A graphical model for sequential teams

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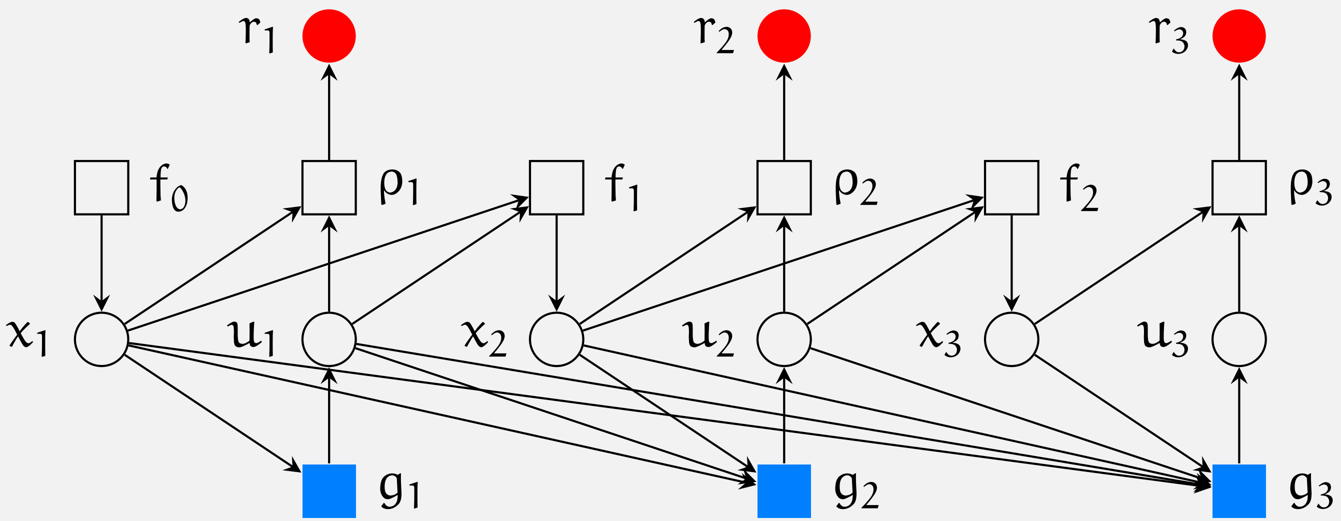
A glimpse of the result

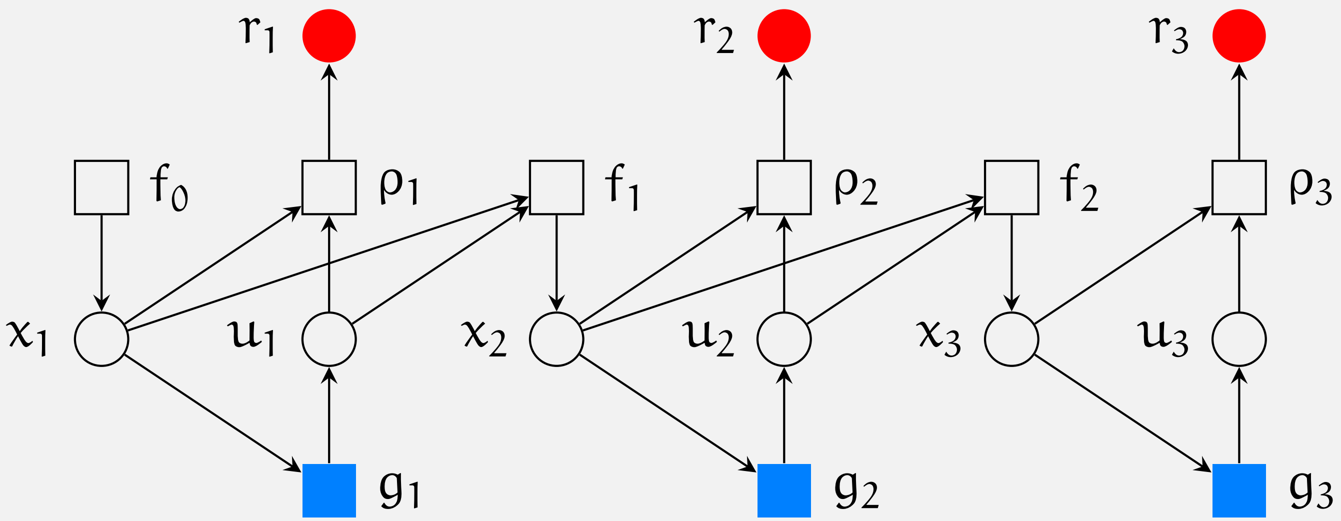
Structural results in sequential teams

- Example: MDP (Markov decision process)
  - Controlled MC: $\Pr(x_t \mid x_1, \dots, x_{t-1}, u_1, \dots, u_{t-1}) = \Pr(x_t \mid x_{t-1}, u_{t-1})$
  - Controller: $u_t = g_t(x_1, \dots, x_t, u_1, \dots, u_{t-1})$
  - Reward: $r_t = \rho_t(x_t, u_t)$
  - Objective: Maximize $\mathbb{E}\left\{ \sum_{t=1}^{T} R_t \right\}$
- Structural results
  - Without loss of optimality, $u_t = g_t(x_t)$

