Multicarrier Modulation Using Perfect Reconstruction DFT Filter Bank Transceivers

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***Abstract*— Multicarrier modulation techniques such as discrete multitone (DMT) have been used for reliable digital data transmission in various applications, i.e. in digital subscriber line (DSL) and Wi-Fi products. It is a well known fact that the IFFT/FFT pair in DMT provides poor spectral selectivity due to the insufficient sidelobe attenuation of the subchannel frequency response. To remedy this problem, we propose to use perfect reconstruction DFT filter banks (achieving perfect inter-symbol interference cancellation) instead of DMT. In this case, spectral selectivity can be greatly improved. Equalization is performed via a zero-padding technique combined with QR factorization of the channel matrix. Experiments indicate that the proposed transceiver exhibits a significant improvement in terms of achievable bit rates in an environment where narrow band noise dominates.**

I. Introduction

Multicarrier modulation (MCM) is currently used for data transmission in many applications such as digital subscriber line (DSL) and Wi-Fi products for Internet access. In essence, MCM divides the available channel bandwidth into M subchannels through the use of M narrow band subcarriers, each of them transmitting different portions of the input bit stream. The MCM scheme deployed in the above-mentioned applications is referred to as discrete multitone (DMT) modulation[1]. In DMT, an inverse fast Fourier transform (IFFT) is employed to modulate a group of QAM symbols, possibly from different constellations. Due to the IFFT, DMT suffers from poor subchannel spectral selectivity since the sidelobes of the subchannel frequency response are poorly attenuated. A narrow band noise (such as HAM radio interference) can thus cause notable damage, since adjacent subchannels of the affected subcarrier are more likely to pick up this interference as well. Poor spectral selectivity can also pose a serious problem in applications where spectral power allocation requirements must be met precisely.

To improve spectral selectivity, more efficient alternatives to the IFFT in DMT modulation should be considered. In this respect, the use of filter banks appears particularly promising [1]–[4]. While there exist many types of filter banks, contrary to the work in [4], we restrict our attention to a class of computationally efficient filter banks, the DFT-modulated filter bank [1]–[3]. In [1], [2], the resulting DFT filter bank transceiver does not achieve complete inter-symbol interference (ISI) cancellation (or, equivalently, they are not characterized by the perfect reconstruction (PR) property). To compensate, decision-feedback equalizers (DFE) must be employed. These equalizers entail a level of post-processing complexity normally not found in DMT, where only simple one-tap frequency-domain equalizers and a cyclic prefix are required to mitigate ISI. More recently, Phoong *et al.* have proposed a design method for DFT filter bank transceivers based on ISI minimization [3]. However, ISI minimization does not necessarily yield ISI-free systems, and the PR property may not hold.

The main objective of this paper is to investigate the use of PR DFT filter banks in transceivers for DSL-like systems. The aim is to provide better spectral selectivity to combat impairments such as narrow band noise. As explained in this paper, by using the PR criterion, DFE are unnecessary, and a simple equalization scheme — analogous to that in DMT — can be employed. In contrast with [3], the design method introduced in this paper guarantees that the PR property is maintained. This ensures complete ISI cancellation as in DMT. Results shown in this paper indicate that the relative improvement in terms of achievable bit rates is very significant when a narrow band noise contaminates the system. However, improving spectral selectivity implies a penalty in bandwidth utilization. In cases where noise is wide band and weak, DMT might be preferable.

This paper is organized as follows. In Section II, we present a method to design perfect reconstruction DFT filter banks, assuming an ideal channel. This assumption is lifted in Section III, and an equalization scheme based on zero-padding and the QR factorization is described. In Section IV, experimental results are shown, and a conclusion is given in Section V.

II. Perfect Reconstruction DFT Filter Banks

A filter bank transceiver is illustrated in Figure 1; the equalization scheme is not shown on this figure and will be discussed in Section III. Parameters N and M represent the upsampling factor and the number of subcarriers[2], respectively. In this paper, we consider *redundant* filter banks, where $N > M$. The transmitted and received symbols are respectively denoted by $a_m[n]$ and $b_m[n]$, $m = 0, \ldots, M-1$. The

[1]In wireless communications, DMT is also known as orthogonal frequency division multiplexing (OFDM).

