented by

$$\mathcal{R} = \{\mathcal{L}s \; : \; \text{all } s \in \mathcal{S}\}. \qquad (33)$$

The orthogonal complement of $\mathcal{R}$ is the *noise subspace* and is denoted by $\mathcal{R}^\perp$. It can be shown that the range of the adjoint operator $\mathcal{L}^+$, when the domain is restricted to the noise subspace $\mathcal{R}^\perp$, is the orthogonal complement of $\mathcal{S}$ which is represented by $\mathcal{S}^\perp$. The above concept of the signal and noise subspaces can be reconciled with the conventional definitions for the point source case [5].

We now derive a MUSIC type algorithm for distributed source parameter estimation. In [4] we have defined the concept of the effective dimension of the signal subspace. For distributed sources the signal component might extend to the whole observation space. For such signals the dimensionality of the signal subspace is p the number of sensors. The effective dimension of the signal subspace has been defined as the dimension of the subspace which contains 95 percent of the total energy of the signal. For localization of distributed sources we use the effective dimension of the signal subspace to estimate the noise eigenvectors.

Suppose that $\mathcal{R}^\perp$ has dimension $p - q_e$ where q_e is the effective dimension of the signal subspace, and we have a basis for $\mathcal{R}^\perp$, say $\mathbf{e}_1, \ldots, \mathbf{e}_{p-q_e}$, and let $\mathbf{E}_n = [\mathbf{e}_1, \ldots, \mathbf{e}_{p-q_e}]$. Since $\mathbf{e}_i$'s are in $\mathcal{R}^\perp$, their transformation under $\mathcal{L}^+$ will be in $\mathcal{S}^\perp$, i.e.

$$\mathcal{L}^+ \mathbf{e}_i = \mathbf{a}^H(\theta)\mathbf{e}_i \in \mathcal{S}^\perp, \qquad i = 1, \ldots, p - q_e. \qquad (34)$$

Thus for all $s(\theta) \in \mathcal{S}$ we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n s(\theta) d\theta = 0. \qquad (35)$$

In (32) the source subspace $\mathcal{S}$ was defined as the span of the functions $s_i(\theta;\psi_i)$. Hence (35) can be written as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n s_i(\theta;\psi_i) d\theta = 0 \qquad (36)$$

for all realizations of $s_i(\theta;\psi_i)$, and for $i = 1, \ldots, q$. Since $s_i(\theta;\psi_i)$ is a random function, this is equivalent to

$$E\left[\left\|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n s_i(\theta;\psi_i) d\theta\right\|^2\right] = 0 \qquad (37)$$

for $i = 1, \ldots, q$. Using (3) and (4) this equation can be expressed as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n p(\theta, \theta'; \psi_i)\mathbf{E}_n^H \mathbf{a}(\theta') d\theta d\theta' = 0 \qquad (38)$$

for $i = 1 \ldots, q$. We propose that the parameter vector be estimated by locating the peaks of

$$\hat{\psi} = \arg\max_{\psi} \frac{1}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}^H(\theta)\mathbf{E}_n p(\theta, \theta'; \psi)\mathbf{E}_n^H \mathbf{a}(\theta') d\theta d\theta'}. \qquad (39)$$

We call this method the distributed signal parameter estimator (DSPE). The spectrum of the DSPE algorithm is searched in an m-dimensional space for q local maxima where m is the dimension of the parameter vector ψ. The maximum points correspond to the estimates of the parameter vector.

5. Some simulation results

In this section we show how the DSPE technique can localize distributed sources. Here we only consider a single ID source with a uniform correlation kernel. The source is arriving from 10 degrees with an extension of 2 degrees at a uniformly spaced array of 20 sensors. Fig. 4 shows the spectrum of the DSPE localizer. The prominent peak in this spectrum corresponds to the distributed source. As it is seen the method successfully localizes the source. In this simulation we have used 2 eigenvalues for the signal subspace. The spectrum of the corresponding MUSIC algorithm can be read in the figure by putting $\Delta = 0$. For a complete set of simulations see [5].