Graphically ... original



Graphically ... structural results



Structural results in sequential teams

- Example: real-time source coding

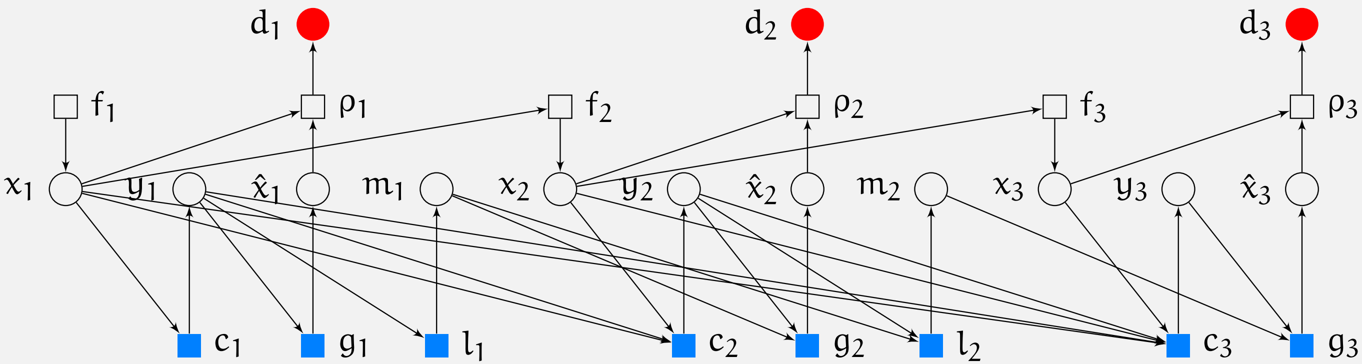
  - Source: First order Markov source $\{x_t,\ t = 1, \ldots\}$
  - Real-time source coder: $y_t = c_t(x_1, \ldots, x_t, y_1, \ldots, y_{t-1})$
  - Finite memory decoder: $\hat{x}_t = g_t(y_t, m_{t-1})$
  - $m_t = l_t(y_t, m_{t-1})$
  - Cost: $d_t = \rho_t(x_t, \hat{x}_t)$

Hans S. Witsenhausen, On the structure of real-time source coders,
Bell Systems Technical Journal, vol 58, no 6, pp 1437-1451, July-August 1979

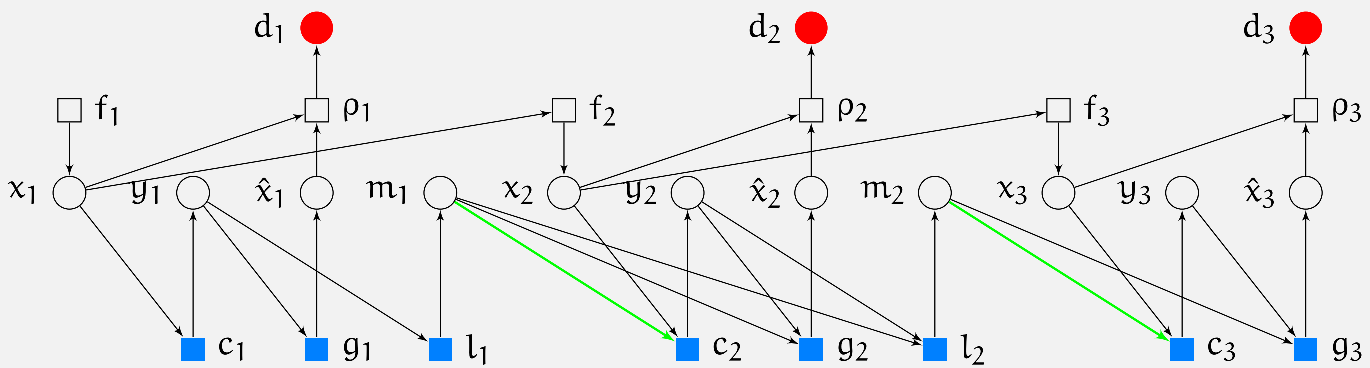
- Structural Results

  - Without loss of optimality, $y_t = c_t(x_t, m_{t-1})$

Graphically . . . original



Graphically . . . structural results



The main idea

- Represent a sequential team as a directed graph
- Simplify the graph

Sequential teams – Salient features

- A team is sequential if and only if there exists a partial order between the system variables.

- There is no loss of optimality in restricting attention to non-randomizing decision makers

- Data available at a DM can be ignored if it is independent of the future rewards conditioned on other data at the DM

- Variables functionally determined from the data available at a DM can be assumed to be observed at the DM.

Graphical models – Salient features

- Any partial order gives rise to a DAG (Directed Acyclic Graph)
- A DAFG can be used to efficiently check for conditional independence using d-separation
- A DAFG can be used to efficiently check for conditional independence with deterministic nodes using D-separation

Match between features of sequential teams and graphical models

The rest is a matter of details . . .

The model

- Components of a sequential team
  - A set N of indices of system variables $\{X_n, n \in N\}$.

    Finite sets $\{\mathcal{X}_n, n \in N\}$ of state spaces of X_n
    - $A \subset N$, variables generated by DM
    - $N \setminus A$, variables generated by nature
    - $R \subset N$, reward variables
  - Information sets $\{I_n, n \in N\}$, such that $I_n \subseteq \{1, \dots, n\}$. $\mathcal{I}_n = \prod_{i \in I_n} \mathcal{X}_i$
  - $F_{N \setminus A} = \{f_n, n \in N \setminus A\}$, where f_n is a conditional PMF $\mathcal{X}_n$ given $\mathcal{I}_n$
  - Design: $G_A = \{g_n, n \in A\}$, where g_n is a decision rule from $\mathcal{I}_n$ to $\mathcal{X}_n$

The model

- Probability measure induced by a design

$$P^{G_A}(X_N) = \prod_{n \in N \setminus A} f_n(X_n | I_n) \prod_{n \in A} I\left[X_n = g_n(I_n)\right]$$

- Optimization problem

Minimize $E\left\{\sum_{n \in R} X_n\right\}$, where the expectation is with respect to P^{G_A}.