[2]In the DMT literature, a subcarrier is often called a "tone".

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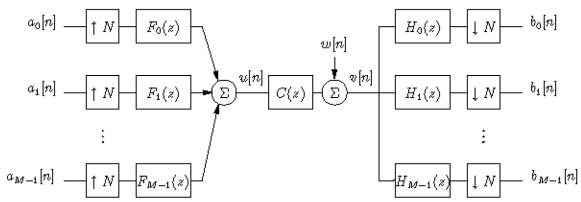

Fig. 1. A filter bank transceiver.

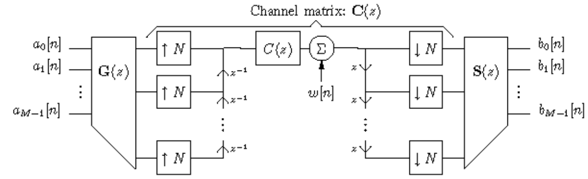

Fig. 2. Polyphase representation of a filter bank transceiver.

symbols $a_m[n]$ come from QAM constellations of possibly different sizes, which are determined by a bit loading algorithm. The channel is modelled by a finite impulse response (FIR) filter of degree Q, i.e. $C(z) = \sum_{n=0}^{Q} c[n]z^{-n}$, and additive noise $w[n]$, which is not necessarily white. Finally, $F_m(z) = \sum_{n=0}^{D-1} f_m[n]z^{-n}$ and $H_m(z) = \sum_{n=0}^{D-1} h_m[n]z^{n}$, $m = 0, \dots, M-1$ represent the synthesis and analysis filters, respectively. Note that, for convenience, $H_m(z)$ is noncausal. Causality can be retrieved simply by adding a delay of $D-1$ samples.

In a DFT filter bank, the filters $F_m(z)$ and $H_m(z)$ are derived from single prototype filters $F(z)$, $H(z)$, respectively, using a complex modulation, i.e.

$$\begin{aligned} F_m(z) &= F(zW^m) \\ H_m(z) &= H(zW^{-m}), \end{aligned}$$

where $W = e^{-j2\pi/M}$. Hence, in a DFT filter bank transceiver, only two filters, $F(z)$ and $H(z)$, have to be designed instead of $2M$ filters. This simplifies the design problem considerably. In addition, the DFT modulation can be implemented efficiently using the FFT algorithm as will be discussed in Section III. To ensure that a real signal $u[n]$ is obtained at the channel input, $a_m[n] = a^*_{M-m}[n]$ for $m = M/2+1, \dots, M-1$, and "QAM" symbols $a_0[n]$ and $a_{M/2}[n]$ are chosen from a constellation of real symbols.

It is very convenient to use the polyphase representation to simplify the analysis of DFT filter banks. In fact, by defining,

$$\begin{aligned} F_m(z) &= \sum_{k=0}^{N-1} z^{-k} G_{k,m}(z^N) \\ H_m(z) &= \sum_{k=0}^{N-1} z^{k} S_{m,k}(z^N), \end{aligned}$$

where[3] $G_{k,m}(z) = \sum_{n=0}^{D/N-1} f_m[Nn+k]z^{-n}$ and $S_{k,m}(z) = \sum_{n=0}^{D/N-1} h_m[Nn+k]z^{n}$, the DFT filter bank transceiver can be represented, with the help of the noble identities, as in Figure 2 [5]. $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are $N \times M$ and $M \times N$ polyphase matrices whose entries are given by $[\mathbf{G}(z)]_{k,m} = G_{k,m}(z)$ and $[\mathbf{S}(z)]_{m,k} = S_{m,k}(z)$, respectively.

