On the infinite-dimensional symmetry group of the Davey–Stewartson equations

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The Lie algebra of the group of point transformations, leaving the Davey–Stewartson equations (DSE's) invariant, is obtained. The general element of this algebra depends on four arbitrary functions of time. The algebra is shown to have a loop structure, a property shared by the symmetry algebras of all known $(2+1)$-dimensional integrable nonlinear equations. Subalgebras of the symmetry algebra are classified and used to reduce the DSE's to various equations involving only two independent variables.

I. INTRODUCTION

The purpose of this paper is to apply the method of symmetry reduction to the Davey–Stewartson equations (DSE's).[1] To do this we first obtain the group of Lie point symmetries leaving the DSE's invariant. We show that this group is infinite dimensional, study its structure, and determine its low-dimensional subgroups. The different subgroups are then used to reduce the DSE's to a lower-dimensional system.

We recall that the DSE's describe the propagation of two-dimensional water waves moving under the force of gravity in water of finite depth. We shall write these equations in the form

$$\begin{aligned} &i\Psi_t + \Psi_{xx} + \epsilon_1 \Psi_{yy} = \epsilon_2 |\Psi|^2 \Psi + \Psi w\,, \\ &w_{xx} + \delta_1 w_{yy} = \delta_2 (|\Psi|^2)_{yy}\,, \end{aligned} \tag{1.1}$$

where $\Psi(x,y,t)$ and $w(x,y,t)$ are a complex and real function, respectively, and δ_1, δ_2, ϵ_1, and ϵ_2 are real constants, with $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. The subscripts in (1.1) denote partial derivatives.

For purely one-dimensional propagation (along the x axis) we have $\Psi_y = 0$ and can consider solutions with $w = 0$. The DSE's (1.1) then reduce to the nonlinear Schrödinger equation

$$i\Psi_t + \Psi_{xx} = \epsilon_2 |\Psi|^2 \Psi\,. \tag{1.2}$$

The DSE's thus have the same relation to the nonlinear Schrödinger equation as the Kadomtsev–Petviashvili equation[2] has to the Korteweg–de Vries one, they provide a two-dimensional generalization in which the basic direction of wave propagation remains a privileged one.

The DSE's belong to the rather limited class of equations in more than $1+1$ dimensions that are exactly integrable[3] by inverse scattering techniques and their generalizations.[4–6] In particular, the DSE's were shown to have soliton and multisoliton solutions.[3]

Some recent work has been devoted to the study of symmetry groups of integrable equations in more than two dimensions.[7–12] Thus the Kadomtsev–Petviashvili equation,[7] the modified Kadomtsev–Petviashvili equation,[8] the potential Kadomtsev–Petviashvili equation,[12] and the integrable three-wave problem[10,11] all have infinite-dimensional symmetry groups. The corresponding infinite-dimensional Lie algebras all have a specific loop algebra structure. They all have Virasoro-type subalgebras and can be embedded into simple classical loop algebras of the $A_n^{(1)}$ type.[13] On the other hand, some of the multidimensional equations of the Jimbo–Miwa series,[14] which are integrable in a conditional sense,[9] have been shown to have infinite-dimensional Lie symmetry algebras that are not loop algebras.[9]

In Sec. II we present the symmetry algebra of the DSE's and exhibit its loop algebra structure by relating it to the algebra sl(7,$\mathbb{C}$). We also obtain the group transformations by integrating the vector fields forming the symmetry algebra. In Sec. III we classify the one- and two-dimensional subalgebras of the DS algebra into conjugacy classes under the action of the DS group. The one-dimensional subalgebras are used in Sec. IV to reduce the DS equations to various integrable systems in $1+1$ dimensions.

II. THE SYMMETRY GROUP OF THE DAVEY–STEWARTSON EQUATIONS

A. The DS symmetry algebra

Standard procedures exist for determining the symmetry algebra of a system of differential equations.[15] They are so algorithmic that they have been successfully programmed using REDUCE,[16] MACSYMA,[8] or other symbolic languages. In order to be able to apply a previously written program,[8] we rewrite the DSE's (1.1) in a real form, setting $\psi = u + iv$. We obtain

$$\begin{aligned} &\Delta_1 \equiv u_t + v_{xx} + \epsilon_1 v_{yy} - \epsilon_2 v(u^2+v^2) - vw = 0, \\ &\Delta_2 \equiv -v_t + u_{xx} + \epsilon_1 u_{yy} - \epsilon_2 u(u^2+v^2) - uw = 0\,, \\ &\Delta_3 \equiv w_{xx} + \delta_1 w_{yy} \\ &\qquad - 2\delta_2[u u_{yy} + (u_y)^2 + v v_{yy} + (v_y)^2] = 0\,. \end{aligned} \tag{2.1}$$

An element of the symmetry algebra of (2.1) is written as

$$V = \eta_1\,\partial_x + \eta_2\,\partial_y + \eta_3\,\partial_t + \phi_1\,\partial_u + \phi_2\,\partial_v + \phi_3\,\partial_w\,, \tag{2.2}$$

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where η_i and ϕ_i $(i=1,2,3)$ are functions of x, y, t, u, v, and w. These functions are obtained by solving the determining equations, that in turn follow from the equations

$$\mathrm{pr}^{(2)}V\cdot\Delta_j(x,y,t,u,v,w)|_{\Delta_i=0}=0,\quad i=1,2,3\,, \tag{2.3}$$

where $\mathrm{pr}^{(2)}V$ is the second prolongation[15] of the vector field V. Applying the program[8] we obtain the determining equations, a relatively simple system of linear partial differential equations for η_i and ϕ_i. By solving them we find that a general element of the symmetry algebra of the DSE's (2.1) can be written as

$$V=X(f)+Y(g)+Z(h)+W(m)\,, \tag{2.4}$$

where

$$\begin{aligned}
X(f)&=f(t)\partial_t+[f'(t)/2](x\,\partial_x+y\,\partial_y-u\,\partial_u\\
&\quad-v\,\partial_v-2w\,\partial_w)-[(x^2+\epsilon_1 y^2)/8]\\
&\quad\times[f''(t)(v\,\partial_u-u\,\partial_v)+f'''(t)\partial_w]\,,\\
Y(g)&=g(t)\partial_x-[x/2]\\
&\quad\times[g'(t)(v\,\partial_u-u\,\partial_v)+g''(t)\partial_w]\,,\\
Z(h)&=h(t)\partial_y-[\epsilon_1 y/2]\\
&\quad\times[h'(t)(v\,\partial_u-u\,\partial_v)+h''(t)\partial_w]\,,\\
W(m)&=m(t)(v\,\partial_u-u\,\partial_v)+m'(t)\partial_w\,.
\end{aligned} \tag{2.5}$$

