ON THE ISSUE OF RANK ESTIMATION IN SUBSPACE TRACKING: THE NA-CSVD SOLUTION *

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ABSTRACT

The issue of rank estimation in subspace tracking algorithms is adressed. In a recent paper, we proposed a subspace tracking algorithm, the NA-CSVD. We now extend the performance of NA-CSVD to rank tracking by coupling it with a recently proposed rank tracking technique. The paper includes an overview of typical rank+subspace tracking algorithms which, along with our proposed algorithm, are tested in various simulation scenarios. The new algorithm tracks efficiently the rank and the signal subspace.

1 INTRODUCTION

Consider the exponentially weighted data matrix defined recursively as

$$A(k) = \begin{bmatrix} \sqrt{\lambda} A(k-1) \\ \mathbf{x}(k)^H \end{bmatrix} \tag{1}$$

where $0 < \lambda < 1$ is a forgetting factor, k is the time index and $\mathbf{x}(k) \in \mathbb{C}^{N \times 1}$ is the incoming measurement vector. The singular value decomposition (SVD) of $A(k)$ is $A(k) = U(k)\Sigma(k)V(k)^H$, where $V(k)$ is a $N \times N$ unitary matrix, $U(k)$ is a $k \times N$ matrix with orthonormal columns which usually needs not to be computed, and $\Sigma(k) = \mathrm{diag}(\sigma_1, \cdots, \sigma_N)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$. This decomposition enables the real-time tracking of various parameters, e.g. direction of arrival (DOA) of narrowband plane-waves impinging on an array of sensors, or frequencies of sinusoids in additive receiver noise.

Let r represent the number of physical data sources which generate the measurement $\mathbf{x}(k)$. Note that in absence of noise, r is equal to the rank of the singular value matrix $\Sigma(k)$, i.e. the number of non-zero diagonal entries of $\Sigma(k)$. Accordingly, the right singular vectors, i.e. the columns of $V(k)$, form the basis vectors of two orthogonal subspaces, $V = [V_S|V_N] = [\mathbf{v}_1, \cdots, \mathbf{v}_r|\mathbf{v}_{r+1}, \cdots, \mathbf{v}_N]$, namely the signal subspace (spanned by V_S) and the noise subspace (spanned by V_N).

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The quality of the parameter estimation depends directly on the quality of the subspace tracking, since the singular vectors $V(k)$ are further used by algorithms like MUSIC or ESPRIT to extract the desired parameters. In practical situations, the number of sources r may not be constant; so, the parameters estimate depends also on the estimation of r. Indeed, if r is over-estimated, the complexity of the tracking algorithm increases, due to the extra processing of unwanted random DOA estimates; and if r is under-estimated, the reduction of the size of V_S generally causes a bias in the parameters estimate. So, a subspace tracking algorithm is really complete only when it tracks both the subspaces and the rank r.

In a recent paper [1], we proposed an $O(Nr)$-complexity QR Jacobi-type subspace tracking algorithm, the Noise Average Cross-terms Singular Value Decomposition (NA-CSVD). We now extend the performance of NA-CSVD to rank tracking by coupling it with a technique recently intoduced by Kavcic *et al.* [2]. On the issue of rank+subspace tracking methods, this paper puts more emphasis on the rank tracking capabilities of the algorithms. We first provide an overview of typical rank+subspace tracking algorithms; then, these algorithms are tested in various simulation scenarios. Simulation experiments show that our improved NA-CSVD tracks efficiently the rank and the signal subspace.

The paper is organized as follows: Section 2 presents an overview of rank+subspace tracking methods, Section 3 presents our new algorithm, simulations results are in Section 4 and final remarks are provided in Section 5.

