Sequential team form and its simplification using graphical models

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Outline

- Sequential team
- Team form
- Simplification of team form
- Representation of team form as a graphical model
- Automated simplification of the graphical model
Multi-agent decentralized systems: a classification

- **Multi-agent systems**
  - Static systems
  - Dynamic systems
    - Sequential
    - Non-seq
      - Classical info. struct.
      - Non-classical info. struct.
  - Teams
  - Games

- Information available to the agents
- Objective
- Order of agents' actions

- Information structures
Multi-agent decentralized systems: a classification

- Multi-agent systems
  - Information available to the agents
    - Static systems
    - Dynamic systems
      - Sequential
        - Classical info. struct.
        - Non-classical info. struct.
      - Non-sequential
    - Objective
      - Teams
      - Games

Sequential multi-stage teams with non-classical information structures
**Notation**

For a set $M$

- **Variables:** $X_M = (X_m : m \in M)$.

- **Spaces:** $X_M = \prod_{m \in M} X_m$

- **σ-algebras:** $\mathcal{F}_M = \bigotimes_{m \in M} \mathcal{F}_m$
Model for a sequential team

- A collection of $n$ system variables, $(X_k, k \in N)$ where $N = \{1, \ldots, n\}$

- A collection $\{(X_k, \mathcal{F}_k)\}_{k \in N}$ of measurable spaces.

- A collection $\{I_k\}_{k \in N}$ of information sets such that $I_k \subset \{1, \ldots, k - 1\}$.

- A set $A \subset N$ of controllers/agents.

- A set $R \subset N$ of rewards.

- The variables $X_{N\setminus A}$ are chosen by nature according to stochastic kernels $\{p_k\}_{k \in N\setminus A}$ where $p_k$ is a stochastic kernel from $(X_{I_k}, \mathcal{F}_{I_k})$ to $(X_k, \mathcal{F}_k)$. 
Objective

- Choose a strategy $\{g_k\}_{k \in A}$ such that the control law $g_k$ is a measurable function from $(X_{I_k}, \mathcal{F}_{I_k})$ to $(X_k, \mathcal{F}_k)$.

- Joint measure induced by strategy $\{g_k\}_{k \in N}$

\[
P(dX_N) = \bigotimes_{k \in N \setminus A} p_k(dX_k | X_{I_k}) \bigotimes_{k \in A} \delta_{g_k(X_{I_k})}(dX_k)
\]

- Choose a strategy to maximize

\[
\mathbb{E}^{g_A}\left\{ \sum_{i \in R} X_i \right\}
\]

This maximum reward is called the value of the team.
Generality of the model

This model is a generalization of the model presented in


which in turn is equivalent to the intrinsic model presented in

Hans S. Witsenhausen, *On information structures, feedback and causality*, SICON-71

which is as general as it gets.
A (sequential) team form is the team problem
where the measurable spaces \( \{(X_k, \mathcal{F}_k)\}_{k \in \mathbb{N}} \) and the
stochastic kernels \( \{p_k\}_{k \in \mathbb{N} \setminus A} \) are not pre-specified.

\[ \mathcal{T} = (N, A, R, \{I_k\}_{k \in \mathbb{N}}) \]: system variables, control variables, reward variables, and
the information sets are specified.
Equivalence of team forms

Two team forms $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$ and $\mathcal{T}' = (N', A', R', \{I'_k\}_{k \in N'})$ are equivalent if the following conditions hold:

1. $N = N'$, $A = A'$, and $R = R'$;
2. for all $k \in N \setminus A$, we have $I_k = I'_k$;
3. for any choice of measurable spaces $\{(X_k, \mathcal{F}_k)\}_{k \in N}$ and stochastic kernels $\{p_k\}_{k \in N \setminus A}$, the values of the teams corresponding to $\mathcal{T}$ and $\mathcal{T}'$ are the same.