The functions $g(t)$, $h(t)$, and $m(t)$ are arbitrary real-valued functions of class C^∞ over some time interval $T\subseteq\mathbf{R}$. The function $f(t)$ satisfies

$$f(t)=\begin{cases}\text{arbitrary}, & \text{if } \delta_1=-\epsilon_1=\pm 1,\\ a+bt+ct^2, & \text{if } \delta_1\neq-\epsilon_1\end{cases} \tag{2.6a}$$

(a, b, and c are arbitrary real constants). The primes in (2.5) denote derivatives with respect to time t. The DSE's have been shown to be integrable precisely in the case when we have

$$\delta_1=-\epsilon_1\,, \tag{2.6b}$$

i.e., when $f(t)$ is allowed to be arbitrary. We shall mainly concentrate on this case. The commutation relations for the DS algebra (2.4), (2.5) are easy to obtain, namely

$$\begin{aligned}
[X(f_1),X(f_2)]&=X(f_1f_2'-f_1'f_2)\,,\\
[X(f),Y(g)]&=Y(fg'-f'g/2)\,,\\
[X(f),Z(h)]&=Z(fh'-f'h/2)\,,\\
[X(f),W(m)]&=W(fm')\,,\\
[Y(g_1),Y(g_2)]&=-W(g_1g_2'-g_1'g_2)/2\,,\\
[Z(h_1),Z(h_2)]&=-\epsilon_1W(h_1h_2'-h_1'h_2)/2,\\
[Y(g),Z(h)]&=[Y(g),W(m)]=[Z(h),W(m)]\\
&=[W(m_1),W(m_2)]=0\,.
\end{aligned} \tag{2.7}$$

We see that the DS Lie algebra L allows a Levi decomposition[17]

$$L=S\mathbin{\supset\!\!\!\!+}N, \tag{2.8}$$

where $S=\{X(f)\}$ is a simple infinite-dimensional Lie algebra and $N=\{Y(g),Z(h),W(m)\}$ is the radical of L, which in this case happens to be a nilpotent ideal.

The "obvious" physical symmetries of the DSE's are obtained by restricting all the functions f, g, h, and m to be first-order polynomials. We then have

$$\begin{aligned}
&P_0=X(1)=\partial_t,\quad P_1=Y(1)=\partial_x,\\
&P_2=Z(1)=\partial_y,\quad R_0=W(1)=v\,\partial_u-u\,\partial_v,\\
&D=X(t)=t\,\partial_t+(x\,\partial_x+y\,\partial_y\\
&\qquad-u\,\partial_u-v\,\partial_v)/2-w\,\partial_w\,,\\
&B_1=Y(t)=t\,\partial_x-x(v\,\partial_u-u\,\partial_v)/2,\\
&B_2=Z(t)=t\,\partial_y-\epsilon_1 y(v\,\partial_u-u\,\partial_v)/2\,,\\
&R_1=W(t)=t(v\,\partial_u-u\,\partial_v)+R_0\,.
\end{aligned} \tag{2.9}$$

We see that P_0, P_1, and P_2 generate translations in the t, x, and y directions, respectively; D corresponds to dilations, B_1 and B_2 to Galilei boosts in the x and y directions, respectively. Finally R_0 corresponds to a rotation in the (u,v) plane, i.e., a constant change of phase of $\Psi(x,y,t)$ and R_1 to a change of phase of Ψ, linear in t, accompanied by constant shift in w (see below).

B. Loop structure of the DS symmetry algebra

Similarly as the algebra of the Kadomtsev–Petviashvili equation,[7] the DS symmetry algebra for $\delta_1=-\epsilon_1$ (and only in this case) can be embedded into a Kac–Moody type loop algebra.[13] To see this, let us restrict f, g, h, and l to be Laurent polynomials in t. A basis for this algebra is provided by the operators

$$\begin{aligned}
X(t^n)&=t^n\,\partial_t+nt^{n-1}\Delta/2-n(n-1)t^{n-2}A_1/4\\
&\qquad-n(n-1)(n-2)t^{n-3}W_1/4\,,\\
Y(t^n)&=t^nX-nt^{n-1}A_2/2-n(n-1)t^{n-2}W_2/2\,,\\
Z(t^n)&=t^nY-\epsilon_1 nt^{n-1}A_3/2-\epsilon_1 n(n-1)t^{n-2}W_3/2\,,\\
W(t^n)&=t^nA_4+nt^{n-1}W_4\,,
\end{aligned} \tag{2.10}$$

where we have introduced the notation

$$\begin{aligned}
&\Delta=x\partial_x+y\partial_y-u\partial_u-v\partial_v-2w\partial_w\,,\\
&X=\partial_x,\quad Y=\partial_y\\
&A_1=\tfrac{1}{2}(x^2+\epsilon_1 y^2)(v\,\partial_u-u\,\partial_v),\quad A_2=x(v\,\partial_u-u\,\partial_v),\\
&A_3=y(v\,\partial_u-u\,\partial_v),\quad A_4=v\,\partial_u-u\,\partial_v,\\
&W_1=\tfrac{1}{2}(x^2+\epsilon_1 y^2)\partial_w,\quad W_2=x\,\partial_w,\\
&W_3=y\,\partial_w,\quad W_4=\partial_w.
\end{aligned} \tag{2.11}$$