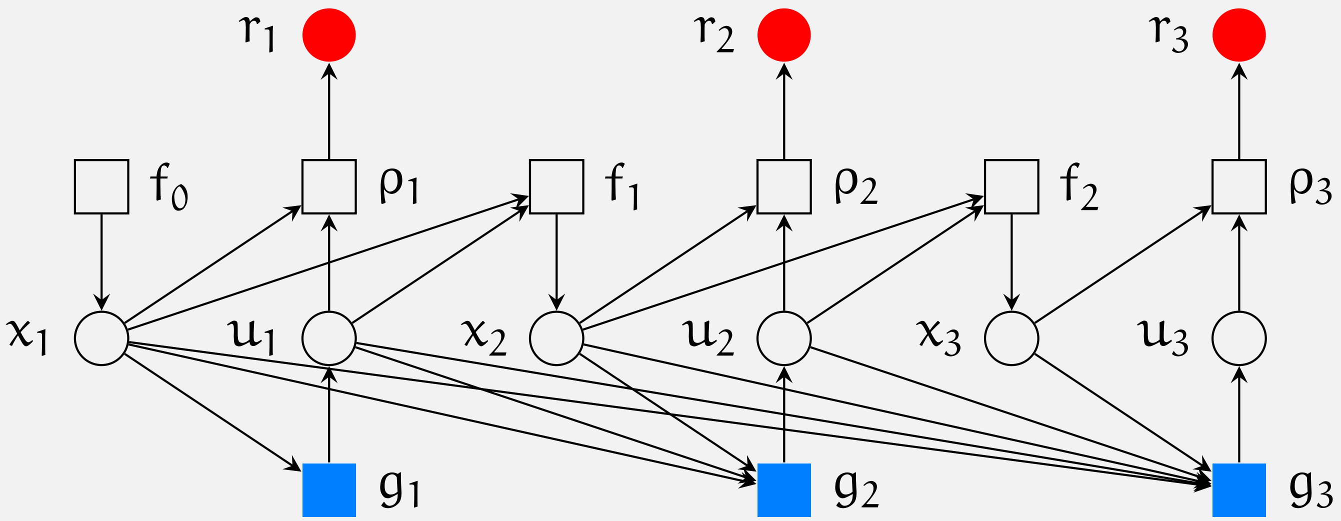
Representation as a graphical model

- Directed Acyclic Factor Graph
- Nodes
  - ▷ Variable node n $\equiv$ system variable X_n
  - ▷ Factor node $\tilde{n}$ $\equiv$ conditional PMF f_n or decision rule g_n
- Edges
  - ▷ $(i, \tilde{n})$, for each $n \in N$ and $i \in I_n$
  - ▷ $(\tilde{n}, n)$, for each $n \in N$
- Acyclic Graph
  - ▷ Sequential team $\Rightarrow$ partial order on variable nodes $\Rightarrow$ acyclic graph

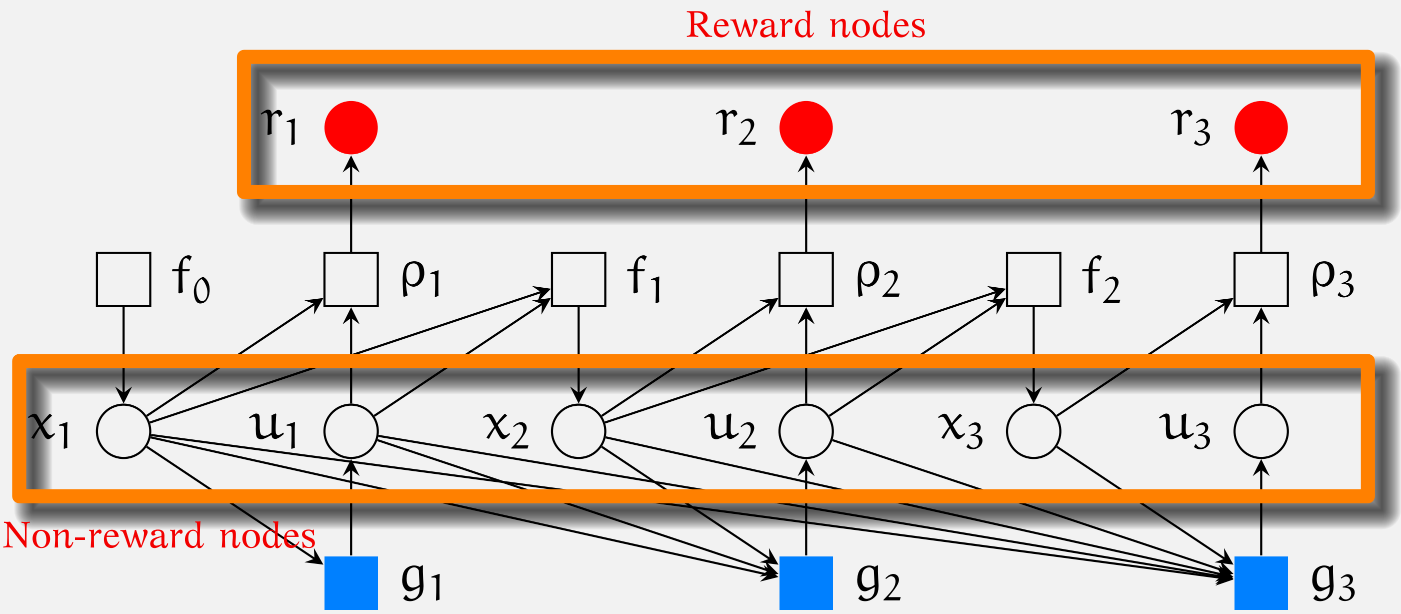
Graphical models – Terminology

- parents(n)
  - $\{m : m \to n\}$
  - Parents of a control (factor) node = data observed by controller
- children(n)
  - $\{m : n \to m\}$
  - Children of a control node = control action
- ancestors(n)
  - $\{m : \exists \text{ directed path from } m \text{ to } n\}$
  - Ancestors of a control node = all nodes that affect the data observed
- descendants(n)
  - $\{m : \exists \text{ directed path from } n \text{ to } m\}$
  - Descendants of a control node = all nodes affected by the control action

