Team Decision Theory and Information Structures in Optimal Control Problems – Part I
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1 Introduction

1.1 Team Decision Problem

- A team is an organization in which there is a single goal or payoff common to all the members. Each member receives certain information \( z_i \) and controls the decision variable \( u_i \).

- The payoff of the decision problem \( (J) \) is defined to be a function of the control action taken by each member.

\[
J = J(\gamma_1, \gamma_2, \ldots, \gamma_N)
\]

- The control action \( u \) is a policy function \( \gamma \) that depends on the observation of each member.

\[
u_i = \gamma_i(z_i) \text{ and } \gamma_i \in \Gamma_i
\]

where \( \Gamma_i \) represents the set of admissible control law.

The objective of the team decision problem is to find \( \gamma_i^* \in \Gamma_i \forall i \) such that, \( J(\gamma_1, \gamma_2, \ldots, \gamma_N) \) is minimized.

1.2 Types of a Team Decision Problem

Team decision problem is of two types: static, and dynamics. In a static team decision problem, the information state depends only on some random variable \( \xi \) which represents the uncertainty of the external world which are not controlled by any member. While in dynamic team decision problem, the present information also depends on the past control actions.

Prior to this paper, only static team decision problems have been considered (Marschak and Radner). This paper by Yu–chi and Chu is a start point for tackling dynamics team decision problem, where they proved that if the information has a special structure (called partially nested), the dynamic problem can be converted into a static one and hence, the previously developed methods can still be employed for solving these particular problems.

1.3 Information Structure

The information \( z_i \) is assumed to be a known linear combination of \( \xi \) and some of the control actions of the predecessor members, i.e.

\[
z_i = H_i \xi + \sum_j D_{ij} u_j \quad \forall i
\]

where \( \xi \in \mathbb{R}^n \) defined on the probability space \( (\mathbb{R}^n, \mathbf{F}, P) \) represents all the uncertainty that cannot be controlled by any member.

Assume that:

- \( \xi \) is Gaussian \( N(0, X) \) with \( X > 0 \) and this probability distribution is know to all the members,

- \( H_i \) and \( D_{ij} \) are known to all members, and

- The system is causal, i.e. if \( D_{ij} \neq 0 \) then \( D_{ji} = 0 \), \( \forall i, j \in I \)

Definition 1. We say member \( j \) is related to member \( i \) i.e. \( jRi \), if \( D_{ij} \neq 0 \)

Definition 2. We say member \( j \) is precedent to member \( i \), denoted by \( j \{i \), iff

\( a \) \( jRi \) or

\( b \) \( \exists r, s, t, \ldots, k \in I \) such that \( jRr, rRs, \ldots, kRi \)

In graphical representation \( j \{i \) is equivalent to the existence of a path from member \( i \) to member \( j \).
Example 1. *In a static team decision problem with* $N$ *members, no precedence exists among members.*

\[
D_{ij} = 0 \quad \forall i, j \in I \\
z_i = H_i\xi \quad \forall i
\]

Figure 1 represent the precedence diagram for the static case.

![Figure 1: Precedence Diagram for a Static Team Decision Problem](image)


\[
x_{i+1} = Fx_i + Gu_i + \omega_i, \quad i = 1, \ldots, N
\]

with the observation at each stage:

\[
y_i = Hx_i + v_i \quad i = 1, \ldots, N
\]

The distribution of $x_1, w_i$, and $v_i$ are independent, zero-mean and Gaussian. The system is assumed to posses perfect memory i.e. $x_i$ = linear in $(x_1, u_1, \ldots, u_{i-1}, \omega_1, \ldots, \omega_{i-1}) \ \forall i$

Thus, the information state can be expressed as:

\[
z_i = H_i\xi + \sum_j D_{ij}u_j
\]

(3)

where $\xi = (x_1, u_1, \ldots, u_{i-1}, \omega_1, \ldots, \omega_{i-1})$. Figure 2 illustrate the fact that $z_j$ is included in $z_i$ if $i < j$. This fact is stressed by the dotted line in Figure 2.

![Figure 2: Precedence Diagram for Multi-stage stochastic control](image)

Example 3. *Consider the following partially nested Linear-Gaussian information structure*

\[
z_1 = H_1\xi \\
z_2 = H_2\xi \\
z_3 = H_3\xi + D_{31}u_1 + D_{32}u_2
\]
\[ z_4 = \begin{bmatrix} H_4' \\ H_3 \end{bmatrix} \xi + \begin{bmatrix} 0 & 0 & D_{43} \\ D_{31} & D_{32} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \]

\[ z_5 = D_{54}u_4 \]

\[ z_6 = H_6\xi + D_{64} \]

\[ z_7 = H_7\xi \]

\[ z_8 = D_{87}u_7 \]

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\end{array} \]

\[ \text{GROUP I} \quad \text{GROUP II} \]

Figure 3: Precedence Diagram for Example 3

Figure 3 represents the precedence diagram of example 3.