The operators (2.11) form the basis of an 11-dimensional solvable Lie algebra. It has a ten-dimensional nilpotent ideal, the nilradical $\mathrm{NR}(L)=\{X,Y,A_1,A_2,A_3,A_4,W_1,W_2,W_3,W_4\}$. In turn the algebra $\mathrm{NR}(L)$ has an eight-dimensional uniquely defined maximal Abelian ideal $\{A_i\,W_i,\,i=1,\dots,4\}$. The algebra (2.11) can be embedded into the simple Lie algebra $\mathrm{sl}(7,\mathbb{C})$. Indeed, consider the $\mathrm{sl}(7,\mathbb{C})$ matrix

$$\left\{\begin{array}{ccccccc} \delta & 0 & x+\sqrt{-\epsilon_1}y & 0 & w_2+\epsilon_1(-\epsilon_1)^{1/2}w_3 & 0 & -2w_4 \\ 0 & -\delta & 0 & x+\sqrt{-\epsilon_1}y & a_2+\epsilon_1(-\epsilon_1)^{1/2}a_3 & 0 & -2a_4 \\ 0 & 0 & 2\delta & 0 & cw_1 & 0 & -w_2+\epsilon_1(-\epsilon_1)^{1/2}w_3 \\ 0 & 0 & 0 & 0 & ca_1 & 0 & a_2+\epsilon_1(-\epsilon_1)^{1/2}a_3 \\ 0 & 0 & 0 & 0 & -2\delta & 0 & x-(-\epsilon_1)^{1/2}y \\ 0 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta \end{array}\right\}. \tag{2.12}$$

Setting all entries but one equal to 0 and the remaining one equal to 1, we obtain 11 matrices having the same commutation relations as the vector fields (2.11) [$\delta=1$ corresponds to the operator Δ, $x=1$ or $y=1$ to X or Y, respectively, $a_i=1$ or $w_i=1$ to A_i or W_i, respectively ($i=1,...,4$)]. This embedding provides us with an identification of the algebra generated by $X(t^n)$, $Y(t^n)$, $Z(t^n)$, and $W(t^n)$ of (2.10). We have obtained an infinite-dimensional subalgebra of the affine loop algebra,

$$\hat{\mathrm{sl}}(7,\mathbb{C})=\left\{[\mathbb{R}(t,t^{-1})\otimes \mathrm{sl}(7,\mathbb{C})]\oplus R(t,t^{-1})\frac{d}{dt}\right\}. \tag{2.13}$$

The vector fields $X(t^n)$ form a simple subalgebra isomorphic to the $\mathbb{Z}$-graded algebra $\{R(t,t^{-1})d/dt\}$, which is in turn isomorphic to the Virasoro algebra (without a central extension).[13] Notice also that each element of (2.10) has a well-defined degree in a natural grading obtained by attributing the degree n to the monomial t^n and the degree μ ($0\leqslant\mu\leqslant 6$) to each element of (2.11), where μ is the distance from the diagonal in the matrix (2.12) to the corresponding element ($\mu=0$ for Δ, $\mu=1$ for A_1, $\mu=2$ for X, Y, and W_1, etc.). The degrees of $X(t^n)$, $Y(t^n)$, $Z(t^n)$, and $W(t^n)$ are thus $n-1$, $n+2$, $n+2$, and $n+5$, respectively.

C. The group transformations

The elements of the connected part of the symmetry group of the DSE's are obtained by integrating the general element of the DS Lie algebra (2.4). We consider separately the cases $f(t)=0$ and $f(t)\neq 0$. In each case we write the vector field V in the form (2.2), where η_i and ϕ_i must be specified, and integrate the equations

$$\frac{d\tilde{x}}{d\lambda}=\eta_1,\quad \frac{d\tilde{y}}{d\lambda}=\eta_2,\quad \frac{d\tilde{t}}{d\lambda}=\eta_3,$$
$$\frac{d\tilde{u}}{d\lambda}=\phi_1,\quad \frac{d\tilde{v}}{d\lambda}=\phi_2,\quad \frac{d\tilde{w}}{d\lambda}=\phi_3, \tag{2.14}$$

where

$$\eta_i=\eta_i(\tilde{x},\tilde{y},\tilde{t},\tilde{u},\tilde{v},\tilde{w}),\quad \phi_i=\phi_i(\tilde{x},\tilde{y},\tilde{t},\tilde{u},\tilde{v},\tilde{w}).$$

The boundary conditions are

$$\tilde{x}|_{\lambda=0}=x,\quad \tilde{y}|_{\lambda=0}=y,\quad \tilde{t}|_{\lambda=0}=t,$$
$$\tilde{u}|_{\lambda=0}=u,\quad \tilde{v}|_{\lambda=0}=v,\quad \tilde{w}|_{\lambda=0}=w. \tag{2.15}$$

The results of this integration are presented below. For each of the two cases mentioned above, we give the transformed variables and the expression for the new solution in terms of the original one.

(i) Case $f(t)=0$,

$$\tilde{x}(\lambda)=x+\lambda g(t),\quad \tilde{y}(\lambda)=y+\lambda h(t),\quad \tilde{t}(\lambda)=t,$$
$$\tilde{\Psi}(\tilde{x},\tilde{y},\tilde{t})=\Psi(\tilde{x}-\lambda g(\tilde{t}),\tilde{y}-\lambda h(\tilde{t}),\tilde{t})\exp i\{(\lambda/2)g'(\tilde{t})[\tilde{x}-(\lambda/2)g(\tilde{t})]$$
$$+(\epsilon_1/2)\lambda h'(\tilde{t})[\tilde{y}-(\lambda/2)h(\tilde{t})]-\lambda m(\tilde{t})\}, \tag{2.16}$$
$$\tilde{w}(\tilde{x},\tilde{y},\tilde{t})=w[\tilde{x}-\lambda g(\tilde{t}),\tilde{y}-\lambda h(\tilde{t}),\tilde{t})]-(\lambda/2)g''(\tilde{t})[\tilde{x}-(\lambda/2)g(\tilde{t})]-(\epsilon_1/2)\lambda h''(\tilde{t})[\tilde{y}-(\lambda/2)h(\tilde{t})]+\lambda m'(\tilde{t}).$$

Setting $g(t)=h(t)=0$ in (2.16), we see that the presence of $W(m)$ in the symmetry algebra simply means that the DSE's are invariant under an arbitrary time dependent change in the phase of Ψ, compensated by an appropriate transformation of w.