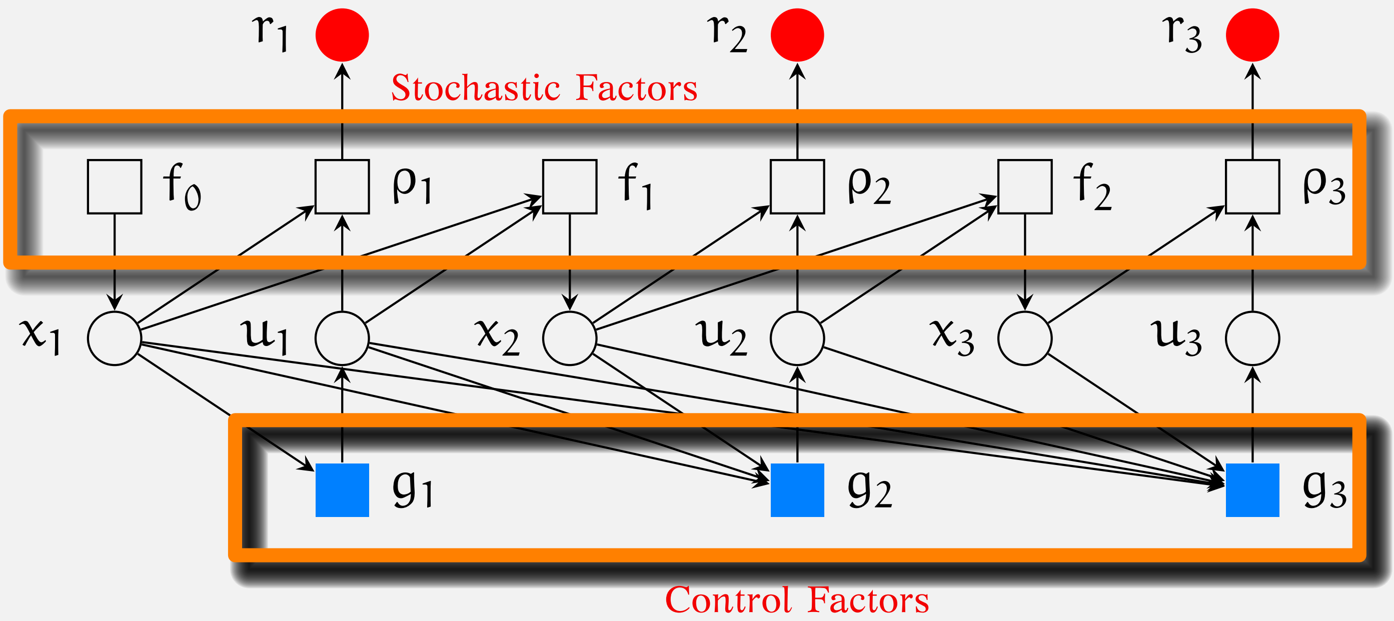
Graphical Models – Example



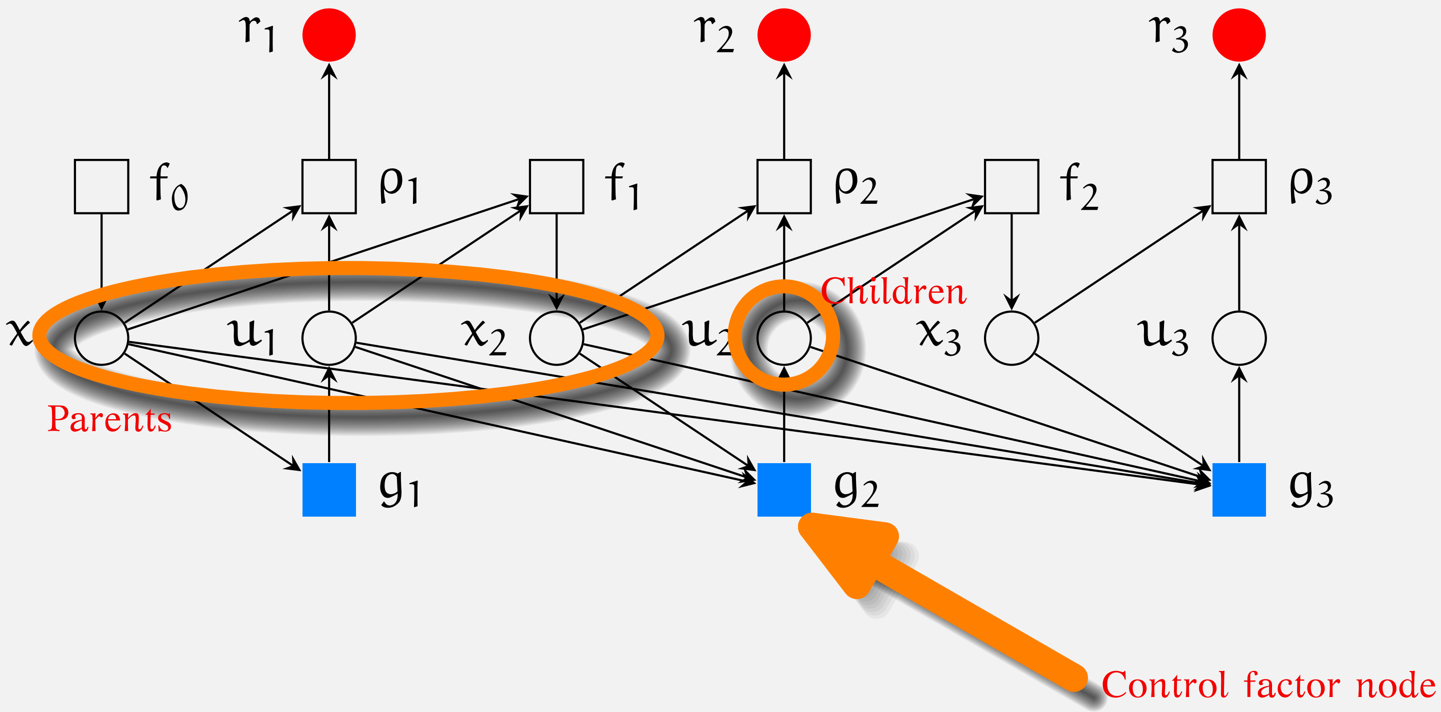
Graphical Models – Variable nodes



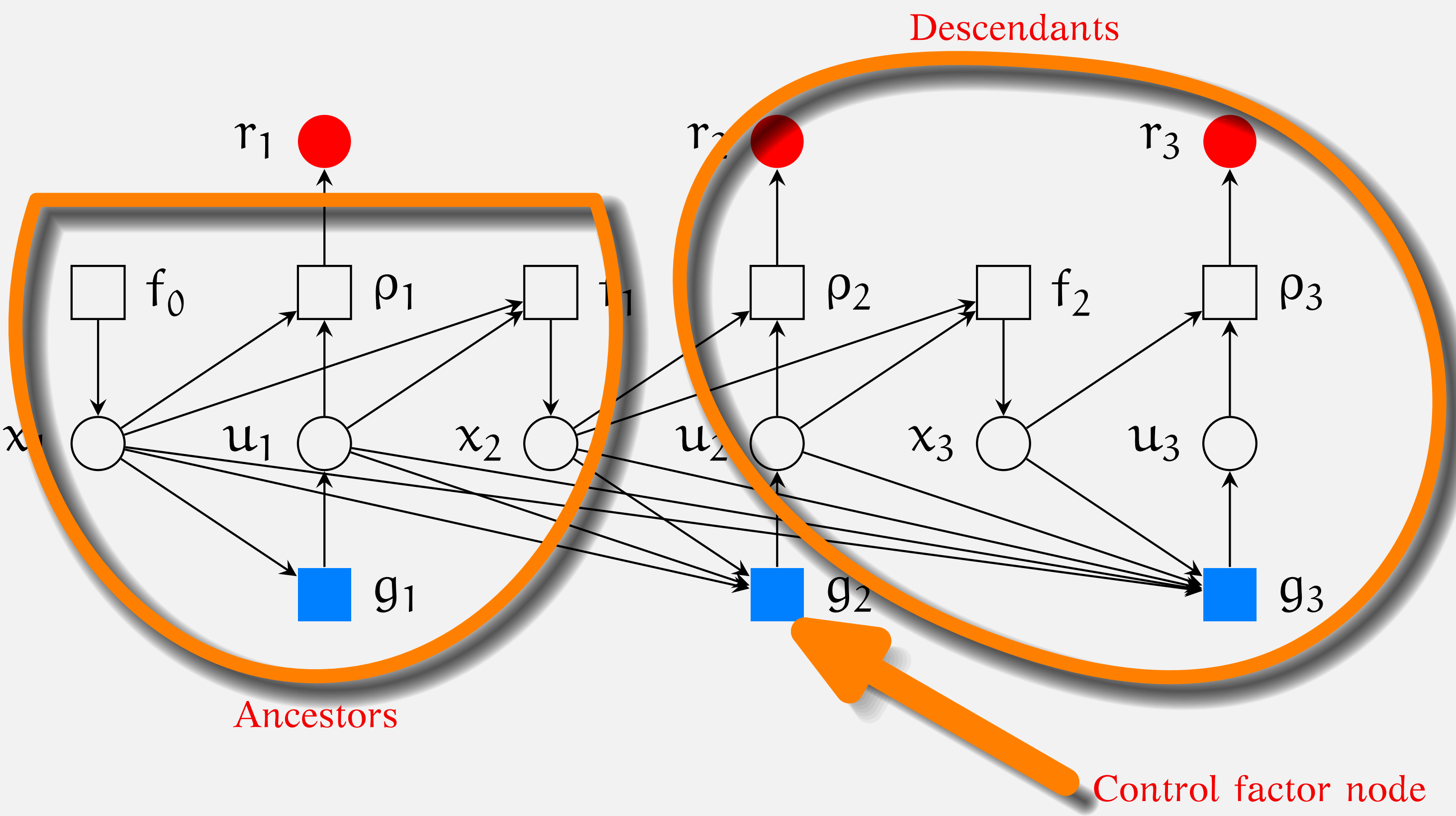
Graphical Models – Factor nodes



Graphical Models – Parents and Children



Graphical Models – Ancestors and descendents



Structural results

- The main idea

  If some data available at a DM is independent of future rewards given the control action and other data at the DM, then that data can be ignored

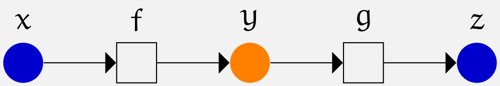
> Can we automate this process?

Struct. result $\equiv$ cond. independence

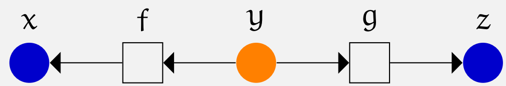
Graphical models can easily
test conditional independence

Conditional independence

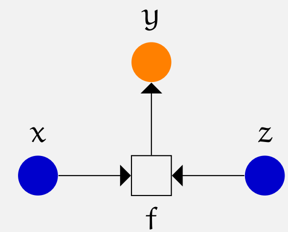
- Three canonical graphs to verify $x \perp\!\!\!\perp z \mid y$




Markov chain




Hidden cause




Explanation

- Blocking of a trail

A trail from a to b is blocked by C if $\exists$ a node v on the trail such that either:

- either $\rightarrow v \rightarrow$, $\leftarrow v \leftarrow$, or $\leftarrow v \rightarrow$, and $v \in C$
- $\rightarrow v \leftarrow$ and neither v nor any of v's descendants are in C.

Conditional independence

- d-separation

  A is d-separated from B by C if all trails from A to B are blocked by C

- Conditional independence

  For any probability measure P that factorizes according to a DAFG,

  $$A \text{ d-separated from } B \text{ by } C \text{ implies}$$
  $$X_A \text{ is conditionally independent of } X_B \text{ given } X_C,\ P \text{ a.s.}$$