For the rest of this section, let us assume that the channel is ideal, i.e. $C(z) = 1$ (this assumption will be lifted in Section III). In the absence of noise (i.e. $w[n] = 0$), the overall transfer function of the DFT filter bank transceiver, from the transmitter to the receiver, is equal to

$$\mathbf{T}_0(z) = \mathbf{S}(z)\mathbf{G}(z). \tag{1}$$

For errorless data transmission, the transfer function should be equal to the $M \times M$ identity matrix, i.e. $\mathbf{T}_0(z) = \mathbf{I}_M$. A system satisfying the above property is called a *perfect reconstruction* (or PR) system. Note that DMT, essentially composed of an IFFT/FFT pair, is a PR system.

PR systems can be designed using *paraunitary* matrices. A paraunitary matrix $\mathbf{G}(z)$ satisfies [5]

$$\widetilde{\mathbf{G}}(z)\mathbf{G}(z) = \mathbf{I}_M, \tag{2}$$

where $\widetilde{\mathbf{G}}(z) = \mathbf{G}^H(1/z^*)$. Here, the tilde operator represents paraconjugation. For such systems, the analysis filter bank is set to $\mathbf{S}(z) = \widetilde{\mathbf{G}}(z)$.

In this work, we force $\mathbf{G}(z)$ to be paraunitary so that the transceiver system is characterized by the PR property. Paraunitaryness can be enforced by noting that, since $\mathbf{G}(z)$ is a polyphase matrix of a DFT filter bank, it can be factorized as follows [6]

$$\begin{aligned} \mathbf{G}(z) &= \begin{bmatrix} \mathbf{I}_N & z^{-1}\mathbf{I}_N & \dots & z^{-L+1}\mathbf{I}_N \end{bmatrix} \times \\ &\quad \begin{bmatrix} U_0(z^L) & 0 & \dots & 0 \\ 0 & U_1(z^L) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_{P-1}(z^L) \end{bmatrix} \begin{bmatrix} \mathbf{I}_M \\ \mathbf{I}_M \\ \vdots \\ \mathbf{I}_M \end{bmatrix} \mathbf{W}^* \\ &\triangleq \mathbf{U}(z)\mathbf{W}^*, \end{aligned} \tag{3}$$

where $U_k(z) = \sum_{n=0}^{D/P-1} f_m[Pn+k]z^{-n}$ is the P-fold polyphase component of $F(z)$ and $\mathbf{W}$ is the $M \times M$ DFT matrix, i.e. $[\mathbf{W}]_{k,m} = (1/\sqrt{M})W^{km}$. The parameter P is defined as the least common multiple between M and N, and $L = P/N$. From (2) and (3), it is easy to prove that $\mathbf{G}(z)$ is paraunitary if and only if $\mathbf{U}(z)$ is paraunitary. Depending on the relationship between M and N, $\mathbf{U}(z)$ can take three different forms[4] [6]. Each form can then be parametrized by the factorization theorem of paraunitary matrices [5]. Such parametrization will be denoted by the function p,

$$\mathbf{x} \xrightarrow{p} \mathbf{U}(z),$$

[3]We assume, without loss of generality, that D in a multiple of N.

[4]Note that when $M = N$, the only possible choice for $\mathbf{U}(z)$ is the identity matrix $\mathbf{I}_M$. This situation is that of DMT, where the "prototype filter" is a rectangular window. Redundancy (i.e. $N > M$) allows for PR non-rectangular windows.

where $\mathbf{x}$ is a vector of real numbers of a proper size. Furthermore, according to (3) a correspondence between the prototype filter $F(z)$ (via its polyphase components $U_k(z)$, $k = 0, \ldots, P-1$) and the matrix $\mathbf{U}(z)$ can be established. In other words, to each paraunitary matrix $\mathbf{U}(z)$, there corresponds a prototype filter $F(z)$ and vice versa. This correspondence can be represented by the mapping q as

$$\mathbf{U}(z) \xrightarrow{q} F(z).$$

In order to provide good spectral selectivity, among all possible paraunitary matrices, it is desirable to select the one that minimizes the stopband energy. The prototype filter design problem can thus be cast as the following unconstrained optimization problem

$$\mathbf{x}_o = \arg\min_{\mathbf{x}} \int_{\pi/M}^{\pi} |F(e^{j\omega}; \mathbf{x})|^2 d\omega, \tag{4}$$

where $F(z; \mathbf{x}) = q\left(p(\mathbf{x})\right)$. This approach is similar to the one suggested in [4]. Note that it is not necessary to constrain the passband of the prototype filter to be flat to obtain a good frequency response. Due to the paraunitaryness of $\mathbf{G}(z)$, one can show, using the power complementary property, that the passband of $F(z)$ will be constant, even if the cost function does not explicitly take this into account [5].