The first two conditions can be verified trivially. There is no easy way to check the last condition.
**Simplification of team forms**

A team form $\mathcal{T}' = (N', A', R', \{I'_k\}_{k \in N'})$ is a simplification of a team form $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$ if

\[
\mathcal{T}' \text{ is equivalent to } \mathcal{T}
\]

and

\[
\sum_{k \in A} |I'_k| < \sum_{k \in A} |I_k|.
\]

$\mathcal{T}'$ is a strict simplification of $\mathcal{T}$ if $\mathcal{T}'$ is equivalent to $\mathcal{T}$, $|I'_k| \leq |I_k|$ for $k \in N$, and at least one of these inequalities is strict.
Given a team form, can we simplify it?

Asking for simplification of a team form is same as asking for structural properties that do not depend on the nature of the process (discrete or continuous values), the specific form of probability measure (Gaussian, uniform, binomial , etc.) and the specific properties of cost function (convex, monotone, etc.)
Some Preliminaries
Partial Orders

A strict partial order $\prec$ on a set $S$ is a binary relation that is transitive, irreflexive, and asymmetric. i.e., for $a$, $b$, $c$ in $S$, we have

1. if $a \prec b$ and $b \prec c$, then $a \prec c$ (transitive)
2. $a \not\prec a$ (irreflexive)
3. if $a \prec b$ then $b \not\prec a$ (asymmetric)

The reflexive closure $\preceq$ of a partial order $\prec$ is given by

$$a \preceq b \text{ if and only if } a \prec b \text{ or } a = b$$
Partial Order

Let \( A \) be a subset of a partially ordered set \((S, \prec)\). Then, the lower set of \( A \), denoted by \( \overset{\lor}{A} \), is defined as

\[
\overset{\lor}{A} := \{ b \in S : b \preceq a \text{ for some } a \in A \}.
\]

By duality, the upper set of \( A \), denoted by \( \overset{\land}{A} \), is defined as

\[
\overset{\land}{A} := \{ b \in S : a \preceq b \text{ for some } a \in A \}.
\]
A team problem is sequential if and only if there is a partial order between the agents
Partial orders can be represented by directed graphs.

So, sequential teams can be represented as directed graphs.
Representing teams using directed graphs

Representing teams using directed graphs


Fig. 3.
Representing teams using directed graphs

Tseneo Yoshikawa, Decomposition of Dynamic Team Decision Problems, TAC-78.

Fig. 1. Precedence diagram.
Representing teams using directed graphs


Figure 1: LIMID representation of the ID version of PIGS. The full previous treatment and test history is available when decisions must be taken.

Descendant of \( D_2 \) and \( D_2 \) is an ancestor of \( D_2 \). The set of descendants and the set of ancestors of \( D_2 \) is denoted \( CS/CT/B4/D2/B5 \) and \( CP/D2/B4/D2/B5 \) respectively.

The arcs in a LIMID have a different meaning depending on the type of node they go into. If chance node \( D_6 \) (connoting random variable) is a parent of chance node \( D_6 \), it indicates that the distribution of (the random variable) \( D_6 \) is specified conditionally on the value of \( D_6 \). A decision node \( CS \) is a parent of chance node \( D_6 \) if the distribution of \( D_6 \) can depend on decision \( CS \). A decision node \( CS \) is a parent of decision node \( CS \) if the choice of alternative for decision \( CS \) is known to the decision maker when decision \( CS \) is taken and may influence that decision. When chance node \( D_6 \) is a parent of a decision node \( CS \) it indicates that the value of \( D_6 \) will be known when decision \( CS \) is taken and might influence that decision. Finally arcs into value nodes represent the decision maker's (expected) utility given the states of its parents. Value nodes cannot have children.