2 OVERVIEW OF RANK+SUBSPACE TRACKING TECHNIQUES

In this paper, we are interested in the case of non-coherent sources only. In practical situations, the presence of noise in the data causes the rank of the singular value matrix $\Sigma(k)$ to be N, so that one generally tracks the effective rank of $\Sigma(k)$. Therefore, the number of sources can be identified by the knee-point where the singular value spectrum falls to the noise floor. The effective rank of $\Sigma(k)$ is particularly hard to identify when the signal-to-noise ratio (SNR) is low. Various techniques have been developped to determine r even in low SNR scenarios: threshold-based, MDL-type, probabilistic ... The following is a broad but not exhaustive overview of rank+subspace tracking algorithms.

A complex rank+subspace tracking algorithm consists in performing an exact $O(N^3)$ SVD followed by the Minimum Description Length (MDL) method [3] to identify the rank. MDL identifies r by minimizing a criteria parametrized by the approximate singular values; therefore, its performance strongly depends on the quality of the tracking of the singular values.

Like many rank+subspace tracking algorithms, Yang's PASTd-MDL [4] is a MDL-type algorithm. It processes separately the subspace and the rank tracking. Indeed, if r is the estimated number of sources, the PASTd algorithm tracks the $r+1$ largest singular values $\{\sigma_1, \cdots, \sigma_{r+1}\}$ and the corresponding singular vectors, plus an average noise singular value $\bar{\sigma}_N$; MDL is used to estimate the rank at each iteration by using the singular value spectrum $\{\sigma_1, \cdots, \sigma_{r+1}, \bar{\sigma}_N, \cdots, \bar{\sigma}_N\}$.

Subspace average 4 (SA4) [5] is another MDL-type algorithm. Let $R(k) = U(k)D(k)U(k)^H$ be the EVD of the sample correlation matrix $R(k)$. SA4 tracks the basis vectors and the average eigenvalues of four sphericalized subspaces, as in $U = [U_1 U_2 U_3 U_4] = [\mathbf{u}_1, \cdots, \mathbf{u}_{r-1} | \mathbf{u}_r | \mathbf{u}_{r+1} | \mathbf{u}_{r+1}, \cdots, \mathbf{u}_N]$. The four average eigenvalues $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ and $\bar{\lambda}_4$ are then used in a modified MDL procedure (SA4-MDL) to estimate r.

The Invariant Subspace Updating (ISU) algorithm [6] tracks the singular vectors by resolving a quadratic matrix equation. Once the subspace basis vectors are estimated at time $k-1$, the parameters, e.g. DOAs, are extracted and the corresponding steering vectors are gathered in a matrix B. If the number of sources has increased between $k-1$ and k, the measurement $\mathbf{x}(k)$ contains a component which can not be expressed as a linear combination of the steering vectors. This is detected by observing the projection of $\mathbf{x}(k)$ on B. The detection of a reduction of r is performed by identifying which steering vector should be removed from B in order to increase the norm of the projection of $\mathbf{x}(k)$ on B.

Stewart's rank revealing URV [7] approximates the SVD with the following decomposition

$$A(k) = U \begin{bmatrix} S & F \\ \mathbf{0} & G \end{bmatrix} V^H, \tag{2}$$

where $V \in \mathbb{C}^{N \times N}$, $U \in \mathbb{C}^{k \times N}$, $S \in \mathbb{C}^{r \times r}$ and $G \in \mathbb{C}^{(N-r)\times(N-r)}$; S and G are upper-triangular. The smallest singular value of S approximates the r^{th} singular value of $A(k)$, and the frobenius norm of $[F^H G^H]$ is compared to two adaptive thresholds to track the variations of r. The updating of an URV decomposition is essentially made by Givens rotations which are applied in order not to destroy that specific structure. The cross-product of the URV with its transpose provides a correlation-based rank-revealing structure, the CRV decomposition [8].