Despite the improved spectral selectivity, the transceiver resulting from the optimization in (4) has a few drawbacks. It obviously requires more computational power than DMT due to the utilization of modulated filters instead of a simple IFFT/FFT pair. The DFT filter banks will also cause a delay which may be problematic for applications with very low latency requirements.

III. Equalization via Zero-Padding and the QR Factorization

We now consider a non-ideal channel, that is, we no longer assume that $C(z) = 1$. The transfer function of the transceiver in (1) thus becomes

$$\mathbf{T}(z) = \mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z), \tag{5}$$

where $\mathbf{C}(z)$ is the $N \times N$ channel matrix, which can be obtained by applying the polyphase identity successively [5], i.e.

$$[\mathbf{C}(z)]_{k,m} = [z^{k-m}C(z)]_{\downarrow N}.$$

Note that $[\cdot]_{\downarrow N}$ denotes the Z-transform of the N-fold decimation of the corresponding time-domain signal. Due to the presence of $\mathbf{C}(z)$ in (5), the PR property of the DFT filter bank developed in Section II no longer holds. However, there exists a way to mitigate the channel matrix in (5) so that the design method (4) can still yield PR systems. Let us modify Figure 2 by adding an equalizer block between the channel and the receiver. The transfer function (5) can then be written as

$$\mathbf{T}'(z) = \mathbf{S}(z)\mathbf{E}(z)\mathbf{C}(z)\mathbf{G}(z). \tag{6}$$

One cannot simply use $\mathbf{E}(z) = \mathbf{C}^{-1}(z)$ to equalize the channel as possibly unstable infinite impulse response (IIR) filters might be obtained. We propose instead to use zero-padding and the QR factorization, as explained below, for equalization purposes. The key is to choose an upsampling factor such that $N \geq M + Q$.

Whenever $N \geq M + Q$ is satisfied, the channel matrix can be partitioned as $\mathbf{C}(z) = \begin{bmatrix}\mathbf{C}_0 & \mathbf{C}_1(z)\end{bmatrix}$, where $\mathbf{C}_0$ is an $N \times (N-Q)$ matrix of real numbers and $\mathbf{C}_1(z)$ is an $N \times Q$ polynomial matrix [7]. Hence, we can formulate (6) as

$$\mathbf{T}'(z) = \mathbf{S}(z)\mathbf{E}(z)\begin{bmatrix}\mathbf{C}_0 & \mathbf{C}_1(z)\end{bmatrix}\mathbf{G}(z). \tag{7}$$

$\mathbf{C}_1(z)$ can be cancelled by transmitting a block of Q zeros after each block of $N-Q$ time-domain samples. This is equivalent to set the last Q rows of $\mathbf{G}(z)$ to zeros, i.e. $\mathbf{G}(z) = \begin{bmatrix}\mathbf{G}_0(z) & \mathbf{0}\end{bmatrix}^T$, where $\mathbf{0}$ denotes a matrix of zeros of an appropriate size. The transfer function (7) can now be written as

$$\begin{aligned}\mathbf{T}'(z) &= \mathbf{S}(z)\mathbf{E}(z)\begin{bmatrix}\mathbf{C}_0 & \mathbf{C}_1(z)\end{bmatrix}\begin{bmatrix}\mathbf{G}_0(z) \\ \mathbf{0}\end{bmatrix} \\ &= \mathbf{S}(z)\mathbf{E}(z)\mathbf{C}_0\mathbf{G}_0(z).\end{aligned} \tag{8}$$