(ii) Case $f(t)\neq 0$,

$$G(t,\tilde{t})=f^{1/2}(t)\int_t^{\tilde{t}}g(s)f^{-3/2}(s)ds,\quad H(t,\tilde{t})=f^{1/2}(t)\int_t^{\tilde{t}}h(s)f^{-3/2}(s)ds,\quad \phi'(t)=\frac{1}{f(t)} \tag{2.17}$$

[$\phi(t)$ can be any antiderivative of $1/f(t)$], we have

$$\tilde{x}(\lambda)=[x+G(t,\tilde{t}(\lambda))][f(\tilde{t}(\lambda))/f(t)]^{1/2},\quad \tilde{y}(\lambda)=[y+H(t,\tilde{t}(\lambda))][f(\tilde{t}(\lambda))/f(t)]^{1/2},\quad \tilde{t}(\lambda)=\phi^{-1}(\phi(t)+\lambda),$$

$$\tilde{\Psi}(\tilde{x},\tilde{y},\tilde{t})=\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2}\tilde{\Psi}(x,y,t)\exp i\left\{(\tilde{x}^2+\epsilon_1\tilde{y}^2)\left\{\frac{[f'(\tilde{t})-f'(t)]}{8f(\tilde{t})}\right\}\right.$$
$$+\frac{1}{2}\tilde{x}[f(t)f(\tilde{t})]^{-1/2}\left[g(\tilde{t})\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2}-g(t)+\frac{1}{2}f'(t)G(t,\tilde{t})\right]$$
$$+\frac{\epsilon_1}{2}\tilde{y}[f(t)f(\tilde{t})]^{-1/2}\left[h(\tilde{t})\left[\frac{f(t)}{f(\tilde{t})}\right]^{1/2}-h(t)+\frac{1}{2}f'(t)H(t,\tilde{t})\right]$$

$$+\frac{1}{2f(t)}[g(t)G(t,\tilde t)+\epsilon_1 h(t)H(t,\tilde t)]-\frac{f'(t)}{8f(t)}[G^2(t,\tilde t)+\epsilon_1 H^2(t,\tilde t)]$$

$$\left.-\int_t^{\tilde t}\frac{g^2(s)+\epsilon_1 h^2(s)+2m(s)f(s)}{2f^2(s)}ds\right\},$$

$$\tilde w(\tilde x,\tilde y,\tilde t)=[f(t)/f(\tilde t)]w(x,y,t)$$

$$-\frac{1}{8}(\tilde x^2+\epsilon_1\tilde y^2)f^{-2}(\tilde t)\left[f''(\tilde t)f(\tilde t)-\frac{1}{2}[f'(\tilde t)]^2-f''(t)f(t)+\frac{1}{2}[f'(t)]^2\right] \tag{2.18}$$

$$-\frac{1}{4}\tilde x f^{-1/2}(t)f^{-3/2}(\tilde t)\left\{[2g'(\tilde t)f(\tilde t)-g(\tilde t)f'(\tilde t)]\left[\frac{f(t)}{f(\tilde t)}\right]^{1/2}\right.$$

$$\left.-[2g'(t)f(t)-g(t)f'(t)]+\left[f''(t)f(t)-\frac{1}{2}[f'(t)]^2\right]G(t,\tilde t)\right\}$$

$$-\frac{\epsilon_1}{4}\tilde y f^{-1/2}(t)f^{-3/2}(\tilde t)\left\{[2h'(\tilde t)f(\tilde t)-h(\tilde t)f'(\tilde t)]\left[\frac{f(t)}{f(\tilde t)}\right]^{1/2}\right.$$

$$\left.-[2h'(t)f(t)-h(t)f'(t)]+\left[f''(t)f(t)-\frac{1}{2}[f'(t)]^2\right]H(t,\tilde t)\right\}$$

$$-\frac{1}{4f(t)f(\tilde t)}\{[2g'(t)f(t)-g(t)f'(t)]G(t,\tilde t)+\epsilon_1[2h'(t)f(t)-h(t)f'(t)]H(t,\tilde t)\}$$

$$+\frac{1}{8f(t)f(\tilde t)}\left[f''(t)f(t)-\frac{1}{2}[f'(t)]^2\right][G^2(t,\tilde t)+\epsilon_1 H^2(t,\tilde t)]$$

$$+\frac{g^2(\tilde t)+\epsilon_1 h^2(\tilde t)+4m(\tilde t)f(\tilde t)}{4f^2(\tilde t)}-\frac{g^2(t)+\epsilon_1 h^2(t)+4m(t)f(t)}{4f(t)f(\tilde t)}.$$

The variables x, y, and t appearing on the right-hand side of the expressions for Ψ and w in (2.18) are to be interpreted as functions of $\tilde x$, $\tilde y$, and $\tilde t$, i.e.,

$$x=[\tilde x+G(\tilde t,t)][f(t)/f(\tilde t)]^{1/2},$$
$$y=[\tilde y+H(\tilde t,t)][f(t)/f(\tilde t)]^{1/2},\quad t=\phi^{-1}(\phi(\tilde t)-\lambda). \tag{2.19}$$

Note that by construction, a one-dimensional Lie subgroup of transformations is generated if one fixes the functions g, h, l in (2.16), or f, g, h, l in (2.18), and then allows the parameter λ to take on arbitrary real values.

The expressions for $\tilde\Psi$ and $\tilde w$ in (2.16) and (2.18) can be used to generate new solutions of the DSE's from known ones. More precisely, if (Ψ,w) is a local solution of the DSE's in the neighborhood of (x,y,t) then $(\tilde\Psi,\tilde w)$ given by (2.16) or (2.18) will be a local solution of the DSE's in the neighborhood of $(\tilde x(\lambda),\tilde y(\lambda),\tilde t(\lambda))$. In particular, the application of the transformations (2.16) and (2.18) to the "trivial" constant solution

$$\Psi(x,y,t)=\Psi_0,\quad w(x,y,t)=-\epsilon_1|\Psi_0|^2, \tag{2.20}$$

provides us with a family of solutions depending on three arbitrary functions of time in the case of (2.16) and four arbitrary functions of time in the case of (2.18).