The Rank Adaptive Fast Subspace Tracking (RAFST) [9] tracks a sphericalized URV decomposition. The estimation of r is done by assuming that the *pdf* of the sum of the noise singular values is

$$\frac{2}{\sigma_{noise}^2} \sum_{i=r+1}^{N} \sigma_i^2 \overset{d}{=} \chi^2_{2(M-r)(N-r)}, \tag{3}$$

where $M = (1-\lambda^k)/(1-\lambda)$ represents the effective window length, σ_{noise}^2 is the noise power, and $\chi^2_{2(M-r)(N-r)}$ is a central chi-squared random variable with $2(M-r)(N-r)$ degrees of freedom. Here, "$\overset{d}{=}$" means equality in terms of *pdf*. Once the *pdf* is known, the probability of false alarm (wrong choice of r) is obtained. r is chosen in order not to get over a specified false alarm rate.

3 THE NA-CSVD/RSST SOLUTION

In [1], we proposed a QR Jacobi-type subspace tracking algorithm: the Noise Average Cross-terms SVD (NA-CSVD). Using an approximate decomposition in which $\Sigma(k)$ is almost diagonal but not specifically upper-triangular, NA-CSVD tracks the signal subspace plus an extra average noise singular value. It uses exclusively Givens rotations to zero the cross-terms entries of $\Sigma(k)$ and thus reduces the interaction between the signal and the noise subspace. The tracking performance of NA-CSVD has already been shown [1]. NA-CSVD was formerly formulated for subspace tracking only, assuming that the rank remains constant.

Kavcic *et. al* [2] recently proposed an adaptive rank estimation technique for spherical subspace trackers which we denote as RSST. It is a threshold-based method which enables the tracking of the number of sources with a parallel procedure. If r is the estimated number of sources at the previous iteration, RSST suggests to track the $r+1$ largest singular values $\sigma_1 \geq \cdots \geq \sigma_{r+1}$ as well as the average of the $N-r-1$ smallest noise singular values $\bar{\sigma} = \frac{1}{N-r-1}\sum_{i=r+2}^{N} \sigma_i \leq \sigma_{r+1}$. At each time iteration, r is estimated by comparing σ_r and σ_{r+1} with an optimal threshold computed so as to maintain a low error probability.

In this paper, we merge NA-CSVD and RSST. To do this, we retain only the QR step and the refinement step of NA-CSVD which we also modify so as to track an extra singular value (σ_{r+1}). The new algorithm, NA-CSVD/RSST, is presented in Table 1. Compared to MDL-based algorithms, NA-CSVD/RSST has the advantage to have a tunable threshold. Indeed, if the sources power are not constant, one must be able to decide for the minimum power over which a source is assumed to be present or not. NA-CSVD/RSST (and ISU as well) offers such a flexibility. NA-CSVD/RSST

Table 1: NA-CSVD/RSST algorithm

Initialization
r, $\{\sigma_1, \cdots, \sigma_{r+1}\}$,$\bar{\sigma}_N$,$V = [\mathbf{v}_1, \cdots, \mathbf{v}_{r+1}]$
$T = \phi\bar{\sigma}_N$

Loop
for $k = 1, \cdots, \infty$
- $\mathbf{v}_N = [I - VV^H]\mathbf{x}(k)$
- $\mathbf{v}_N \leftarrow \mathbf{v}_N / \|\mathbf{v}_N\|$
- updating with the QR step and the refinement step of NA-CSVD
  $[V, \mathbf{v}_N] \overset{update}{\rightarrow} [V, \mathbf{v}'_N]$
  $\{\sigma_1, \cdots, \sigma_{r+1}, \bar{\sigma}_N\} \overset{update}{\rightarrow} \{\sigma_1, \cdots, \sigma_{r+1}, \bar{\sigma}'_N\}$
- $\bar{\sigma}_N = \frac{\bar{\sigma}'_N + (N-r+2)\sqrt{\lambda}\bar{\sigma}_N}{N-r-1}$
- Dimension updating
  if $\sigma_r < T$
    $\bar{\sigma}_N = \frac{\sigma_{r+1} + (N-r-1)\bar{\sigma}_N}{N-r}$
    $V \leftarrow [\mathbf{v}_1, \cdots, \mathbf{v}_r]$
    $r \leftarrow r - 1$
  elseif $\sigma_{r+1} > T$
    $V \leftarrow [V, \mathbf{v}'_N]$
    $\sigma_{r+2} = \bar{\sigma}_N$
    $r \leftarrow r + 1$
  else
    r is unchanged
  end
- $T = \phi\bar{\sigma}_N$
end

tracks efficiently both the signal subspace and the rank, as shown in the next Section. The complexity of NA-CSVD/RSST is $O(Nr)$.