- Efficient algorithms to verify d-separation

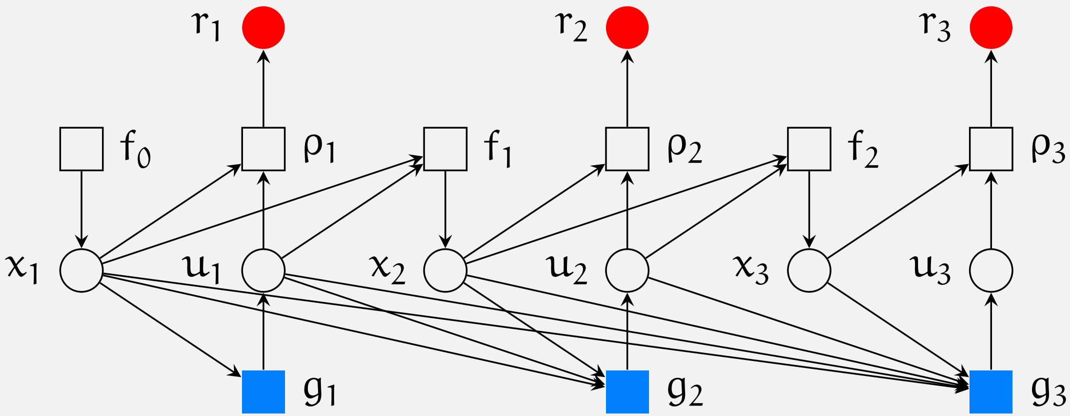
  ▷ Moral graph ▷ Bayes Ball

Automated Structural results

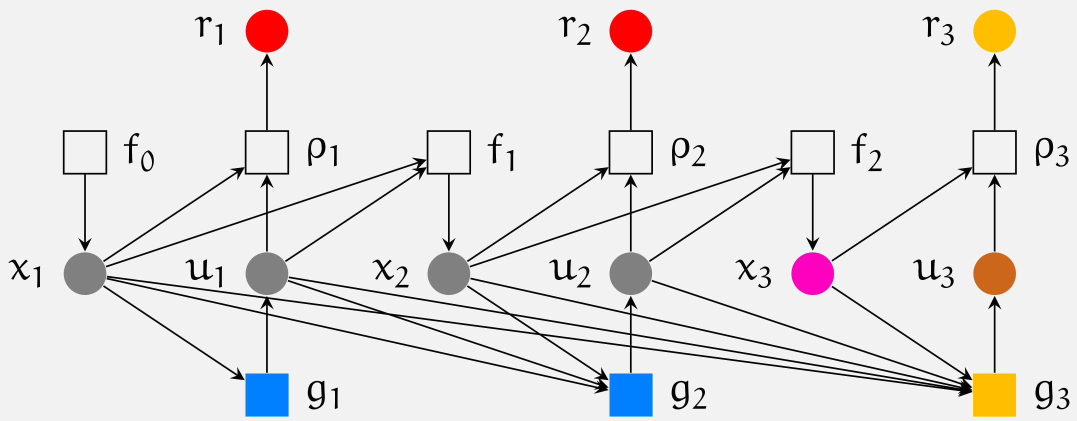
- First attempt
  - Dependent rewards: $R_d(\tilde{n}) = R \cap \text{descendants}(\tilde{n})$
  - Irrelevant data: At a control node $\tilde{n}$, and parent i is irrelevant if $R_d(\tilde{n})$ is d-separate from i given $\text{parents}(\tilde{n}) \cup \text{children}(\tilde{n}) \setminus \{i\}$
  - Requisite data: All parents that are not irrelevant
- Structural result
  - Without loss of optimality, we can remove irrelevant data.

$$u_n = g_n(\text{requisite}(\tilde{n}))$$

Structural Results for MDP – Step 1

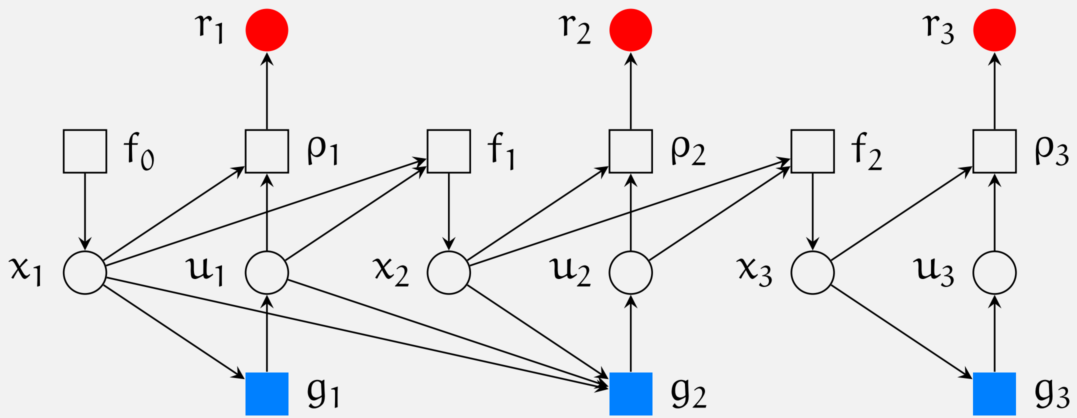


Structural Results for MDP – Step 1

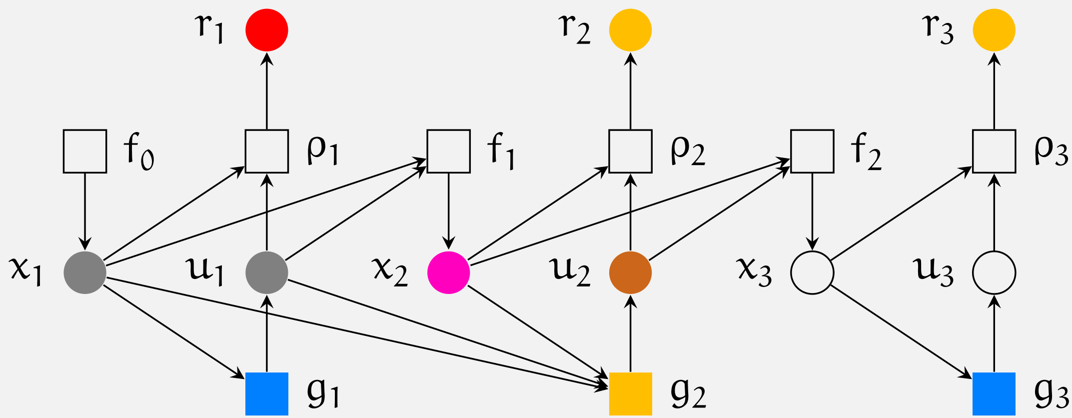


- Pick node g_3.
  - ▷ Original $u_3 = g_3(x_1, x_2, x_3, u_1, u_2)$
  - ▷ requisite(g_3) = $\{x_3\}$
  - ▷ Thus, $u_3 = g_3(x_3)$

Structural Results for MDP – Step 2

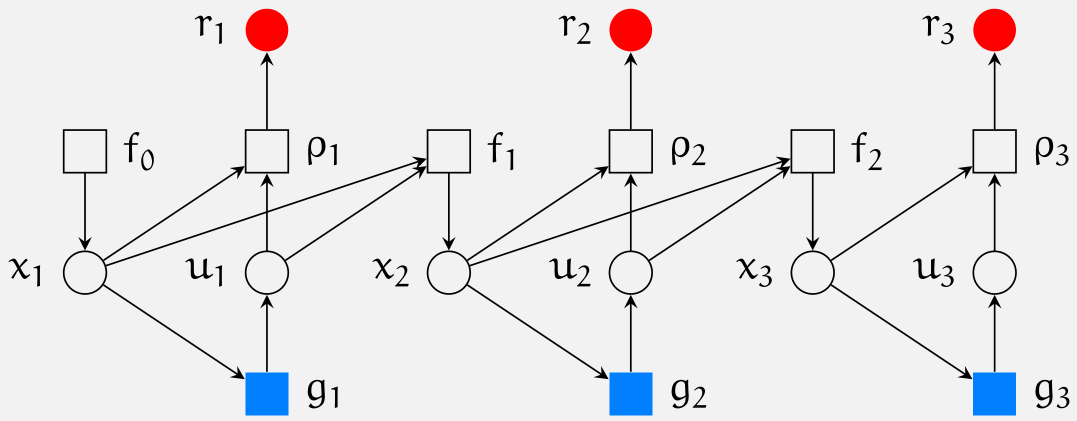


Structural Results for MDP – Step 2



- Pick node g_2.
  - ▷ Original $u_2 = g_2(x_1, x_2, u_1)$
  - ▷ requisite(g_2) = $\{x_2\}$
  - ▷ Thus, $u_2 = g_2(x_2)$

Structural Results for MDP – Simplified



$$u_n = g_n(\mathrm{requisite}(\tilde{n}))$$

Does not work for all problems . . .

