ADAPTIVE SUBSPACE TRACKING USING A CONSTRAINED LINEARIZATION APPROACH

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ABSTRACT

In this paper, we develop and evaluate a new algorithm for tracking the eigenvalue decomposition (EVD) of a time-varying data covariance matrix. The EVD tracking problem is first formulated as a normalized rank-one EVD update problem. An approximate solution to the latter is obtained using a modified linearization approach which maintains the constraint of orthonormality on the updated eigenvectors. The new EVD tracking algorithm has several interesting features. In particular, the eigenvalue update is non-iterative and the matrix of eigenvectors is updated via a finite sequence of Givens rotations. The algorithm is also well suited for implementation on parallel processors. Computer experiments are presented that demonstrate the applicability of the new algorithm in array processing.

1. INTRODUCTION

In recent years, several new algorithms have been developed for efficiently estimating and tracking the eigenvalue decomposition (EVD) of a time-varying data covariance matrix (e.g., [2, 3, 4] and references therein). This problem finds its origin in the application of high-resolution subspace-based signal analysis methods such as MUSIC to non-stationary data sequences. One important example of this is the estimation of the direction of arrivals (DOA's) of multiple moving sources with an array of sensors.

In this paper, we propose a new algorithm for estimating and tracking the EVD of a time-varying data covariance matrix. In Section 2, we first formulate EVD tracking as a normalized rank-one EVD update problem. In Section 3, an approximate solution to the latter is obtained using a modified linearization approach in which a constraint of orthonormality is imposed on the updated eigenvectors through an appropriate parametrization. In Section 4, further approximations to this solution lead to the new algorithm in which the matrix of eigenvectors is updated via a finite sequence of Givens rotations. Section 5 presents the results of computer experiments that demonstrate the applicability of this algorithm in array processing.

This work was supported in part by a grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

2. THE EVD TRACKING PROBLEM

2.1. Formulation

Let $\mathbf{x}(k) \in \mathcal{C}^{L\times 1}$ denote a random column vector of L complex observations made at the kth sampling instant. In a typical narrow-band array processing application, the elements of $\mathbf{x}(k)$ are the Fourier coefficients of the outputs of an L-sensor array at a particular frequency during the kth integration interval, or snapshot. The sequence of random vectors $\mathbf{x}(k)$ is modeled as a zero-mean stochastic process with covariance matrix

$$R(k) = E[\mathbf{x}(k)\mathbf{x}(k)^H], \qquad (1)$$

where $E[.]$ denotes statistical expectation and the superscript ^{H} denotes conjugate transposition. It is implicitely assumed in (1) that the process $\mathbf{x}(k)$ can be non-stationary. We denote by $\lambda_i(k)$ and $\mathbf{q}_i(k)$, $i = 1, ..., L$, the eigenvalues and corresponding orthonormalized eigenvectors of the matrix $R(k)$. That is,

$$R(k) = Q(k)\Lambda(k)Q(k)^H, \qquad (2)$$

where

$$\Lambda(k) = \mathrm{diag}(\lambda_1(k), ..., \lambda_L(k)), \qquad (3)$$

$$Q(k) = [\mathbf{q}_1(k), ..., \mathbf{q}_L(k)]. \qquad (4)$$

It will be convenient to assume that the eigenvalues are arranged in non-increasing order, i.e.: $\lambda_1(k) \geq \lambda_2(k) \geq ... \geq \lambda_L(k) \geq 0$.

The EVD tracking problem can be formulated as follows: for time $k = 0, 1, ...$, we want to compute estimates of the true EVD of the covariance matrix $R(k)$, i.e. estimates of $Q(k)$ and $\Lambda(k)$, which are functions of the sequence of observation vectors $\mathbf{x}(l)$, from time $l = 0$ up to time $l = k$, and possibly some initial conditions. Let

$$\Gamma(k) = \mathrm{diag}(\gamma_1(k), ..., \gamma_L(k)), \qquad (5)$$

$$U(k) = [\mathbf{u}_1(k), ..., \mathbf{u}_L(k)], \qquad (6)$$

denote the desired estimates of $\Lambda(k)$ and $Q(k)$, respectively. In this paper, we shall seek estimates which approximately satisfy the recursive relation:

$$U(k)\Gamma(k)U(k)^H = \alpha U(k-1)\Gamma(k-1)U(k-1)^H + (1-\alpha)\mathbf{x}(k)\mathbf{x}(k)^H, \qquad (7)$$