4 COMPUTATION EXPERIMENTS

In all our experiments, we deal with the problem of estimating the DOAs of plane waves impinging on an array of $N = 16$ sensors. This is an application in which the tracking of the exact number of sources is important.

We implemented the following rank+subspace tracking algorithms: Exact SVD/MDL, PASTd-MDL, SA4, ISU, rank-revealing URV, RAFST and NA-CSVD/RSST, as described in Sections 2 and 3. All algorithms have the same initial conditions and in each experiment, the internal parameters of each algorithm (forgetting factor, thresholds, *a priori* noise power ...) are individually tuned to obtain the best *rank tracking* performance.

In the first experiment, we estimate the delays of detection in sudden changes of the number of sources. The simulation scenario consists in adding new sources and later removing them. We compute the average delays to detect the presence of new sources (δ_{up}) and the loss of sources (δ_{down}). The average detection delays are listed in Table 2, and the DOAs and rank estimates are in Fig 1. A general comment is that we generally have $\delta_{up} < \delta_{down}$. NA-CSVD/RSST has the shortest detection delays: $\delta_{up} = 0$ and $\delta_{down} = 8$. As observed experimentally, δ_{down} depends not only on the forgetting factor, but also on the length of time during which the sources have been present in the data set: the older the source, the longer the detection delay.

Table 2: Delays of detection of sudden changes of r

Algorithm	δ_{up}	δ_{down}
NA-CSVD/RSST	0	8
Exact-MDL	0	41
URV	0.67	73
SA4-MDL	2	20
RAFST	2	42
ISU	4.33	38
PASTd-MDL	6	241

The next simulation tests the case of crossing sources. It is well-known that when two sources get closer, the effective rank of $\Sigma(k)$ drops by one. We would like here to estimate the period of time during which the number of sources remain under-estimated. Two sources at $\theta_1 = (10 + 0.01k)$ and $\theta_2 = (20 - 0.01k)$ cross at $k = 500$. The gap of the "turbulence region" for each algorithm are listed in Table 3, and the DOAs and rank estimates are in Fig 2.

Table 3: gap of the "turbulence region" in the case of crossing sources

Algorithm	crossing (gap)
SA4-MDL	392
NA-CSVD/RSST	101
RAFST	260
PASTd-MDL	132
Exact-MDL	132
ISU	0
URV	—

The main advantage of ISU is to be able to maintain the number of sources to $r = 2$ in the crossing region, even though one of the two estimated DOAs acts randomly. This is due to the fact that ISU does not test directly the presence of sources, but evaluates the relative closeness between the estimated angles and their previous estimation. In the case of URV, the many glitches in the estimation of r are due to the difficulty to tune the thresholds. An acceptable tuning of the thresholds is obtained only after numerous guesses. As observed experimentally, the wrong tuning of URV thresholds either causes rank revealing URV to be unstable, or not to update the subspaces frequently. Finally, An appropriate tuning of λ and ϕ would enable NA-CSVD/RSST to maintain $r = 2$ in the crossing region, but at the expense of a decrease in the quality of the subspace tracking