The matrix $\mathbf{C}_0$ in (8) can be mitigated using the QR factorization. In this case, we obtain [8]

$$\mathbf{C}_0 = \mathbf{Q}\begin{bmatrix}\mathbf{R}_0 \\ \mathbf{0}\end{bmatrix},$$

where $\mathbf{Q}$ is orthogonal (i.e. $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_N$) and $\mathbf{R}_0$ is an $(N-Q) \times (N-Q)$ upper triangular matrix. Now, if we let $\mathbf{E}(z) = \mathbf{E} = \left[\begin{smallmatrix}\mathbf{R}_0^{-1} & 0 \\ 0 & 0\end{smallmatrix}\right]\mathbf{Q}^T$, we have

$$\mathbf{T}'(z) = \mathbf{S}(z)\begin{bmatrix}\mathbf{I}_{N-Q} \\ \mathbf{0}\end{bmatrix}\mathbf{G}_0(z). \tag{9}$$

By partitioning $\mathbf{S}(z)$ as $\mathbf{S}(z) = \begin{bmatrix}\mathbf{S}_0(z) & \mathbf{S}_1(z)\end{bmatrix}$, where $\mathbf{S}_0(z)$ and $\mathbf{S}_1(z)$ are respectively $M \times (N-Q)$ and $M \times Q$ matrices, (9) becomes

$$\mathbf{T}'(z) = \mathbf{S}_0(z)\mathbf{G}_0(z). \tag{10}$$

From (9) and (10), we can observe that the last Q samples of each frame of N samples are dropped at the receiver.

Notice the similarity between (1) and (10). The only difference involves the sizes of the respective matrices, $\mathbf{G}(z)$ being $N \times M$ in (1) and $\mathbf{G}_0(z)$ being $(N-Q) \times M$ in (10). Therefore, if the equalization scheme as discussed above is employed, the PR design method presented in Section II can be applied just as it is, except that N needs to be replaced by $N-Q$.

A complete implementation of the proposed transceiver is illustrated in Figure 3. Note that the factorization of $\mathbf{U}(z)$ in (3) can be exploited to derive an efficient implementation [9]. Moreover, the QR decomposition needs only to be evaluated once if the channel is time-invariant (as would be the case for most twisted-pair channels). Since $\mathbf{R}_0$ is upper triangular, one does not need to compute $\mathbf{R}_0^{-1}$ explicitly; instead, the output of the corresponding block in Figure 3 can be obtained efficiently by backward substitution [8].

It is important to underline that redundancy in the system shown in Figure 3 comes from two factors:

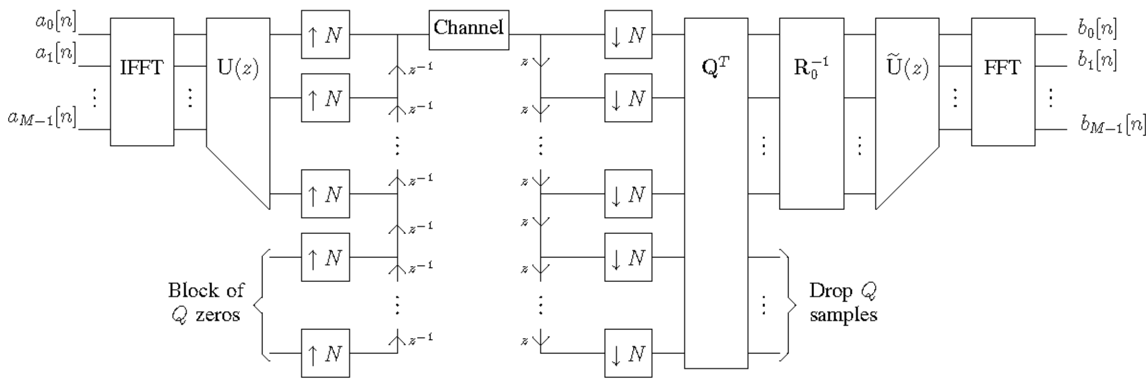

Fig. 3. Implementation of the proposed DFT filter bank transceiver.