1.4 Control Laws

The set of admissible control laws for the \(i^{th}\) decision maker, \(\Gamma_i\), is defined as the the set of all Borel-measurable functions \(\gamma_i : \mathbb{R}^{q_i} \to \mathbb{R}^{k_i}\) [1]. If we take \(u_i \in U_i = \mathbb{R}^{k_i}\), then we have a \(\sigma\)-algebra \(F_i\) on \(U_i\) such that \(\gamma_i^{-1}(F_i) = Z_i\). It is important to note that except for example 1, \(F_i\) and \(Z_i\) \(\forall i\), are generally dependent on the choice of \(\gamma = [\gamma_1, ..., \gamma_N]\).

1.5 Payoff Function

The goal is to minimize \(J(\gamma_1, ..., \gamma_N)\). This goal is the same for all the members in the team. Here we are only considering quadratic utility functions i.e.

\[ J(\gamma_1, ..., \gamma_N) = E[\xi] = E[\frac{1}{2}u^T Qu + u^T S\xi + u^T c], \] (4)

where \(u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}\), \(Q > 0\), \(Q^T = Q\) and the expectation is taken with respect to a priori \(\xi\). Matrices, \(Q, S\), and vector \(c\) are known to all the members.

2 Necessary Conditions for optimality

A necessary condition for optimality is the so-called "member-by-member optimal" condition which is defined by finding \(\gamma_i(z_i) \in \Gamma_i\) such that \(u_i = \gamma_i(z_i)\) satisfies:

\[ \min_{u_i} E[\xi\{J(\gamma_1(z_1), ..., \gamma_{i-1}^*(z_{i-1}), \gamma_i(z_i), \gamma_{i+1}^*(z_{i+1}), ..., \gamma_N^*(z_N)|z_i}\}] \] (5)
For the considered quadratic utility function (4), this is equivalent to:

$$\min_{u_i} E\left\{ \frac{1}{2} u^T Qu + u^T S\xi + u^T c|z_i \right\} = \min_{u_i} J_i$$

(6)

where, $u^T = [\gamma_i^1(z_1), ..., \gamma_i^{q_i-1}(z_{i-1}), \gamma_i(z_i), \gamma_i^{q_i+1}(z_{i+1}), ..., \gamma_i^{q_i}(z_N)]$ for fixed $/\gamma_j$.

Taking partial derivatives of $J_i$ with respect to $u_i$ and keeping in mind that for $j \neq i$ the terms involving $u_j$ are dependent on $u_i$ due to causality, the necessary condition for optimality becomes:

$$Q_{ii}\gamma_i + \sum_{j \neq i} Q_{ij} E(\gamma_j|z_i) + S_i E(\xi|z_i) + c_i + \sum_{j \neq i} [u_i^T Q_{ij} \frac{\partial}{\partial u_i} E(\gamma_j|z_i) + \frac{\partial}{\partial u_i} E(\gamma_j^T S_j \xi|z_i) + \frac{\partial}{\partial u_i} E(\gamma_j^T c_j)] + \sum_{k \neq i} \frac{\partial}{\partial u_i} E(\gamma_k^T Q_{kj} \gamma_j|z_i)] = 0$$

(7)

The last four partial derivative terms of (7) depend explicitly on the form of controls of the following members. This is rather unsatisfactory in as much as we are attempting to solve for those control laws through the consideration of (7).

3 Static Team

As shown by Radner [2], in the case of static teams the quadratic utility function is strictly convex in all $\gamma_i$, and hence the necessary condition for member-by-member optimization is sufficient for optimality.

In static teams, each member’s information is a linear function of $\xi$ only i.e.

$$D_{ij} = 0 \text{ in (3)}$$

$$z_i = H_i \xi \quad \forall i$$

where $H_i$ is $q_i \times n$ with $n > q_i$ and it is of maximal rank i.e. $\text{rank}(H_i) = q_i$.

Thus, the necessary condition for optimality (7) simplifies to:

$$Q_{ii}\gamma_i + \sum_{j \neq i} Q_{ij} E(\gamma_j|z_i) + S_i E(\xi|z_i) + c_i = 0 \quad \forall z_i \quad \text{and} \quad \forall i$$

(8)

since all partial derivatives in (7) vanish. Also, the optimal solution for (4) is unique in a static team decision problem.