By introducing new functions of time, it is possible to obtain a much more simple expression than (2.18) for the elements of the symmetry group of the DSE's when $f(t)\neq 0$. Let $a(t)$, $b(t)$, $c(t)$, and $e(t)$ be arbitrary real-valued functions of class C^∞ with the restriction that $a(t)\neq 0$. Then, for a fixed value of the parameter λ_0, take $f(t)$, $g(t)$, $h(t)$, and $l(t)$ in (2.17) and (2.18) to be the solutions of the following system of functional equations:

$$a(\tilde t)=f(\tilde t)/f(t),$$

$$\frac{b(\tilde t)}{a(\tilde t)}=[f(t)f(\tilde t)]^{-1/2}\left\{g(\tilde t)\left[\frac{f(t)}{f(\tilde t)}\right]^{1/2}-g(t)+\frac{1}{2}f'(t)G(t,\tilde t)\right\},$$

$$\frac{c(\tilde t)}{a(\tilde t)}=[f(t)f(\tilde t)]^{-1/2}\left\{h(\tilde t)\left[\frac{f(t)}{f(\tilde t)}\right]^{1/2}-h(t)+\frac{1}{2}f'(t)H(t,\tilde t)\right\},$$

$$\int_t^{\tilde t}\frac{b^2(s)+\epsilon_1 c^2(s)+2e(s)a(s)}{a^2(s)}ds=\int_t^{\tilde t}\frac{g^2(s)+\epsilon_1 h^2(s)+2m(s)f(s)}{f^2(s)}ds \tag{2.21}$$

$$-\frac{1}{f(t)}[g(t)G(t,\tilde t)+\epsilon_1 h(t)H(t,\tilde t)]+\frac{f'(\tilde t)}{4f(t)}[G^2(t,\tilde t)+\epsilon_1 H^2(t,\tilde t)],$$

$$\tilde t=\phi^{-1}(\phi(t)+\lambda_0),\quad \phi'(t)=1/f(t).$$

Under these conditions, it can be verified that for appropriate constants of integration c_1, c_2, c_3, and c_4, the transformations (2.18) reduce to the following transformations when $\lambda=\lambda_0$ (and only for this value of λ):

$$\tilde x=\left[x+\int b(\tilde t)a^{-3/2}(\tilde t)d\tilde t+c_1\right]a^{1/2}(\tilde t),$$

$$\tilde y=\left[y+\int c(\tilde t)a^{-3/2}(\tilde t)d\tilde t+c_2\right]a^{1/2}(\tilde t),$$

$$\tilde{t}=\zeta^{-1}(t),\quad \zeta(t)=\int\frac{dt}{a(t)}+c_3,$$

$$\tilde{\Psi}(\tilde{x},\tilde{y},\tilde{t})=a^{-1/2}\Psi(x,y,t)\exp i\left\{\frac{a'}{8a}(\tilde{x}^2+\epsilon_1\tilde{y}^2)+\frac{b}{2a}\tilde{x}+\frac{\epsilon_1 c}{2a}\tilde{y}-\frac{1}{2}\int\frac{b^2+\epsilon_1 c^2+2ea}{a^2}\,d\tilde{t}-c_4\right\},\tag{2.22}$$

$$\tilde{w}(\tilde{x},\tilde{y},\tilde{t})=a^{-1}w(x,y,t)-\frac{1}{8}\left(\frac{aa''-\frac{1}{2}a'^2}{a^2}\right)(\tilde{x}^2+\epsilon_1\tilde{y}^2)-\frac{1}{4}\left(\frac{2b'a-ba'}{a^2}\right)\tilde{x}-\frac{\epsilon_1}{4}\left(\frac{2c'a-ca'}{a^2}\right)\tilde{y}+\frac{1}{4}\left(\frac{b^2+\epsilon_1c^2+4ea}{a^2}\right).$$

The functions a, b, c, e, and the derivatives of these functions appearing in the right-hand side of the expressions for $\tilde{\Psi}$ and $\tilde{w}$ in (2.22) are all evaluated at $\tilde{t}$. Moreover, the variables x, y, t appearing in the argument of Ψ and w in the same expressions are to be interpreted as functions of $\tilde{x}$, $\tilde{y}$, $\tilde{t}$, i.e.,

$$\begin{aligned}x&=\tilde{x}a^{-1/2}(\tilde{t})-\int b(\tilde{t})a^{-3/2}(\tilde{t})d\tilde{t}-c_1,\\ y&=\tilde{y}a^{-1/2}(\tilde{t})-\int c(\tilde{t})a^{-3/2}(\tilde{t})d\tilde{t}-c_2,\\ t&=\zeta(\tilde{t}).\end{aligned}\tag{2.23}$$

It should be pointed out that the constants c_1, c_2, c_3, and c_4 can be omitted when using (2.22) since they can always be removed by applying the transformation $\exp\{-X(c_3)-Y(c_1)-Z(c_2)-W(c_4)\}$ prior to the application of (2.22).

Finally, let us mention that the DSE's (1.1) are also invariant under a group of discrete transformations, generated by the transformations

$$\begin{array}{lllll}x\to -x, & y\to y, & t\to t, & \Psi\to\Psi, & w\to w,\\ x\to x, & y\to -y, & t\to t, & \Psi\to\Psi, & w\to w,\\ x\to x, & y\to y, & t\to t, & \Psi\to -\Psi, & w\to w,\\ x\to x, & y\to y, & t\to -t, & \Psi\to\Psi^*, & w\to w.\end{array}\tag{2.24}$$

III. ONE- AND TWO-DIMENSIONAL SUBALGEBRAS OF THE DAVEY–STEWARTSON SYMMETRY ALGEBRA

In order to perform symmetry reduction for the DSE's in a systematic manner, we need to know all subgroups of the symmetry group having generic orbits of codimension 1 and 2 in the $\{x,y,t\}$ space. This is equivalent to performing a classification of all one- and two-dimensional subalgebras of the DS algebra into conjugacy classes under the adjoint action of the DS group, i.e., the group leaving the equations invariant.

The method is exactly the same as the one employed recently for the Kadomtsev–Petviashvili equation,[7] and is an adaptation of methods developed earlier for classifying subalgebras of finite-dimensional Lie algebras.[18,19]

The first step is to classify subalgebras of the factor algebra $S=\{X(F)\}$ in the Levi decomposition (2.8) For this we can use results obtained earlier[7] for an isomorphic algebra. Thus every nontrivial one-dimensional subalgebra of S is conjugate to $\{X(1)\}$ and every two-dimensional subalgebra to $\mathrm{aff}(1,\mathbf{R})=\{X(1),X(t)\}$.

