Sufficient Conditions for the Unique Localization of Distributed Sources

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ABSTRACT

The array output for a distributed source can be approximated by the superposition of the array response to a large number of closely spaced point sources. In the limit, a distributed source corresponds to an infinite number of point sources. In this approximation, the number of free parameters increases with the number of point sources. In this paper, we show that if the point sources (approximation of a distributed source) are related through some parametric constraints, then for any observation at the array output, almost surely, there is a unique solution for the localization problem, provided that the dimensionality of the parameter space satisfies a certain bound. We show this for both coherently and incoherently distributed sources.

1 Introduction

Recent literature in array processing shows a growing interest in detection and localization of distributed sources [4] [7] [5] [2]. Distributed source modeling is invoked in many practical situations. For instance, the lateral variation of sound speed in water may cause energy distribution over an angular volume. In an undersea echo beam sounder, the scattered signal from the lower layers is modeled as a distributed source [4]. Other examples are acoustic sources in a reverberant room, tropospheric or ionospheric propagation of radio waves, reflection of low radio link signal from ground, and so on. A distributed source can be approximated by a large number of closely spaced point sources [4]. The approximation error decreases by increasing the number of point sources and decreasing their spacing. In the limit, a distributed source corresponds to an infinite number of point sources. In this approximation, the number of free parameters increases with increasing the number of point sources. If a classical point source localization method, such as MUSIC, is applied to localize the point sources, a unique solution may not be obtained due to a limited number of sensors. In fact, for a unique solution, the number of point sources must be smaller than the number of sensors [9]. This is an inherent ambiguity of the distributed source modeling. Moreover, determining spatial extension using the point source location estimate is not clear.

In [7], we present a parametric method for localization of distributed sources in which the source subspace, the signal subspace, and the noise subspace are generalized and a two-dimensional MUSIC-type spatial spectrum is defined. The parameters of the sources are estimated by locating the prominant peaks of this spectrum. There, we assume that the number of parameters is known. However, we do not discuss when the solution to the localization problem is unique. We simply assume that the number of parameters is small enough so that the estimator provides a unique solution.

In the present work, we derive the sufficient conditions for a unique localization of spatially distributed sources. In [7], we propose that a MUSIC-type algorithm might be used for distributed source localization if the angular kernel of the distributed source belongs to a parametric class of functions. Here, we derive bounds on the number of parameters required to represent a distributed source. This bound can be used to select a proper class for the angular density of a distributed source. We use the concept of the *topological dimension* of a set which is defined as the number of free (real) parameters required to describe all the elements of that set [3].

2 Problem Formulation

Consider an array of p sensors exposed to q spatially distributed sources. The output of ith sensor is given by

$$x_i = \sum_{j=1}^{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a_i(\theta) s(\theta, \psi_j) d\theta + n_i \qquad (1)$$

where $a_i(\theta)$ is the response of the ith sensor to a unit energy source at direction θ, $s(\theta, \psi_j)$ is the angular density of the jth source, ψ_j is the jth source location parameter vector, and n_i is the additive zero-mean noise at the ith sensor.
For uncorrelated sources the array covariance matrix is

$$\mathbf{R}_{xx} = \sum_{j=1}^{q} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \mathbf{a}(\theta) \rho(\theta, \theta'; \psi_j) \mathbf{a}^H(\theta') d\theta d\theta' + \sigma_n^2 \mathbf{I} \qquad (2)$$

where

$$\rho(\theta, \theta'; \psi_j) \triangleq \mathrm{E}\{s(\theta, \psi_j) s(\theta', \psi_j)\} \qquad (3)$$

is the *angular cross correlation* of source m.
A source is called *coherently distributed* (CD), if $s(\theta, \psi_j)$ is a random multiple of a deterministic function $g(\theta, \psi_j)$ [7], i.e.