Example 2 (Diagrams for PIGS) To represent the ID version of PIGS by a LIMID, we let \( CW \), \( (CX/BP/BD/BN/BM/BM/BN/BG) \) denote the (chance) variables which indicate whether the pig is healthy or unhealthy in the \( CX \)th month and \( D_8 \), \( (CX/BP/BD/BN/BE/BN/BF) \) represent the corresponding test results, which are said to be positive if they indicate presence of the disease, and otherwise negative. The nodes \( CS/CX \), \( (CX/BP/BD/BN/BE/BN/BF) \) indicate presence of the disease, and otherwise negative. The nodes \( CS/CX \), \( (CX/BP/BD/BN/BE/BN/BF) \)
None of these fit our requirements perfectly. So, we use DAFG (Directed Acyclic Factor Graphs)
A graphical model for sequential team forms
A graphical model for sequential team forms

Directed Acyclic Factor Graph $\mathcal{G} = (V, F, E)$ for $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$

$$V = N \times \{0\}, \quad F = N \times \{1\}$$

$$E = \{(k^1, k^0) : k \in N\} \cup \{(i^0, k^1) : k \in N, i \in I_k\}$$

- **Vertices**

  Variable Node $k^0 \equiv$ system variable $X_k$

  Factor node $k^1 \equiv$ stochastic kernel $p_k$ or control law $g_k$.

- **Edges**

  $(k^1, k^0)$ for each $k \in N$

  $(i^0, k^1)$ for each $k \in N$ and $i \in I_k$
An Example: Real-time communication


First order Markov source \( \{S_t, t = 1, \ldots, T\} \).

Real-Time Encoder: \( Y_t = c_t(S^t, Y^{t-1}) \)

Real-Time Finite Memory Decoder: \( \hat{S}_t = g_t(Y_t, M_{t-1}) \)

\[
M_t = l_t(Y_t, M_{t-1})
\]

Instantaneous distortion \( \rho(S_t, \hat{S}_t) \)

Objective: minimize \( E\left\{ \sum_{t=1}^{T} \rho(S_t, \hat{S}_t) \right\} \)
An Example: Real-time communication

\[ D_1 \bullet \quad D_2 \bullet \quad D_3 \bullet \]

\[ \square \ p_{f_1} \quad \square \ p_{\rho_1} \quad \square \ p_{f_2} \quad \square \ p_{\rho_2} \quad \square \ p_{f_3} \quad \square \ p_{\rho_3} \]

\[ S_1 \circ \quad \hat{S}_1 \circ \quad S_2 \circ \quad \hat{S}_2 \circ \quad S_3 \circ \quad \hat{S}_3 \circ \]

\[ \blacksquare \ c_1 \quad \blacksquare \ g_1 \quad \blacksquare \ c_2 \quad \blacksquare \ g_2 \quad \blacksquare \ c_3 \quad \blacksquare \ g_3 \]

\[ Y_1 \circ \quad M_1 \circ \quad Y_2 \circ \quad M_2 \circ \quad Y_3 \circ \]

\[ \blacksquare \ l_1 \quad \blacksquare \ l_2 \]
An Example: Real-time communication
An Example: Real-time communication
Checking conditional independence

Dan Geiger, Thomas Verma, and Judea Pearl, Identifying independence in Bayesian networks, Networks-90.

Conditional independence can be efficiently checked on a directed graph.

Given a DAFG $\mathcal{G} = (V, F, E, D)$ and sets $A, B, C \subseteq V$, $X_A$ is irrelevant to $X_B$ given $X_C$ if $X_A$ is independent to $X_B$ given $X_C$ for all joint measures $P(dX_V)$ that recursively factorize according to $\mathcal{G}$.

Data irrelevant to $X_A$ given $X_C$ is

$$R^{-}_\mathcal{G}(X_A|X_C) = \{k \in C : X_k \text{ is irrelevant to } X_A \text{ given } X_C\}$$
Back to simplification of team forms
Completion of a team

A team form $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$ is complete if for $k, l \in A$, $k \neq l$, such that $I_k \subseteq I_l$ we have $X_k \subseteq I_l$. (If $l$ knows the data available to $k$, then $l$ also knows the action taken by $k$).