One-dimensional subalgebras of the entire DS algebra will thus have the form $\{X(1)+Y(g)+Z(h)+W(m)\}$, or $\{Y(g)+Z(h)+W(m)\}$. Using the transformations (2.16)–(2.19) we can show that every subalgebra of the first type is conjugate to $X(1)$.

The subalgebras of the second type split into several classes depending on which of the functions $g(t)$, $h(t)$, and $m(t)$ are nonzero (in the considered t interval). We drop all details and present representatives of each conjugacy class of one-dimensional subalgebras of the DS algebra in Table I. The classification is under the entire DS group including the discrete transformations (2.24).

In column 1 we introduce a name for each class of subalgebras. In column 2 we give the basis element for each representative subalgebra. In column 3 we present the normalizer of each subalgebra in the DS algebra, i.e., the maximal subalgebra $L_0\subset L$ satisfying

$$[X,X_0]=\lambda X_0,\quad X\in L_0,\tag{3.1}$$

where $\lambda\in\mathbf{R}$ is a constant and X_0 is the corresponding basis element in column 2. In column 4 we give the conditions

TABLE I. One-dimensional subalgebras of the Davey–Stewartson algebra ($a\geqslant 0$ and $\lambda\in\mathbf{R}$ are constants, h, F, H, G, and L are functions of t).

No.	Basis element	Normalizer	Characterization of conjugacy class
$L_{1,1}$	$X(1)$	$X(t)$, $X(1)$, $Y(1)$, $Z(1)$, $W(1)$	$f\neq 0$
$L^a_{1,2}$	$Y(1)+aZ(1)$	$X(t)$, $X(1)$, $Y(-\epsilon_1 aH)+Z(H)$ $Y(1)$, $Z(1)$, $W(L)$	$f=0$, $h=\pm ag\neq 0$
$L_{1,3}(h)$ $h'\neq 0$	$Y(1)+Z(h)$	$-\epsilon_1 Y[\int_0^t(hH'-h'H)ds]+Z(H)$, $Y(1)$, $Z(h)$, $W(L)$	$f=0$, $g\neq 0$ $h\neq\lambda g$
$L_{1,4}$	$Z(1)$	$X(t)$, $X(1)$, $Y(G)$, $Z(1)$, $W(L)$	$f=g=0$, $h\neq 0$
$L_{1,5}$	$W(t)$	$X(t)$, $Y(G)$, $Z(H)$, $W(L)$	$f=g=h=0$, $m\neq 0$
$L_{1,6}$	$W(1)$	$X(F)$, $Y(G)$, $Z(H)$, $W(L)$	$f=g=h=0$ $m=\lambda\neq 0$

under which a general element of the form (2.4) can be transformed into a constant multiple of the element in column 2.

Two isomorphy classes of two-dimensional Lie algebras $\{X_1,X_2\}$ exist, Abelian ($2A_1$) and non-Abelian (A_2), with commutation relation

$$[X_1,X_2]=0 \quad \text{or} \quad [X_1,X_2]=X_1\,, \tag{3.2}$$

respectively. To obtain all such algebras we let X_1 run through all the standard forms of Table I. The other element X_2 must then lie in the normalizer nor$\{X_1\}$ and can be further simplified using the Lie group Nor$\{X_1\}$ corresponding to the algebra nor$\{X_1\}$.

The results are summarized in Table II. Certain redundancies have been left in Tables I and II. Thus two one-dimensional subalgebras $L_{1,3}(h_1)$ and $L_{1,3}(h_2)$ are conjugate to each other if there exist two constants λ and μ, such that

$$h_2(t)=h_1(\lambda t+\mu)\,. \tag{3.3}$$

Similar redundancies exist in Table II and can be removed, e.g., by fixing the values of the function $h(t)$ and its derivative at some point $t=t_0$. Since this has no consequences for symmetry reduction, we shall not dwell on it here.

IV. SYMMETRY REDUCTION FOR THE DAVEY–STEWARTSON EQUATIONS

We shall now use the results of the previous sections to reduce the DSE's to a system of equations involving two independent variables only. To do this we make use of the one-dimensional subalgebras of the DS algebra, listed in Table I. The method is standard and quite simple. We consider an auxiliary function $F(x,y,t,u,v,w)$ and request that it be annihilated by the elements of the one-dimensional subalgebra $\{X\}$:

$$XF=0. \tag{4.1}$$

Equation (4.1) implies that F is a function of five variables only, namely the invariants of the Lie group generated by X. Two invariants ξ and η can be chosen to depend on x, y, and t only, these are the new symmetry variables. The remaining invariants yield the dependence of u, v, and w (i.e., ψ and w) on the symmetry variables.

Only vector fields involving derivatives with respect to the independent variables yield reductions. Hence we shall only use the subalgebras $L_{1,1},...,L_{1,4}$ of Table I. We shall perform the reduction using the "standard" basis elements of Table I. The result for a general vector field (2.4) is obtained from the results for a simplified one by applying a general group transformation (2.16)–(2.24).

A. The algebra $L_{1,1}$

The equation $X(1)F(x,y,t,u,v,w)=0$ tells us that the invariants of $\exp X(1)$ are x, y, u, v, and w. The reduction is hence obtained by setting

$$\Psi(x,y,t)=\phi(\xi,\eta),\quad \xi=x,\quad \eta=y,$$
$$w(x,y,t)=Q(\xi,\eta)\,. \tag{4.2}$$