$$s(\theta, \psi_j) = \gamma g(\theta, \psi_j) \qquad (4)$$

where γ is a random variable and $g(\theta, \psi_j)$ is called the *deterministic angular signal density.* Equation (4) indicates that the components of received signal from a CD source at different angles are the delayed and scaled replicas of each other.
If different rays of signal which arrive at the array are uncorrelated with each other, the source is called an *incoherently distributed* (ID) source. For an ID source, we have

$$\mathrm{E}\{s(\theta, \psi_j) s^*(\theta', \psi_j)\} = p(\theta, \psi_j) \delta(\theta - \theta') \qquad (5)$$

where $p(\theta, \psi_j)$ is the *angular power density* of the source.
We will discuss the uniqueness problem separately for the CD and ID source models. For each case, a *legitimate set* is found which contains all the signals that are chosen from the paramteric class of the angular correlation kernels. Every element in the legitimate set can be a candidate for the localization problem. The *ambiguity set* is a subset of the legitimate set that contains all the signals that can generate nonunique solutions to the localization problem. The objective here is to find the conditions under which the ambiguity set has a smaller dimension than the legitimate set. The approach is similar to the one proposed in [6].

2.1 CD sources

Let the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ be quantized into a grid of $\tilde{q}$ points. It is assumed that a distributed source with the angular signal density $g(\theta; \psi)$, where ψ is an m-dimensional parameter vector, is discretized so that it can take values on the quantized grid. Initially, we consider a single source scenario.
The output of an array of p sensors in a noise-free environment for N snapshots can be represented by

$$\mathbf{X} = \mathbf{A}\mathbf{S}(\psi) \qquad (6)$$

where $\mathbf{A}$ is the $p \times \tilde{q}$ location matrix of the array, $\mathbf{S}(\psi)$ is the $\tilde{q} \times N$ source signal matrix, and $\mathbf{X}$ is the $p \times N$ observation matrix. The signal matrix can be expressed as $\mathbf{S}(\psi) = [\mathbf{s}_1(\psi)\, \mathbf{S}_2(\psi)]$ where $\mathbf{s}_1(\psi)$ is a $\tilde{q} \times 1$ vector and $\mathbf{S}_2(\psi)$ is a $\tilde{q} \times (N-1)$ matrix. Similar to [8], we can show that it suffices to solve the uniqueness problem only for

$$\mathbf{x}_1 = \mathbf{A}\mathbf{s}_1(\psi). \qquad (7)$$

The source signal matrix $\mathbf{s}_1(\psi)$ can be represented by

$$\mathbf{s}_1(\psi) = \gamma_1 \mathbf{g}(\psi) \qquad (8)$$

where $\mathbf{g}(\psi)$ is a $\tilde{q} \times 1$ vector with the ith component equal to the value of $g(\theta; \psi)$ computed at the location of the ith quantized DOA, and γ_1 is the square root of the power. The vectors that satisfy (8) for all ψ, generate a set which is called the *legitimate set* and is denoted by $\mathcal{G}$. Since it is assumed that there is a one-to-one relationship between $\mathbf{g}(\psi)$ and ψ, we will need $m+2$ real parameters to determine $\mathbf{s}_1(\psi)$. Thus, the dimensionality of $\mathcal{G}$ is equal to $m+2$.

A nonunique solution for the localization problem can be found if

$$\mathbf{x}_1 = \mathbf{A}\mathbf{s}_1(\psi) = \mathbf{A}\mathbf{s}'(\psi') \tag{9}$$

or

$$\mathbf{A}[\gamma_1 \mathbf{g}(\psi) - \gamma_1' \mathbf{g}(\psi')] = \mathbf{0}. \tag{10}$$

The legitimate vectors that satisfy this equality for any ψ and ψ', form the *ambiguity set* which is represented by $\mathcal{D}$. To represent each vector in the form of $(\gamma_1 \mathbf{g}(\psi) - \gamma_1' \mathbf{g}(\psi'))$, we need to determine $2(m+2)$ real parameters. However, (10) shows that for the vectors in the ambiguity set $2p$ constraints should be applied to their parameters. Thus, the total number of parameters that can be freely set to satisfy (10) is equal to $2(m+2) - 2p$. This is the dimensionality of $\mathcal{D}$.

Since $\mathbf{s}_1(\psi)$ is a random vector, a unqiue solution for the localization problem can be found, *almost surely*, if

$$\dim\{\mathcal{D}\} < \dim\{\mathcal{G}\} \tag{11}$$

where $\dim\{.\}$ is the dimension operator. This criterion is equal to

$$m < 2p - 2. \tag{12}$$

Note that (12) is independent of $\tilde{q}$ the number of quantized sources. Thus, an infinite number of point sources (a distributed source) are localizable if they are related through some parametric constraints.