Because for a jointly Gaussian random variables the best estimator is linear, we seek control laws of the form:

$$u_i = \gamma_i(z_i) = A_i z_i + b_i \forall i$$

(9)

Solving (8) using (9) we get the following for $b$ and $A$:

$$b_i^T = c_t^T Q^{-1}$$

$$\sum_j Q_{ij} A_j (H_j X H_i^T) = -S_i X H_i^T, \forall i$$

(10)

(11)

The coefficient of the elements of $A_i$ in (11) form a positive definite matrix, hence each matrix $A_i$ is uniquely solved from (11).

As a conclusion (Radner’s Theorem) [2], the control law (9) with (10) and (11) is optimal for static-team optimization problem.

4 Dynamic Team with Partially Nested Information Structure

Based on Eq.(7), in a general case of dynamic teams neither the payoff function is necessarily convex in $\gamma_i$ even though it is quadratic in $u$ with $Q > 0$, nor are $z_i$ Gaussian even though $\xi$ is given as Gaussian. However, Ho
and Chu [1] proved that for a particular information structure called partially nested, there exists a transformed information structure \( \tilde{z}_i \) for which the utility function is strictly convex. Thus, the unique optimal control law is linear in \( \tilde{z}_i \).

**Definition 3:** An information structure (3) is called partially nested if \( j \{i \) implies \( Z_j \subset Z_i \) for all \( i, j, \) and \( \gamma \in \Gamma \).

**Theorem 1.** In a dynamic team with the partially nested information structure,

\[
z_i = H_i \xi + \sum_{j \{i} D_{ij} u_j
\]

is equivalent to an information structure in static form for any fixed set of control laws

\[
\tilde{z}_i = [H_j \xi \mid j \{i \ or \ j = i].
\]

**Proof.** We partition the N-member into the following disjoint sets:

\( N_1 = \{ \text{set of starting members} \} \)

\( N_2 = \{ \text{set of members having } i \text{ as precedent, where } i \in N_1 \} \)

. . .

\( N_j = \{ \text{set of members having } i \text{ as a precedent, where } i \in N_{j-1} \} \).

It is clear that \( \tilde{z}_i = z_i = H_i \xi \) for all \( i \in N_1 \). Now let

\[
\tilde{z}_i = z_i - \sum_{j \{i} D_{ij} \gamma_j(z_i) = H_i \xi \quad \forall i \in N_2 \quad \text{and} \quad j \in N_1
\]

since \( Z_i \supset Z_j \), then \( \tilde{z}_i \) will be \( Z_i \) measurable. Conversely, knowing \( \tilde{z}_i \), we compute

\[
z_i = \tilde{z}_i + \sum_{j \{i} D_{ij} \gamma_j(\tilde{z}_i) \quad \forall i \in N_2 \quad \text{and} \quad j \in N_i
\]

Now, by recursion we can calculate \( z_i \) from \( \tilde{z}_i \) or vice versa for \( i \in N_3, N_4, \ldots \). \( \square \)

**Remark 1.** The reduction of (26) to (27) is possible only when we are considering pure strategy solution exclusively.

**Remark 2.** Note that the validity of Theorem 1 is independent of the nature of the criterion function \( J \). Furthermore, \( z_i \) need not be linear function of \( \xi \) and \( u_j \) for \( j \{i \); nor does \( \xi \) have to be Gaussian.

**Example 4.** A partially nested information structure

\[
\begin{align*}
z_1 &= H_1 \xi \\
z_2 &= H_2 \xi \\
z_3 &= [H_1 \ H_2] \xi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ D_{32} \end{bmatrix} u_2 \\
z_4 &= [H_1 \ H_2] \xi + \begin{bmatrix} 0 \end{bmatrix} u_1 \\
\end{align*}
\]
\[ z_5 = \begin{bmatrix} H_1 \\ H_2 \\ H_3^\prime \\ H_4^\prime \\ H_5 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}^\prime \\ D_{51}^\prime \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{52}^\prime \\ 0 \\ D_{53}^\prime \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_3 \]

\[ z_6 = H_6 \xi \]

\[ z_7 = \begin{bmatrix} H_6 \\ H_7^\prime \end{bmatrix} \xi + \begin{bmatrix} 0 \\ D_{76}^\prime \end{bmatrix} u_6 \]

\[ z_8 = H_8 \xi \]

Figure 4: Precedence Diagram for Example 4

Figure 4 shows the precedence diagram of the above example. From Theorem 1 we can convert this dynamic team problem to a static one with the information structure as:

\[ \hat{z}_i = H_i \xi \]

**Theorem 2.** In dynamic team with partially nested information structure, the optimal control for each member exits, is unique, and linear in \( z_i \).