Substituting into the DSE's (1.1) we obtain the reduced system

$$\phi_{\xi\xi}+\epsilon_1\phi_{\eta\eta}=\epsilon_2|\phi|^2\phi+\phi Q\,, \tag{4.3a}$$

$$Q_{\xi\xi}+\delta_1 Q_{\eta\eta}=\delta_2(|\phi|^2)_{\eta\eta}\,. \tag{4.3b}$$

Applying a general DS group transformation to a solution of (4.3) we obtain a class of solutions of the DSE's, depending on four arbitrary functions $f(t)$, $g(t)$, $h(t)$, and $l(t)$. Thus assuming $f(t)\neq 0$, we obtain

$$\Psi=\phi(\xi,\eta)f^{-1/2}\exp i\left[\frac{1}{8}(x^2+\epsilon_1 y^2)\frac{f'}{f}+\frac{1}{2f}(xg+\epsilon_1 yh)-\frac{1}{2}\int\frac{\epsilon_1 h^2+g^2+2mf}{f^2}ds\right],$$

$$W=Q(\xi,\eta)\frac{1}{f}-\frac{1}{8f^2}\left(ff''-\frac{1}{2}f'^2\right)(x^2+\epsilon_1 y^2)-\frac{x}{4f^2}(2g'f-gf')-\frac{\epsilon_1 y}{4f^2}(2h'f-hf')+\frac{1}{4}\frac{g^2+\epsilon_1 h^2+4mf}{f^2}, \tag{4.4}$$

$$\xi=xf^{-1/2}-\int_0^t g(s)[f(s)]^{-3/2}ds\,,$$

$$\eta=yf^{-1/2}-\int_0^t h(s)[f(s)]^{-3/2}ds\,.$$

Substituting (4.4) into the DSE's (1.1) we find that $\phi(\xi,\eta)$ and $Q(\xi,\eta)$ must satisfy Eqs. (4.3a) and

$$8[Q_{\xi\xi}+\delta_1 Q_{\eta\eta}-\delta_2(|\phi|^2)_{\eta\eta}]=(\delta_1\epsilon_1+1)[2ff_{tt}-(f_t)^2], \tag{4.5}$$

which reduces to (4.3b) if $\delta_1=-\epsilon_1$ or if $f(t)=(a+bt)^2$ [see (2.6)].

TABLE II. Two-dimensional subalgebras of the DS algebra ($a\geqslant 0$, $b\in\mathbf{R}$, $k=0$, are constants, h, H, and m are functions of t).

No.	Type	Basis element
$L_{2,1}^{a,k}$	$2A_1$	$X(1),\ Y(1)+aZ(1)+kW(1)$
$L_{2,2}^{k}$	$2A_1$	$X(1),\ Z(1)+kW(1)$
$L_{2,3}$	$2A_1$	$X(1),\ W(1)$
$L_{2,4}^{a,b,h}$	$2A_1$	$Y(1)+aZ(1),\ Y(-\epsilon_1 ah)+Z(h)+bZ(1)$
$L_{2,5}^{a,h,m}$	$2A_1$	$Y(1)+aZ(1),\ Y(-\epsilon_1 ah)+Z(h)+W(m)$ $[m=0$ if $(a^2,\epsilon_1)\neq(1,-1)]$
$L_{2,6}^{h,H}$, $h'\neq 0$	$2A_1$	$Y(1)+Z(h),\ -\epsilon_1 Y[\int_0^t(hH'-h'H)ds]+Z(H)$
$L_{2,7}^{h,m}$, $h'\neq 0$	$2A_1$	$Y(1)+Z(h),\ W(m)$
$L_{2,8}^{m}$	$2A_1$	$Z(1),\ W(m)$
$L_{2,9}^{m,}$	$2A_1$	$W(t),\ W(m)$
$L_{2,10}^{k}$	A_2	$X(1),\ X(t)+kW(1)$
$L_{2,11}^{a,}$	A_2	$Y(1)+aZ(1),\ 2X(t)$
$L_{2,12}$	A_2	$Z(1),\ 2X(t)$
$L_{2,13}$	A_2	$W(t),\ -X(t)$

B. The algebra $L^a_{1,2}$

The equation $[Y(1)+aZ(1)]F=0$ implies a reduction obtained by setting

$$\psi(x,y,t)=\Omega(\xi,\zeta),\quad w(x,y,t)=Q(\xi,\zeta),\quad \xi=t,\quad \zeta=y-ax\,. \tag{4.6}$$

By substituting into the DS equations we obtain the reduced system

$$i\Omega_\xi+(a^2+\epsilon_1)\Omega_{\zeta\zeta}=\epsilon_2|\Omega|^2\Omega+Q\Omega\,, \tag{4.7a}$$

$$(a^2+\delta_1)Q_{\zeta\zeta}=\delta_2(|\Omega|^2)_{\zeta\zeta}\,. \tag{4.7b}$$

This system can be further simplified. We solve the second equation (choosing $a^2\neq-\delta_1$):

$$Q(\xi,\zeta)=[\delta_2/(a^2+\delta_1)]|\Omega|^2+\alpha(\xi)\zeta+\beta(\xi)\,, \tag{4.8}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are arbitrary functions. Expression (4.8) can be substituted back into Eq. (4.7a) and we obtain an equation for $\Omega(\xi,\zeta)$ alone. A transformation of the dependent and independent variables can be found that takes (4.7a) into the nonlinear Schrödinger equation. The final result is

$$\begin{aligned}\Psi(x,y,t)&=\{\epsilon_4(a^2+\delta_1)/[\epsilon_2(a^2+\delta_1)+\delta_2]\}^{1/2}\phi(\xi,\eta)\\&\quad\times\exp i[(y-ax)F(t)+G(t)]\,,\\ w(x,y,t)&=\{\delta_2\epsilon_4/[\epsilon_2(a^2+\delta_1)+\delta_2]\}|\phi(\xi,\eta)|^2\\&\quad+\alpha(t)(y-ax)+\beta(t)\,,\\ F(t)&=-\int\alpha(t)dt,\\ G(t)&=-\int[(a^2+\epsilon_1)F^2(t)+\beta(t)]dt\,,\\ H(t)&=-2[\epsilon_3(a^2+\epsilon_1)]^{1/2}\int F(t)dt,\\ \epsilon_4&=\operatorname{sgn}\frac{a^2+\delta_1}{\epsilon_2(a^2+\delta_1)+\delta_2}\,,\\ \xi&=\epsilon_3t,\quad \eta=[\epsilon_3(a^2+\epsilon_1)]^{-1/2}(y-ax)+H(t),\\ \epsilon_3&=\operatorname{sgn}(a^2+\epsilon_1)\,.\end{aligned} \tag{4.9}$$

Here $\alpha(t)$ and $\beta(t)$ are arbitrary functions of time, a is a constant, and $\phi(\xi,\eta)$ satisfies the nonlinear Schrödinger equation

$$i\phi_\xi+\phi_{\eta\eta}=\epsilon_3\epsilon_4\phi|\phi|^2\,. \tag{4.10}$$

We shall not present the more general solution, obtained by applying a general DS group element to the solution (4.9).