6. Summary

This paper is a continuation of a previous work for distributed source localization [4]. It has been assumed that the spatial correlation kernel of the signals is chosen from a parametric class of functions. We have generalized the MUSIC algorithm so that it can localize distributed sources. The new technique has been called a distributed source parameter estimator (DSPE). The new algorithm can localize the extended sources by minimizing a scalar product between the back-transform of the noise eigenvectors and a basis for the source subspace. Furthermore, we have studied the performance of a conventional beamformer by finding the array gain for a uniform linear array. We have shown that the gain of a conventional beamformer is bounded with increasing the number of sensors.

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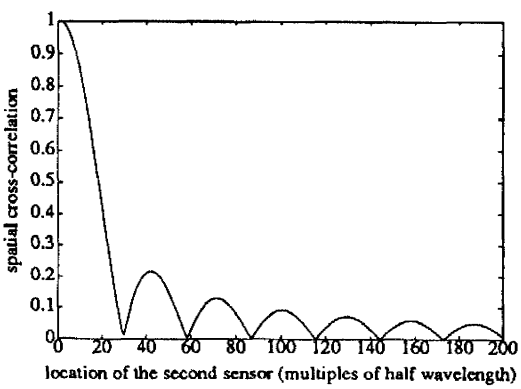

Fig. 1 Spatial cross-correlation for a uniform CD source. (The first sensor is positioned at the phase reference point.)

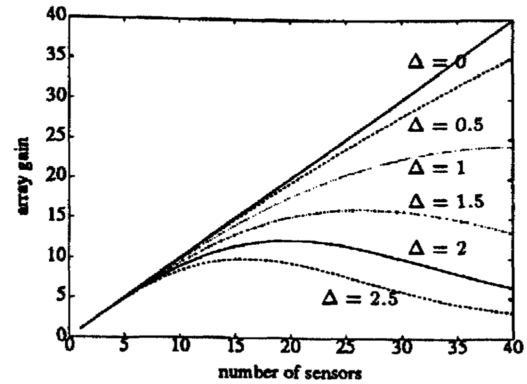

Fig. 2 Array gain for a uniform CD source for different distribution widths, Δ, in degrees.

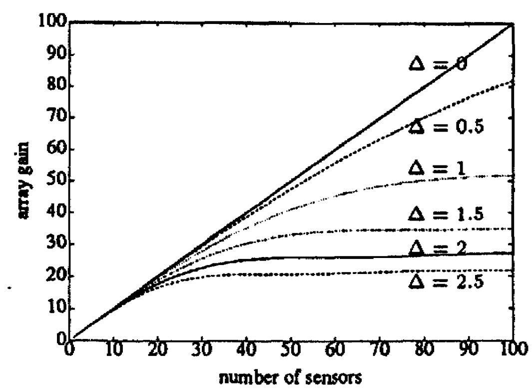

Fig. 3 Array gain for a uniform ID source for different distribution widths, Δ, in degrees.

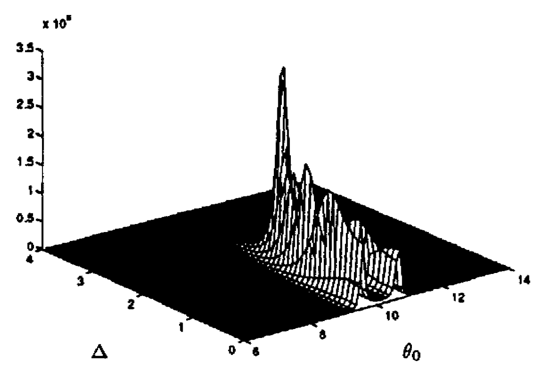

Fig. 4 The spectrum of the DSPE algorithm. The angles are in degrees.