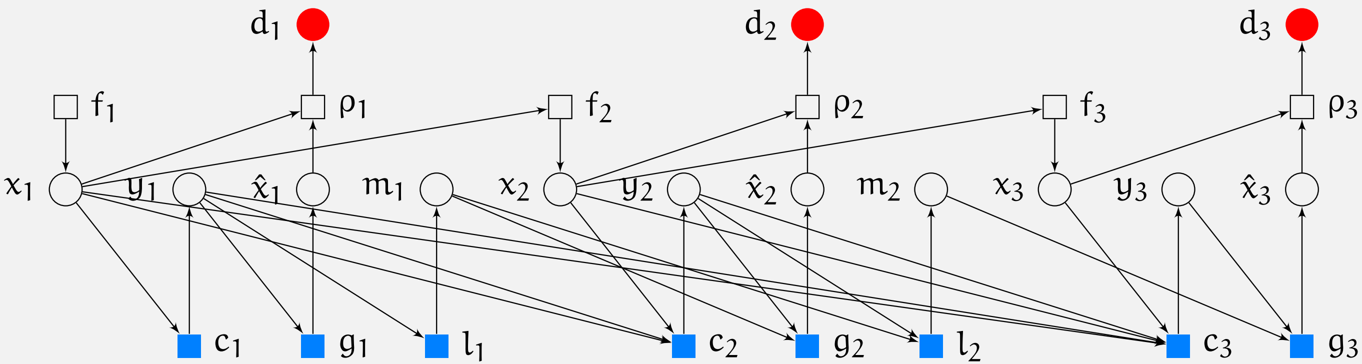
even when structural simplification is possible

A real-time source coding problem

Hans S. Witsenhausen, On the structure of real-time source coders,
Bell Systems Technical Journal, vol 58, no 6, pp 1437-1451, July-August 1979

- Mathematical Model
  - Source: First order Markov source $\{x_t,\ t = 1, \dots\}$
  - Real-time source coder: $y_t = c_t(x(1{:}t), y(1{:}t-1))$
  - Finite memory decoder: $\hat{x}_t = g_t(y_t, m_{t-1})$
  - $m_t = l_t(y_t, m_{t-1})$
  - Cost: $d_t = \rho_t(x_t, \hat{x}_t)$

Model for real-time comm – Does not simplify



Need to take care of deterministic variables!

Functionally determined nodes

- Functionally determined
  - ▷ X_B is functionally determined by X_A if $X_B \perp\!\!\!\perp X_N \mid X_A$
- Conditional independence with functionally determined nodes
  - ▷ Can be checked using D-separation
  - ▷ Similar to d-sep: in the defn of blocking change "in C" by "is func detm by C"
- Blocking of a trail (version that takes care of detm nodes)

  A trail from a to b is blocked by C if $\exists$ a node v on the trail such that either:

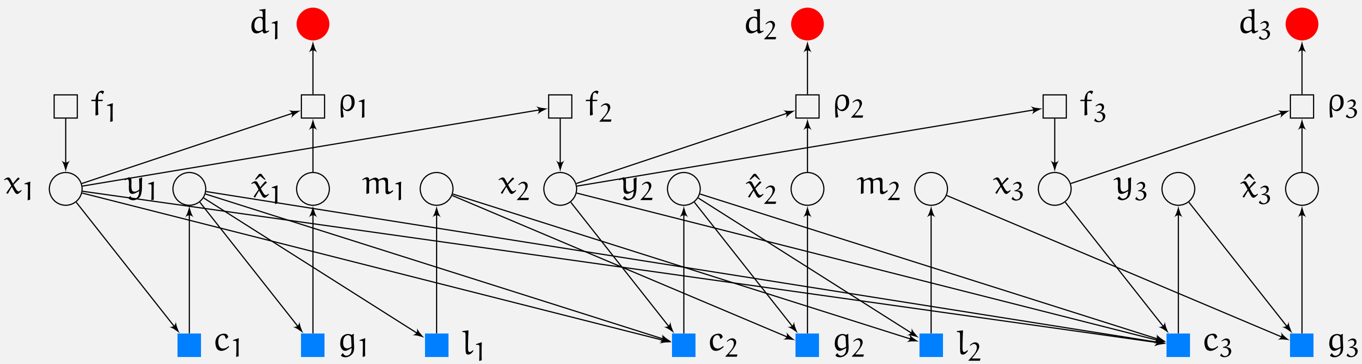
  - either $\rightarrow v \rightarrow$, $\leftarrow v \leftarrow$, or $\leftarrow v \rightarrow$, and v is functionally determined by C
  - $\rightarrow v \leftarrow$ and neither v nor any of v's descendants are in C.