C. The algebra $L_{1,3}(h)$

We have

$$[Y(1)+Z(h)]F=\{\partial_x+h\,\partial_y-(\epsilon_1/2)y[h'(v\,\partial_u-u\,\partial_v)+h''\,\partial_w]\}F=0\,. \tag{4.11}$$

The characteristic system for (4.11) is

$$\frac{dx}{1}=\frac{dy}{h}=-\frac{2\,du}{\epsilon_1yh'v}=\frac{2\,dv}{\epsilon_1yh'u}=\frac{2\,dw}{\epsilon_1yh''}\,. \tag{4.12}$$

By solving (4.12) we obtain

$$\Psi=\phi(\xi,\eta)\exp[i(\epsilon_1/4)(h'/h)y^2],\quad \xi=t,\quad W=Q(\xi,\eta)-(\epsilon_1/4)(h''/h)y^2,\quad \eta=y-h(t)x\,, \tag{4.13}$$

where the DSE's imply

$$i\phi_\xi+(\epsilon_1+h^2)\phi_{\eta\eta}+\frac{ih'}{h}\eta\phi_\eta+\frac{i}{2}\frac{h'}{h}\phi=\epsilon_2|\phi|^2\phi+\phi Q\,, \tag{4.14a}$$

$$(h^2+\delta_1)Q_{\eta\eta}-\frac{\epsilon_1\delta_1}{2}\frac{h''}{h}=\delta_2(|\phi|^2)_{\eta\eta}\,. \tag{4.14b}$$

The system (4.14) can be further simplified. Solving (4.14b) and substituting into (4.14a), we find

$$Q=\frac{\delta_2}{h^2+\delta_1}|\phi|^2+\frac{\epsilon_1\delta_1}{4(h^2+\delta_1)}\frac{h''}{h}\eta^2+\alpha(t)\eta+\beta(t)\,, \tag{4.15}$$

$$\begin{aligned}&i\phi_\xi+(\epsilon_1+h^2)\phi_{\eta\eta}+\frac{ih'}{h}\eta\phi_\eta\\&\quad+\left(\frac{i}{2}\frac{h'}{h}-\frac{\epsilon_1\delta_1}{4(h^2+\delta_1)}\frac{h''}{h}\eta^2-\alpha\eta-\beta\right)\phi\\&\quad=\left(\epsilon_2+\frac{\delta_2}{h^2+\delta_1}\right)|\phi|^2\phi\,.\end{aligned} \tag{4.16}$$

Equation (4.16) can be reduced to a nonlinear Schrödinger equation with variable coefficients. To see this, set

$$Q=A(t)\Omega(\xi,\zeta)\exp[i(\eta^2H+\eta F+G)]\,,\quad \zeta=\gamma(t)\eta+K(t)\,. \tag{4.17}$$

We choose $H(t)$ to satisfy a Riccati equation

$$H'+4(\epsilon_1+h^2)H^2+\frac{\epsilon_1\delta_1}{4(h^2+\delta_1)}\frac{h''}{h}+2\frac{h'}{h}H=0 \tag{4.18}$$

and the other functions in (4.17) to satisfy

$$\begin{aligned}&F'+4(\epsilon_1+h^2)HF+(h'/h)F+\alpha=0\,,\\&G'+(\epsilon_1+h^2)F^2+\beta=0\,,\\&A=h^{-1/2}\exp\left[-2\int(\epsilon_1+h^2)H\,dt\right],\\&K=-2\int(\epsilon_1+h^2)\gamma F\,dt,\quad \gamma=A^2.\end{aligned} \tag{4.19}$$

The function Ω in (4.17) then satisfies the equation

$$i\Omega_\xi+(\epsilon_1+h^2)A^4\Omega_{\xi\xi}=(\epsilon_2+\delta_2/(h^2+\delta_1))A^2|\Omega|^2\Omega\,. \tag{4.20}$$

For $\delta_1=-\epsilon_1$ a particular solution of the Riccati equation (4.18) is

$$H=h'/4(h^2-\epsilon_1)h\,. \tag{4.21}$$

D. The algebra $L_{1,4}$

The algebra generated by $Z(1)=\partial_y$ leads in a simple manner to the nonlinear Schrödinger equation. Indeed a straightforward reduction with $\psi=\Omega(x,t)$, $w=Q(x,t)$ yields

$$i\Omega_t + \Omega_{xx} = \epsilon_2 |\Omega|^2 \Omega + \Omega Q, \quad Q_{xx} = 0 . \tag{4.22}$$

Putting

$$\Psi = \phi(t, \xi) e^{i[Fx + G]}, \quad w = \alpha(t) x + \beta(t), \quad \xi = x + H(t) , \tag{4.23}$$

with

$$F(t) = -\int \alpha(t) dt, \quad G(t) = -\int (F^2 + \beta) dt,$$

$$H(t) = -2 \int F\, dt ,$$

we find that $\phi(t,\xi)$ satisfies the nonlinear Schrödinger equation (1.2).

The algebras of Table II could be used to reduce the DSE's to various systems of nonlinear ordinary differential equations. These are easy to obtain and we shall not go into them here.

V. CONCLUSIONS

We have shown that the Davey–Stewartson equations (1.1) have an infinite-dimensional symmetry group. Moreover, for the integrable case when $\delta_1 = -\epsilon_1$ in (1.1), the symmetry Lie algebra has a loop algebra structure, similar to that of all other known integrable nonlinear differential equations in 2 + 1 dimensions.[7–12]

One-dimensional subalgebras of the symmetry algebra have been used in Sec. IV to reduce the DSE's to one of three two-dimensional systems. These are the system (4.3), the nonlinear Schrödinger equation (4.10) and Eq. (4.16). Large classes of solutions of the nonlinear Schrödinger equation are known (solitons, multisolitons, background radiation, quasiperiodic solutions).[4,5] The system (4.3) and Eq. (4.16) have, to our knowledge, not been studied in the literature. They merit a separate investigation and we plan to return to them in the future.

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