where α is a forgetting factor with $0 < \alpha < 1$. In addition to (7), we shall further require that the structure of the matrices $\Gamma(k)$ and $U(k)$ be the same as that of $\Lambda(k)$

and $Q(k)$, respectively. In the case of $\Gamma(k)$, this implies that

$$\gamma_1(k) \geq \gamma_2(k) \geq \ldots \geq \gamma_L(k) \geq 0, \quad (8)$$

while for $U(k)$, we have the following constraint:

$$U(k)^H U(k) = I_L, \quad (9)$$

where I_L denotes the $L \times L$ identity matrix. In this way, $U(k)$ and $\Gamma(k)$ provide a legitimate EVD at all time.

In the area of matrix analysis, the computational problem specified by (7)-(9) is known as a rank-one EVD update. An algorithm which can provide an exact solution to to this problem has been presented in [1]. Its use in signal procesing as a means to solve the EVD tracking problem has been proposed in [4, 3]. However, this approach is computationnaly expensive and suffers form several drawbacks.

For these reasons, new EVD tracking algorithms have been proposed recently in which (7) is only approximately satisfied at each iteration [2]. The underlying motivation for this approach follows from the observation that (7) is generally not an optimal EVD estimator. Hence, seeking modifications to (7) which may result in more efficient EVD tracking algorithms without affecting significantly the quality of the estimation is perfectly legitimate. This is the general philosophy that we follow in this paper to derive a new EVD tracking algorithm. However, before proceeding with the derivation, we need to recast the updating problem (7)-(9) in a normalized form which will simplify our work. We refer to this as the "preprocessing".

2.2. Preprocessing

To simplify the notations, we first make the following identifications:

$$U \equiv U(k-1), \quad \Gamma \equiv \Gamma(k-1), \quad \mathbf{x} \equiv \mathbf{x}(k), \quad (10)$$

and we denote by $F(U, \Gamma, \mathbf{x})$ the right-hand-side of (7):

$$F(U, \Gamma, \mathbf{x}) = \alpha U \Gamma U^H + (1-\alpha)\mathbf{x}\mathbf{x}^H. \quad (11)$$

We now describe four simplification steps. The ith step ($i = 1, \ldots, 4$) consists in expressing $F(.)$ in the form

$$F(U, \Gamma, \mathbf{x}) = U_i[\Gamma_i + \xi_i \xi_i^H] U_i^H, \quad (12)$$

where the unitary matrix U_i, the diagonal matrix Γ_i and the vector ξ_i are appropriately defined, so that the modified rank-one EVD problem which appears within brackets in (12) is simplified at each step.

1. Diagonalization and scaling: Transform (7)-(9) into the equivalent rank-one EVD update of a diagonal matrix. This is achieved by using

$$\xi_1 = \sqrt{(1-\alpha)} U^H \mathbf{x}, \quad U_1 = U, \quad \Gamma_1 = \alpha \Gamma. \quad (13)$$

2. Mapping into real vector space: Map $\xi_1 \in \mathcal{C}^{L \times 1}$ into $\xi_2 \in \mathcal{R}^{L \times 1}$ so that the updating problem only involves real quantities. To this end, define

$$D = \text{diag}(\xi_{1,i}/|\xi_{1,i}|) \quad (14)$$

where $\xi_{1,i}$ denotes the ith entry of ξ_1, and let

$$\xi_2 = D^H \xi_1, \quad U_2 = U_1 D, \quad \Gamma_2 = \Gamma_1. \quad (15)$$

Step	Operation
1	$\xi \leftarrow \sqrt{(1-\alpha)} U^H \mathbf{x}$ $\Gamma \leftarrow \alpha \Gamma$
2	$D = \text{diag}(\xi_i/\|\xi_i\|)$ $\xi \leftarrow D^H \xi$ $U \leftarrow UD$
3	H = block Householder matrix $\xi \leftarrow H^T \xi$ $U \leftarrow UH$
4	P_1 = permutation matrix $\xi \leftarrow P_1^T \xi$ $U \leftarrow U P_1$ $\Gamma \leftarrow P_1^T \Gamma P_1$

Table 1: Summary of preprocessing steps.