1) The filter bank transformation (where M samples are encoded into $N-Q$ samples).
2) Zero-padding (where Q zeros are concatenated to a frame of $N-Q$ samples, yielding a frame of N samples in total).

The first kind of redundancy is used to provide an improved spectral selectivity as explained in Section II, whereas the second kind handles ISI mitigation. In DMT, only the second kind of redundancy exists. Therefore, a tradeoff can be made between spectral selectivity and bandwidth utilization. We should use PR DFT filter banks in environments that benefit from better spectral selectivity so that bandwidth is not lost unnecessarily.

IV. Experimental Results

The PR DFT filter bank transceiver considered in this section is characterized by the following parameters: $M = 64$, $N = 80$, $Q = 8$, and $D = 1600$. The sampling rate is set to $F_s = 2.208$ MHz. The prototype filter was designed by solving (4) and the resulting frequency response is illustrated in Figure 4. Even though the filter is not "ideal", the system is still characterized by the PR property due to the underlying paraunitary structure. A delay of 0.83 ms will be incurred by such prototype filter. Also shown in the figure is the frequency response of the DMT rectangular window. Notice that, with DMT, the first sidelobe is attenuated by about 13 dB, whereas the attenuation of the PR DFT filter bank is more than 38 dB.

Next, we simulated both systems in two different noisy environments. Performance was measured in terms of achievable bit rates, as obtained by the rate adaptive bit loading algorithm proposed in [10]. The algorithm first determines which subcarriers should be disabled. Whenever the allocated power would be negative with respect to the optimal water-filling power distribution, the proper subcarrier is turned off. The available power budget is then allocated equally among the remaining subcarriers. The number of bits which can be loaded in each enabled subcarrier is given by [1], [10]

$$\beta_m = \begin{cases} \frac{1}{2}\log_2\left(1+\frac{\mathrm{SNR}_m \cdot \gamma_{\mathrm{code}}}{\Gamma \cdot \gamma_{\mathrm{margin}}}\right) & \text{if } m = 0 \text{ or } M/2 \\ \log_2\left(1+\frac{\mathrm{SNR}_m \cdot \gamma_{\mathrm{code}}}{\Gamma \cdot \gamma_{\mathrm{margin}}}\right) & \text{otherwise,} \end{cases}$$

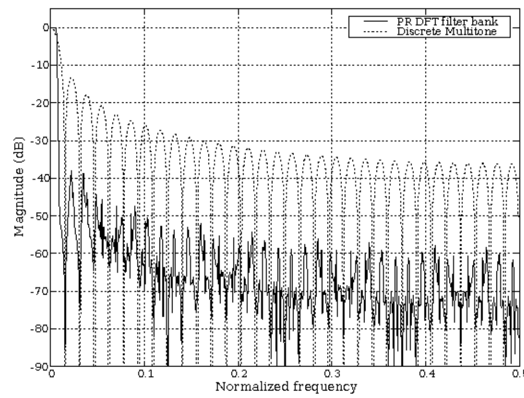

Fig. 4. Frequency responses of the PR DFT filter prototype filter ($M = 64$, $N - Q = 72$ and $D = 1600$) and DMT.

where SNR_m is the signal-to-noise ratio of the m-th subcarrier, γ_{code} is the coding gain, γ_{margin} is the additional noise margin, and Γ is the SNR gap, representing the difference between the actual modulation scheme (QAM) and channel capacity for a given error probability. The achievable bit rate is obtained by multiplying the modulation rate by the total number of bits loaded in each subcarrier [1], i.e.

$$R = \frac{F_s}{N}\sum_{m=0}^{M/2}\beta_m,$$

where F_s is the sampling rate (in Hz). The result is expressed in bits per second (bps). In our experiments, we select $\gamma_{\mathrm{code}}/\gamma_{\mathrm{margin}} = 1$ and $\Gamma = 9.8$ dB (corresponding to an error probability of 10^{-7}).