If a team is not complete, it can be completed by sequentially adding “missing links”

Depending on the order in which we proceed, we can end up with different completions. However,

all completions of a team form are equivalent.
Completion of a team form
Completion of a team

\[
\begin{align*}
D_1 & \rightarrow D_2 & D_3 \\
\text{pf}_1 & \rightarrow \text{p}_\rho_1 & \text{pf}_2 & \rightarrow \text{p}_\rho_2 & \text{pf}_3 & \rightarrow \text{p}_\rho_3 \\
S_1 & \rightarrow \hat{S}_1 & S_2 & \rightarrow \hat{S}_2 & S_3 & \rightarrow \hat{S}_3 \\
c_1 & \rightarrow \text{g}_1 & c_2 & \rightarrow \text{g}_2 & c_3 & \rightarrow \text{g}_3 \\
Y_1 & \rightarrow M_1 & Y_2 & \rightarrow M_2 & Y_3 \\
l_1 & \rightarrow l_2
\end{align*}
\]
Simplification of team forms

Step 1: Complete the team form.

(Note: All completions of a team form are equivalent to the original)
Removing irrelevant nodes

Recall Given a DAFG $\mathcal{G} = (V, F, E, D)$ and sets $A, B, C \subset V$, $X_A$ is irrelevant to $X_B$ given $X_C$ if $X_A$ is independent to $X_B$ given $X_C$ for all joint measures $P(dX_V)$ that recursively factorize according to $\mathcal{G}$ and

$$R^-(\mathcal{G})_A(X_A|X_C) = \{k \in C : X_k \text{ is irrelevant to } X_A \text{ given } X_C\}$$

For any $k \in A$ in a team form $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$, replacing $X_{I_k}$ by $X_{I_k} \setminus (R^-(\mathcal{G})_R \cap \overrightarrow{X_k} | X_{I_k}, X_k) \setminus X_k)$

does not change the value of the team.
Removing irrelevant nodes
Removing irrelevant nodes

\[
\begin{align*}
\hat{S}_1 &\quad \hat{S}_2 &\quad \hat{S}_3 \\
S_1 &\quad S_2 &\quad S_3 \\
c_1 &\quad c_2 &\quad c_3 \\
Y_1 &\quad Y_2 &\quad Y_3 \\
l_1 &\quad l_2
\end{align*}
\]
Removing irrelevant nodes
Removing irrelevant nodes
Removing irrelevant nodes
Removing irrelevant nodes
Removing irrelevant nodes

\[ \begin{align*}
D_1 & \quad D_2 & \quad D_3 \\
p_f_1 & \quad p_{\rho_1} & \quad p_f_2 & \quad p_{\rho_2} & \quad p_f_3 & \quad p_{\rho_3} \\
S_1 & \quad \hat{S}_1 & \quad S_2 & \quad \hat{S}_2 & \quad S_3 & \quad \hat{S}_3 \\
c_1 & \quad g_1 & \quad c_2 & \quad g_2 & \quad c_3 & \quad g_3 \\
Y_1 & \quad M_1 & \quad Y_2 & \quad M_2 & \quad Y_3 \\
l_1 & \quad l_2
\end{align*} \]
Removing irrelevant nodes
Removing irrelevant nodes

\[ Y_t = c_t(S_t, M_{t-1}) \]
Simplification of team forms

Step 1: Complete the team form.

(Note: All completions of a team form are equivalent to the original)

Step 2: At control factor node k, remove incoming edges from nodes irrelevant to $X_R \cap \overrightarrow{X}_k$ given $(X_{I_k}, X_k)$

(Note: The resultant team form is equivalent to the original)
Does not always work
**Another Example: Shared randomness**

Plant:  
\[ S_{t+1} = f_t(S_t, A^1_t, A^2_t, W_t) \]

Shared Randomness:  
\[ \{Z_t, t = 1, \ldots, T\} \]

independent of plant disturbance and observation noise.