**Proof.** As shown by Theorem 1, a team with information structure (12) is equivalent to one with static structure (13). Therefore, by Radner’s theorem [2], the optimal control law exists, is unique and linear in \( \hat{z}_i \). Since \( \hat{z}_i = z_i \) for \( i \in N_1 \), \( \gamma_i \) is linear in \( z_i \) for \( i \in N_1 \) also. By (14) we deduce that \( \gamma_i \) is also linear in \( z_i \) for \( i \in N_2 \). Repeated application of Theorem 1 and (14) yields the desired result [1].

### 5 Applications

**Application 1. Linear Quadratic Gaussian Control Problem**

For an LQG control problem (example 2), the information \( z_i \) at each time \( t \) is linear and nested. Because the utility function (16):

\[ J = E[\frac{1}{2} x_N^T S x_{N+1} + \frac{1}{2} \sum_{k=1}^N (x_k^T H^T H x_k + u_k^T b u_k)] \] (16)

can be converted to the form:

\[ J = E[\frac{1}{2} u^T Q u + u^T S \xi] + \text{ terms independent of } u \] (17)

By theorem 2, the optimal controls for all time instants exist, are unique, and linear in \( z_i \) i.e.

\[ u_i^* = A_i z_i + b_i \quad \forall i \] (18)
Remark 3. The conclusion concerning the linearity of the control law is independent of the correlation between the initial state, process noise, and observation noise. It only requires that they are jointly Gaussian.

Application 2. One-step Communication Delay Control Systems

Suppose we have two coupled linear-discrete time-dynamic system controlled by $u_1(t)$ and $u_2(t)$, $t = 1, 2, ..., N$, with the usual Gaussian distribution and noise setup. $y_1(t), y_2(t)$ are the noise observations for the system. Suppose

$$z_1(t) = \{y_1(\tau), y_2(k) | \tau = 1, 2, ..., t; k = 1, ..., t - 1\}$$

$$z_2(t) = \{y_2(\tau), y_1(k) | \tau = 1, 2, ..., t; k = 1, ..., t - 1\}$$

i.e., two controllers share all the past information with one step communication delay. Figure 5(a) represents the precedented diagram, which show that the information structure is partially nested. Hence, if the criterion is quadratic in the state and control variables, then by Theorem 2, the optimal solution is linear. Furthermore, if there is a third linear system coupled to the first system via the second system as shown in Figure 5b, then we may conclude that the first system can tolerate a two-step delay in sharing information with the third system.

Application 3. Hierarchical Control System

Suppose an information structure diagram is that, of Figure 6, which informally represents a chain of commands. Then under linear-quadratic-Gaussian assumptions, the optimal solution is a gain linear without the need for lateral communication.

Roughly speaking, the implication of Theorem 2, is that, if a DMS action affects our information, then knowing what he knows will yield linear optimal solutions.

Application 4. Two Person Zero-Sum Multi-stage Games.

The result of Theorem 2 can be employed to show that in LQG zero-sum decision games with perfect memory for each player and no cooperation, linear strategy for both players is optimal.
6 Conclusion

1. A dynamic team problem can be reduced to a static one if the information structure is partially nested.

2. In addition, if

   - the uncertainty in the system $\xi$ is Gaussian
   - the information structure $z_i$ is linear in $\xi$ and $u_j$ for $j \{i$ and it has a partially nested structure
   - the payoff function is quadratic

   then the optimal control exists, is unique, and linear in $z_i$.

Although these results provide an approach for solving some team problems, the following limitations still hold:

   - Partially nested information structure is not always the case for many team decision problems. In the part II of this paper Chu [4] showed that even if an information structure is not partially nested, under some conditions the optimal solution can be found from an auxiliary problem with partially nested information structure. However this is still not general.

   - Generally, uncertainty may not be Gaussian/ may not be normally distributed.

   - The result is limited to quadratic utility functions.

   - For a decision maker with a large number of precedent the size of information is very large which makes the calculation too heavy.

As we learned in the course, in the case of having Markov property (which holds for many problems) there are more efficient ways to calculate the optimal decision. For instance in the LQG problem in application 1 dynamic programming using Kalman Filter and for application 2 using a coordinator can reduce the amount of calculations required to find the optimal strategy.
References