3. Deflation: Reduce the dimensionality of the problem whenever some of the eigenvalues $\gamma_i(k)$ are repeated. Scecifically, assume that the number of distinct eigenvalues is $K \leq L$. Then, by using an appropriate block Householder matrix $H = \text{diag}(H_1, \ldots, H_K)$ (see [1]) it is possible to zero out $L-K$ components of the vector ξ_2 without affecting the diagonal matrix Γ_2. Thus, we have

$$\xi_3 = H^T \xi_2, \quad U_3 = U_2 H, \quad \Gamma_3 = \Gamma_2. \quad (16)$$

where the superscript ^{T} denotes transposition.

4. Reordering: Using an appropriate permutation matrix P_1, let

$$\xi_4 = P_1^T \xi_3, \quad U_4 = U_3 P_1, \quad \Gamma_4 = P_1^T \Gamma_3 P_1, \quad (17)$$

so that the last $L-K$ components of ξ_4 are zero, while the first K components of Γ_4 are in decreasing order.

These steps are summarized in Table 1 where the notation $\xi \leftarrow f(\xi)$ is used to indicate that the result of some operation $f(\xi)$ is overwritten on ξ.

3. SOLUTION VIA CONSTRAINED LINEARIZATION

Let ξ, Γ ans U respectively denote the transformed data vector, diagonal eigenvalue matrix and unitary eigenvector matrix following the application of steps 1 to 4 in Table 1. Observe that ξ and Γ can be partitionned as

$$\xi^T = [\xi_u^T, \mathbf{0}^T], \quad \Gamma = \text{diag}(\Gamma_u, \Gamma_l) \quad (18)$$

where $\xi_u = [\xi_1, \ldots, \xi_K]^T$ with $\xi_i > 0$, $\Gamma_u = \text{diag}(\gamma_1, \ldots, \gamma_K)$ with $\gamma_1 > \gamma_2 > \ldots > \gamma_K \geq 0$, $\Gamma_l = \text{diag}(\gamma_{K+1}, \ldots, \gamma_L)$ with $\gamma_i \in \{\gamma_1, \ldots, \gamma_K\}$ for $i = K+1, \ldots, L$. With this partitionning of ξ and Γ, $F(U, \Gamma, \xi)$ (12) (after step 4) can now be written as

$$F(U, \Gamma, \mathbf{x}) = U \begin{bmatrix} \Gamma_u + \xi_u \xi_u^T & \mathbf{0}_{K \times (L-K)} \\ \mathbf{0}_{(L-K) \times K} & \Gamma_l \end{bmatrix} U^H. \quad (19)$$

Hence, the general rank-one EVD update problem (7)-(9) over $\mathcal{C}^{L\times L}$ has been simplified to the rank-one EVD update of a diagonal matrix over $\mathcal{R}^{L\times L}$, namely:

$$V\tilde{\Gamma}_u V^T = \Gamma_u + \xi_u \xi_u^T, \quad (20)$$

$$\tilde{\Gamma}_u = \mathrm{diag}(\tilde{\gamma}_1, ..., \tilde{\gamma}_K), \quad (21)$$

$$V^T V = I_K. \quad (22)$$

As explained earlier, our interest in this paper lies in approximate solutions to the simplified EVD update problem (20)-(22) which naturally lend themselves to efficient numerical implementations. In this section, as a first step towards this goal, we use a constrained linearization approach to derive an approximate solution to (21)-(23). In the next section, further simplifications to this "generic" solution will provide us with a new EVD tracking algorithm with interesting properties.

At the basis of our derivation is the observation that in most practical applications of EVD tracking, the forgetting parameter α is close to one. If we further note that in the limit $\alpha \to 1$, one has $\tilde{\Gamma}_u \to \Gamma_u$ and $V \to I_K$ and that for $1-\alpha$ sufficiently small, the modified EVD can be analytically connected to the unmodified one [2], we are lead to conclude that when α is close to one, the modifications to the EVD must be small.

To emphasize this point, let us write $\tilde{\Gamma}_u$ in the form

$$\tilde{\Gamma}_u = \Gamma_u(I_K + \Omega), \quad (23)$$

$$\Omega = \mathrm{diag}(\omega_1, ..., \omega_K). \quad (24)$$

The unknown parameter ω_i $(i = 1, ..., K)$ represents the relative variation in the ith modified eigenvalue and is expected to be small when α is close to 1.