Figures 5 and 6 show the achievable bit rates of the proposed PR DFT filter bank and DMT systems with respect to the transmitted signal power. Results are obtained by averaging the bit rates over 100 random channels of degree $Q = 8$. Such channels are generated by using exponentially damped independent Gaussian random variables (with zero mean and unit variance). To mitigate ISI, a block of 8 samples was used either as a suffix of zeros in the case of PR DFT filter banks

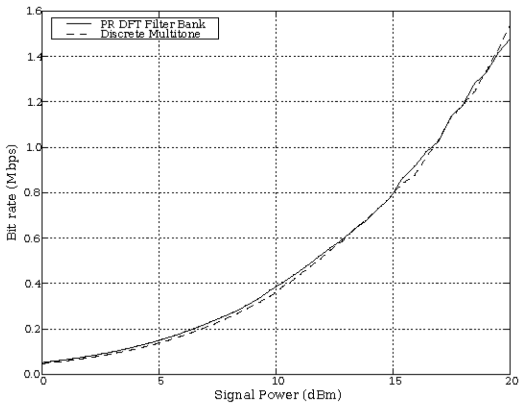

Fig. 5. Bit rates obtained with an additive white Gaussian noise of -50 dBm/Hz.

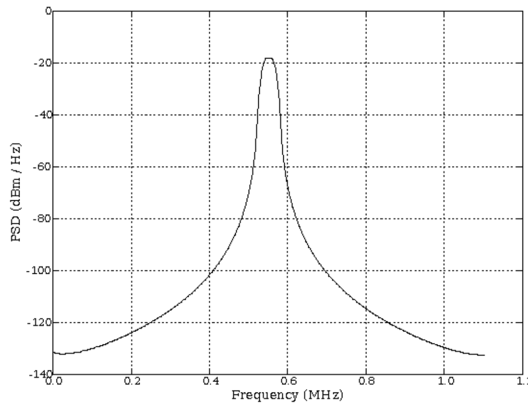

Fig. 7. Power spectral density of the additive narrow band noise.

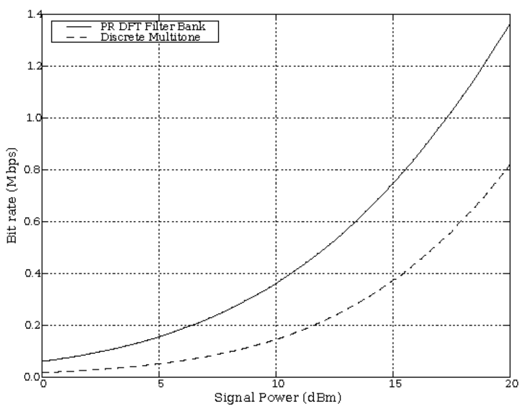

Fig. 6. Bit rates obtained with a narrow band noise whose PSD is given in Figure 7.

or as a cyclic prefix in the case of DMT [7]. In Figure 5, a Gaussian white noise with a flat power spectral density (PSD) of -50 dBm/Hz was added to the received signal. In Figure 6, we added a Gaussian narrow band noise, whose PSD is shown in Figure 7.

Under an additive white noise at low SNR, as shown in Figure 5, a marginal gain in terms of bit rate can be noted by using a PR DFT filter bank transceiver instead of a DMT one; while at higher SNR, DMT appears to be preferable. However, Figure 6 clearly illustrates that the PR DFT filter bank system offers a better immunity against narrow band noise. An improvement of about 0.6 Mbps can be noted for a transmitted power of 20 dBm. This represents a relative gain of almost 75%. In this case, trading bandwidth to improve spectral selectivity provides huge benefits. This flexibility is not possible with DMT transceivers.

V. Conclusion

In this paper, we have considered ISI-free (or PR) DFT filter bank transceivers. The prototype filter is designed using paraunitary matrices to ensure that the PR property is satisfied. A minimization of the stopband energy is then applied which increases spectral selectivity considerably. The resulting transceiver can be equalized via a zero-padding algorithm combined with QR factorization. Compared to DMT, experiments show that the PR DFT filter bank system achieves a much higher bit rate under a narrow band noise. As the additional bandwidth required to improve spectral selectivity can be controlled, the proposed PR DFT filter bank system offers tradeoff possibilities which are not feasible with DMT.

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