A multi-source case can be treated similarly with an angular signal density equal to the addition of the angular signal density of the single sources. For a multi-source scenario, the dimensionalities of $\mathcal{G}$ and $\mathcal{D}$ are equal to $q(m+2)$ and $2q(m+2) - 2p$, respectively, where q is the number of CD sources. Thus, the uniqueness constraint implies that

$$q < \frac{2p}{m+2}. \tag{13}$$

Note that for the point source case, $m = 1$ and we have $q < 2p/3$, which is the well known sufficient condition for unique localization of coherent point sources [9].

2.2 ID sources

The true correlation matrix is the limit of the sample correlation matrix when the observation time tends to infinity. The sample correlation matrix is a Hermitian random matrix with jointly Wishart distributed elements [1]. In the sequel, we find the dimensionality of the true correlation matrix, keeping in mind that it is the limit of a random matrix. The error between the true and the sample correlation matrices can be arbitrarily reduced by increasing the observation time. We consider a subset of the sample correlation matrices which generate ambiguous solutions for the localization problem. Then, we show that this set converges into a set that has a smaller dimension than the set of all possible correlation matrices.

Such as for the CD case, assume that the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is uniformly sampled into a grid of $\tilde{q}$ points. A distributed source with the angular correlation kernel $\rho(\theta, \theta'; \psi)$, where ψ is an m-dimensional parameter vector, takes its values on this grid in a noise-free environment. Again, initially we assume a single source in a noise-free environment. The correlation matrix of the array output is shown as

$$\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H \tag{14}$$

where $\mathbf{A}$ is the $p \times \tilde{q}$ dimensional location matrix of the array and $\mathbf{R}_s$ is the $\tilde{q} \times \tilde{q}$ correlation matrix of the point sources. Since the point sources are the samples of the distributed source, their cross-correlation matrix satisfies

$$\mathbf{R}_s = \mathbf{P}(\psi) \qquad \text{for some} \quad \psi \in \Psi \tag{15}$$

where Ψ is the parameter set and the components of $\mathbf{P}(\psi)$ are the values of the angular correlation kernel of the distributed source, $\rho(\theta, \theta'; \psi)$, computed on the grid. All the correlation matrices $\mathbf{R}_x$ that satisfy (14) with the constraint (15) form the *legitimate set* $\mathcal{G}$. Since $\mathbf{P}(\psi)$ is a function of $m+1$ free (real) parameters, the topological dimension of the legitimate set is $m+1$.
Let us define

$$\mathbf{F} = \mathbf{A}\mathbf{P}(\psi)\mathbf{A}^H - \mathbf{A}\mathbf{P}(\psi')\mathbf{A}^H \qquad (16)$$

for some ψ and ψ'. The set of all matrices which can be represenetd by (16) has dimensionality $2(m+1)$. A nonunique solution for the DOA estimation problem can be found if

$$\mathbf{F} = \mathbf{0}. \qquad (17)$$

Note that for ID sources, $\mathbf{P}(\psi)$ is a diogonal matrix. Thus, (17) provides p^2 complex constraints with only p of them being independent. The number of parameters that can be chosen freely to satisfy (17) is equal to $2(m+1) - p$. Define the *ambiguity set* as

$$\mathcal{D} = \{\mathbf{R}_x \,|\, \mathbf{A}\mathbf{P}(\psi)\mathbf{A}^H = \mathbf{A}\mathbf{P}(\psi')\mathbf{A}^H\}. \qquad (18)$$

The elements of $\mathcal{D}$ produce nonunique solutions for the DOA estimator. The topological dimension of $\mathcal{D}$ is equal to $2(m+1) - p$. A unique solution can be, *almost surely*, found for the localization problem if

$$2(m+1) - p < m+1 \quad \text{or} \quad \mathrm{m} < \mathrm{p} - 1. \qquad (19)$$

This suggests that distributed sources are unqiuely resolvable if they are chosen from a parametric class of angular correlation kernels with the dimension of the parameter vector smaller than the number of sensors.
A multi-source case can be treated similarly with an angular correlation kernel equal to the addition of the angular correlation kernels of the single sources. In a multi-source case with q uncorrelated sources, the dimensionality of $\mathcal{G}$ and $\mathcal{D}$ are equal to $q(m+1)$ and $2q(m+1) - p$, respectively. The uniqueness criterion is then given by

$$q(m+1) < p. \qquad (20)$$

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