The introduction of a similar representation for V in terms of small parameters requires additional care because of the orthogonality constraint (22). To derive such a representation, we first note that $\det(V) = \pm 1$ as a result of (22), where det(.) denotes the determinant of its matrix argument. Without loss of generality, we shall assume that $\det(V) = +1$. This amounts to multiplying one of the modified eigenvectors (i.e., any column of V) by -1. With this additional restriction, V now belongs to the group of unimodular orthogonal matrices, also known as proper rotations. An important result in group theory states that any proper rotation V can be expressed as

$$V = \exp(\Theta) \quad (25)$$

where $\Theta = (\theta_{ij})$ is a skew-symmetric matrix in $\mathcal{R}^{K\times K}$ (i.e., $\Theta^T = -\Theta$, or equivalently, $\theta_{ji} = -\theta_{ij}$), and exp(.) is the matrix exponential function, defined as

$$\exp(\Theta) = \Sigma_{k=0}^{\infty} \Theta^k / k! \quad (26)$$

The above parametrization of V in terms of $K(K-1)/2$ real parameters is of particular interest to us. Indeed, when Θ is small, so is $V - I_K$ and in particular, when $\Theta = \mathbf{0}_{K\times K}$, then $V = I_K$. Using this parametrization in connection with (20), we thus expect the matrix Θ to be small whenever α is close to 1.

Within the framework of the above parametrization, the proposed constrained linearization approach amounts to the following steps: (i) substitute (23) and (25) in (20), perform the necessary expansions and retain only linear terms (i.e., degree equal zero or one) in Ω and Θ; (ii) solve for Ω and Θ; and (iii) substitute the solutions in (23) and (25), respectively. The resulting matices $\tilde{\Gamma}_u$ and V so obtained are the desired approximate solutions to the simplified rank-one EVD update problem (20)-(22).

The first step leads to the following equation:

$$\Gamma_u \Omega + \Theta \Gamma_u + \Gamma_u \Theta = [\xi_u \xi_u^T - \Gamma_u] \quad (27)$$

Note that this equation is linear in Ω and Θ. By considering independently the diagonal and off-diagonal entries in (27), we easily arrive at

$$\gamma_i \omega_i = (\xi_i^2 - \gamma_i), \quad i = 1, ..., K, \quad (28)$$

$$\theta_{ij} = \xi_i \xi_j / (\gamma_j - \gamma_i), \quad 1 \le i < j \le K. \quad (29)$$

Note that $\theta_{ii} = 0$ while the remaining elements of Θ are obtained by symmetry. Also note that in the simplified EVD update problem (20), the original eigenvalues are distinct so that division by $\gamma_j - \gamma_i$ is permitted. Upon substitution of (28) in (23)-(24), we obtain:

$$\tilde{\Gamma}_u = \Gamma_u + \mathrm{diag}(\xi_1^2, ..., \xi_K^2). \quad (30)$$

Similarly, knowledge of the parameters θ_{ij} from (29) can be used in (25)-(26) to construct the matrix of modified eigenvectors V. In the next section, we propose a computationally efficient approach for implementing this computation.

4. ALGORITHMIC REALIZATION BASED ON PLANAR ROTATIONS

Recall the mathematical derivation in Section III is based on the assumption that the matrices of rotation and scaling parameters, i.e. Ω and Θ, respectively, are small. In this section, the assumption of a small Θ matrix will be further exploited to derive a computationally efficient approximation for the updated eigenvector matrix $V = \exp(\Theta)$ (25) as a sequence of small planar rotations. When used in connection with (19) and (20), this will result in a new EVD tracking algorithm with interesting properties.

To begin with, let Ψ_{ij} denote the $K \times K$ matrix obtained from Θ by setting all its entries to zero, except for the ij-element and the ji-element, which are left unchanged. It is then possible to express Θ in terms of the Ψ_{ij} as

$$\Theta = \Sigma_{j>i} \Psi_{ij} \quad (31)$$

Assuming that the matrix Θ is small, it is possible to arrive at the following approximation for V (25):

$$V \approx \Pi_{j>i} \exp(\Psi_{ij}). \quad (32)$$

Hence, the matrix V has been expressed has a product of simpler orthogonal matrices.