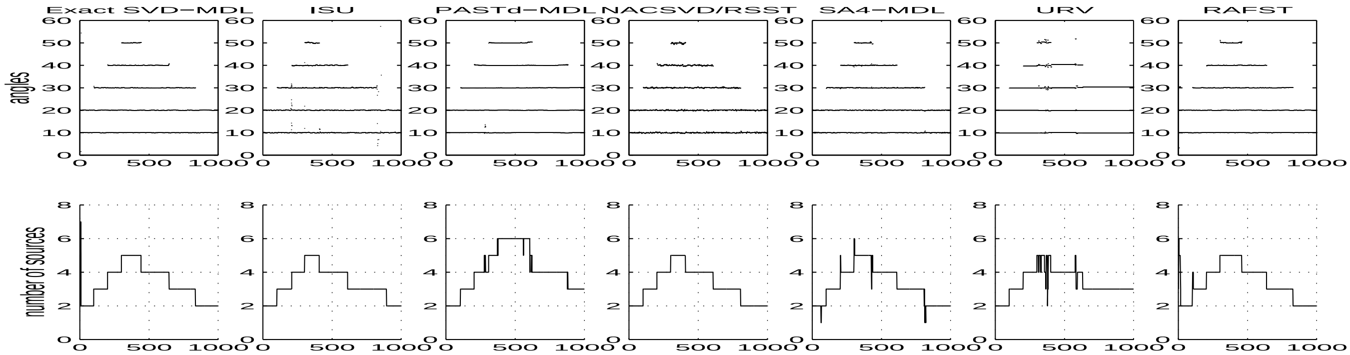

Figure 1: Detection delays of sudden rank changes.

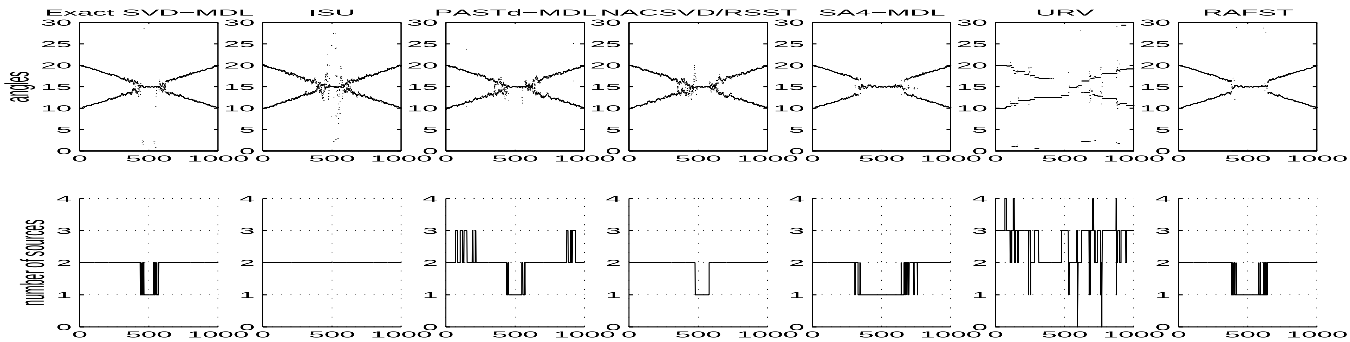

Figure 2: Rank+subspace tracking of crossing sources

performance. Nevertheless, NA-CSVD/RSST provides a shorter crossing region than Exact-SVD/MDL.

5 CONCLUSION

Separating the rank tracking performance from the subspace tracking performance of an algorithm is not an easy task. The rank tracking performance also depends on the tuning of various parameters: forgetting factor (all algorithms), thresholds (ISU,URV,NA-CSVD/RSST), a priori estimation of the noise power (URV, RAFST). RAFST provides a good subspace+rank tracking, but at the expense of a higher complexity due to the computation of a complex distribution function. PASTd-MDL usually over-estimates the rank if it is not well initialized, while ISU is unable to deal with the case of $r = 0$. Also, like the URV, ISU suffers from the fact that two different thresholds have to be tuned. The stability of URV rank estimate is hardly maintained. The tracking performance of SA4 is altered by the average of eigenvalues particularly in the case of fast moving sources. Finally, NA-CSVD/RSST performs efficiently in all the simulated scenarios, requiring no *a priori* knowledge of the noise power and having only one easy-to-tune threshold.

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