Control Station 1:  
\[ A^1_t = g^1_t(S^t, A^{1,t-1}, Z^t) \]

Control Station 2:  
\[ A^2_t = g^2_t(S^t, A^{2,t-1}, Z^t) \]

Instantaneous cost:  
\[ \rho_t(S_t, A^1_t, A^2_t) \]
Another Example: Shared randomness
Another Example: Shared randomness (Step 1)
Coordinator for a subset of agents

For $a, b \in A$, consider a coordinator that observes $X_C := X_{I_a} \cap X_{I_b}$ and chooses partial functions $\hat{g}_a : X_{I_a \setminus C} \to X_a$ and $\hat{g}_b : X_{I_b \setminus C} \to X_b$.

Agent $a$ and $b$ simply carry out the computations prescribed by $\hat{g}_a$ and $\hat{g}_b$.

Remove irrelevant incoming edges at the coordinator!

Equivalently, at agents $a$ and $b$, remove edges from nodes that are irrelevant to $X_R \cap \overrightarrow{X}_{\{a,b\}}$ given $(X_C, X_{\{a,b\}})$. 
Coordinator for a subset of agents

For any $B \subset A$ in a team form $\mathcal{T} = (N, A, R, \{I_k\}_{k \in N})$ and any $b \in B$, let $X_C = \bigcap_{b \in B} X_{I_b}$. Then, replacing $X_{I_b}$ by $X_{I_b} \setminus (R_0^{-}(X_R \cap \bigvee_{B} X_{C} \setminus X_{B}) \setminus X_{B})$ does not change the value of the team.
Simplification of team forms

Step 1: Complete the team form.
(Note: All completions of a team form are equivalent to the original)

Step 2: At control factor node k, remove incoming edges from nodes irrelevant to $X_R \cap \overrightarrow{X}_k$ given $(X_{I_k}, X_k)$
(Note: The resultant team form is equivalent to the original)

Step 3: At all nodes of any subset B of A, remove incoming edges from nodes irrelevant to $X_R \cap \overrightarrow{X}_B$ given $(\bigcup_{b \in B} X_{I_b}, X_B)$.
(Note: The resultant team form is equivalent to the original. Furthermore, this computation can be carried out efficiently on a lattice of shared information.)
Another Example: Shared randomness (Step 3)
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Another Example: Shared randomness (Step 3)
Another Example: Shared randomness (Step 2)
Another Example: Shared randomness (Step 1)

\[ A_t^1 = g_t^1(S_t) \]
\[ A_t^2 = g_t^2(S_t, A_t^1) \]
**Simplification of team forms**

**Step 1:** Complete the team form.

*(Note: All completions of a team form are equivalent to the original)*

**Step 2:** At control factor node $k$, remove incoming edges from nodes irrelevant to $X_R \cap \overrightarrow{X}_k$ given $(X_{I_k}, X_k)$

*(Note: The resultant team form is equivalent to the original)*

**Step 3:** At all nodes of any subset $B$ of $A$, remove incoming edges from nodes irrelevant to $X_R \cap \overrightarrow{X}_B$ given $(\bigcup_{b \in B} X_{I_b}, X_B)$.

*(Note: The resultant team form is equivalent to the original. Furthermore, this computation can be carried out efficiently on a lattice of shared information.)*
Conclusion

Team form for sequential teams, equivalence and simplification of team forms.

Representing a team form as a DAFG

Carrying out the simplification of the team form on the DAFG. This process can be automated.

Future Directions

Sequential decomposition of a team form on a DAFG (The sequential decomposition of Witsenhausen’s standard form can be carried out efficiently on a DAFG).

Adding belief states / information states (need to study conditional independence properties and define an appropriate notion of simplification)
Thank you