Now consider the matrix $\exp(\Psi_{ij})$, which is the basic building block in the product (32). Using the definition (26) of the matrix exponential function, one can verify that

$$\exp(\Psi_{ij}) = G_{ij}(\theta_{ij}), \quad (33)$$

Step	Operation
1	$\mathbf{x} \leftarrow \mathbf{x}(k)$ $U \leftarrow U(k-1)$ $\Gamma \leftarrow \Gamma(k-1)$
2	Preprocessing (see Table 2)

3	for $i = 1 : K-1$
	  for $j = i+1 : K$
	    $\theta \leftarrow \xi_i \xi_j / (\gamma_j - \gamma_i)$
	    $U \leftarrow U \begin{bmatrix} G_{ij}(\theta) & 0_{K\times(L-K)} \\ 0_{(L-K)\times K)} & I_{L-K} \end{bmatrix}$
	  end
	end

4	$\Gamma \leftarrow \Gamma + \mathrm{diag}(\xi_i^2)$
5	P_2 = permutation matrix $U \leftarrow U P_2$ $\Gamma \leftarrow P_2^T \Gamma P_2$
6	$U(k) \leftarrow U$ $\Gamma(k) \leftarrow \Gamma$

Table 2: The new EVD tracking algorihtm.

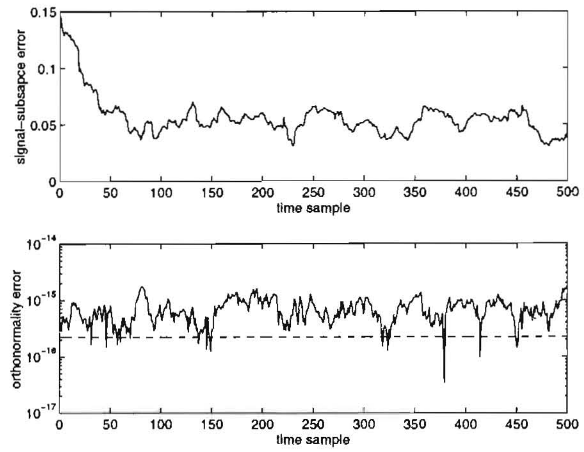

Figure 1: (a) Distance between estimated and true signal-subspace; (b) Orthonormality of signal-subspace eigenvectors.

where $G_{ij}(\theta) \in \mathcal{R}^{K\times K}$ is the well known planar (or Givens) rotation. Combining (33) and (34), we obtain

$$V \approx \Pi_{j>i} G_{ij}(\theta_{ij}), \quad (34)$$

where the rotation parameters θ_{ij} can be easily computed from (29). Note that these parameters are proportional to $1-\alpha$ and are thus relatively small in most practical applications. In summary, we have shown that for α close to one, the orthogonal matrix V, as given by (25) and (29), can be approximated as the product of $K(K-1)/2$ small planar rotations. Furthermore, to the first degree of approximation in the rotation parameters θ_{ij}, the order in which these rotations are applied is totally arbitrary.

A complete EVD tracking algorithm based on the above approximation of the modified eigenvector matrix V is described in Table 2. In the absence of *a priori* knowledge, the algorithm can be initialized by using $U(0) = \Gamma(0) = I_L$. Other initialization approach are also possible, such as using the EVD of an initial estimate of the covariance matrix $R(k)$ (2). The new algorithm requires $12L^3 + O(L^2)$ flops per iteration. However, for applications in which only a signal or noise-subspace of dimension $M < L$ is needed, modifications can be easily made to the algorithm so as to reduce the complexity to $12LM^2 + O(LM)$ flops.

5. COMPUTER EXPERIMENTS

Due to lack of space, we only describe a limited set of experiments aimed at evaluating the convergence behavior of the new algorithm in a stationary environments. We consider a uniform linear array of $L = 8$ sensors with half-wavelength spacing. The wavefield consists of two narrow-band plane wave signals with DOAs of 9^o and 12^o (w.r.t. broadside) in white background noise. The SNR of each source at the sensor level is 20dB. The new algorithm is applied to the data so generated and relevant performance measures are computed. For the sake of comparaison, exact EVD of the recursive exponential covariance matrix estimate is also computed.

Fig. 1(a) (top) shows the distance between the signal-subspace estimated with the new algorithm and the true signal-subspace as a function of time, averaged over 50 runs. Results for the exact EVD approach (not shown) are identical. Thus, the new algorihtm exhibits the same convergence behavior as the much more costly "exact EVD approach". This provides a practicle justification for the constrained linearization approach used here. Fig. 1(b) shows the quantity $||U_s(k)^H U_s(k) - I_2||_2$ versus time for a single run, where $U_s(k) = [\mathbf{u}_1(k), \mathbf{u}_2(k)]$ contains the estimated signal-subspace eigenvectors. The dashed curve in the figure represents the relative accuracy of numbers on our computer system. These results indicate that the new algorithm is very effective in preserving the orthogonality of the estimated eigenvectors.

6. REFERENCES

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