Automated Structural results

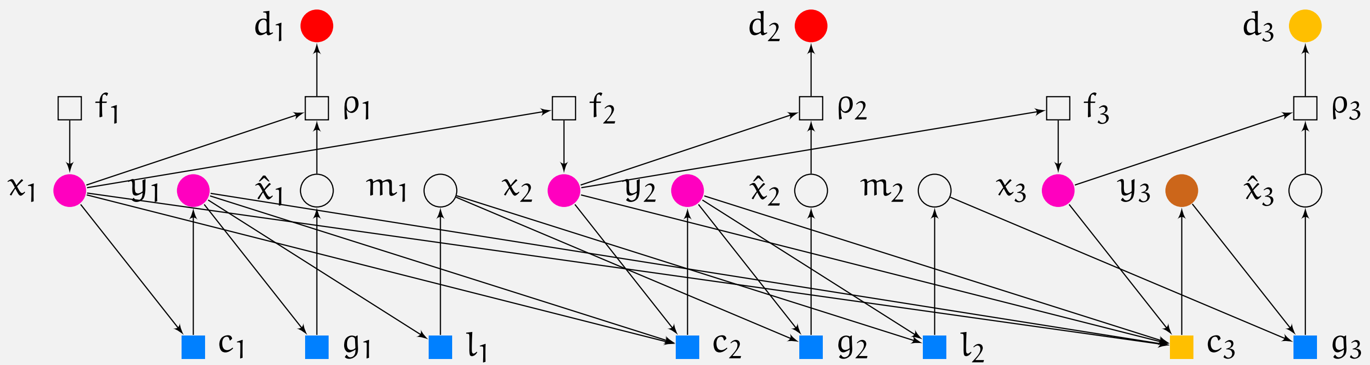
- Second attempt
  - Irrelevant data: Change d-separation by D-separation
  - Requisite data: All parents that are not irrelevant
- Structural result
  - Without loss of optimality, we can remove irrelevant data and add appropriate functionally determined data

$$u_n = g_n(\text{requisite}(\tilde{n}), \text{functionally\_detm}(\tilde{n}) \cap \text{ancestors}(R_d(\tilde{n})))$$

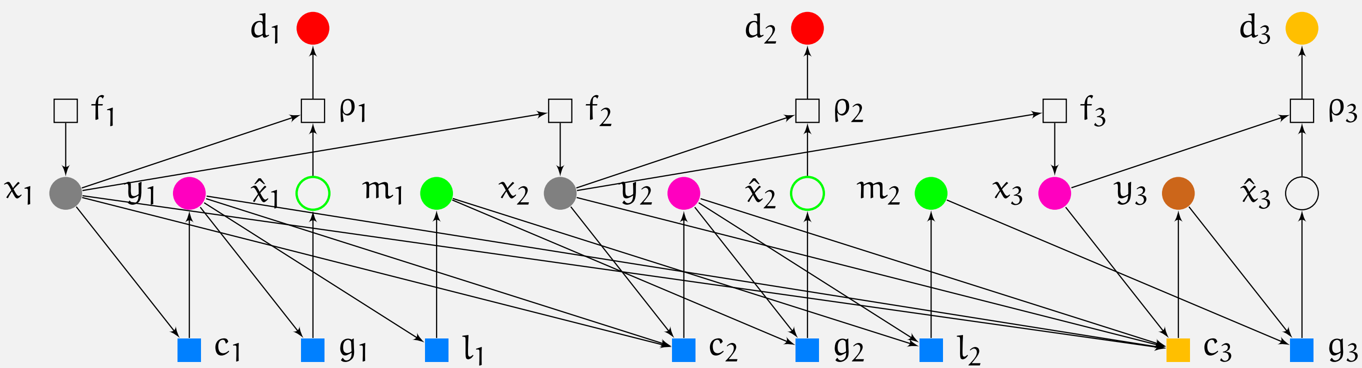
Lets try this!



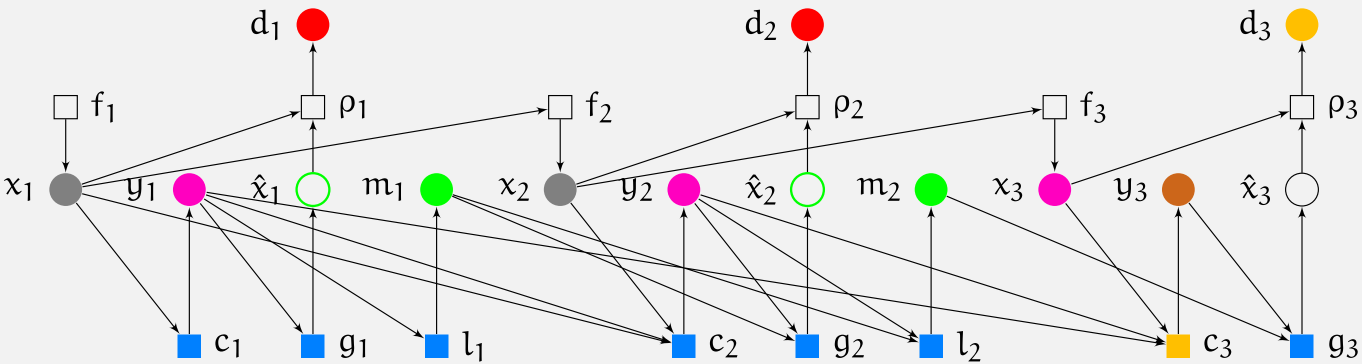
Structural Results for Dec MDP – Step 1



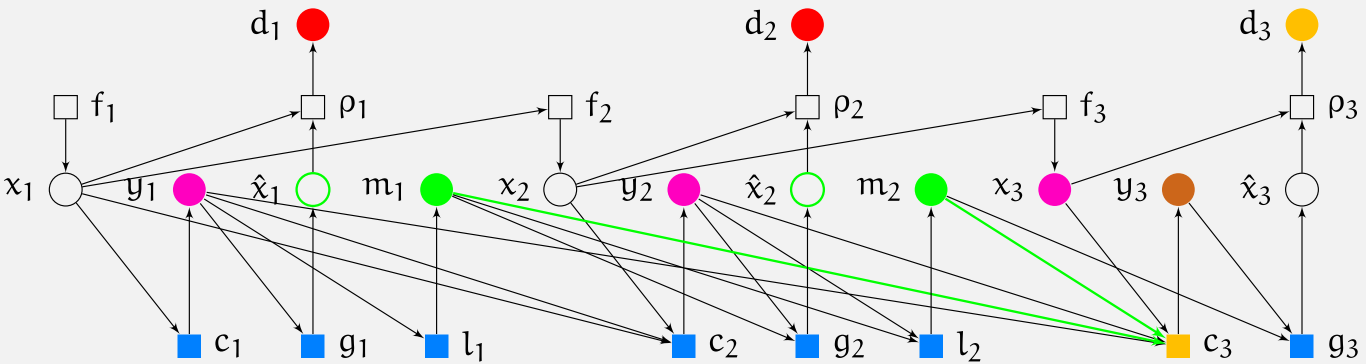
Structural Results for Dec MDP – Step 2



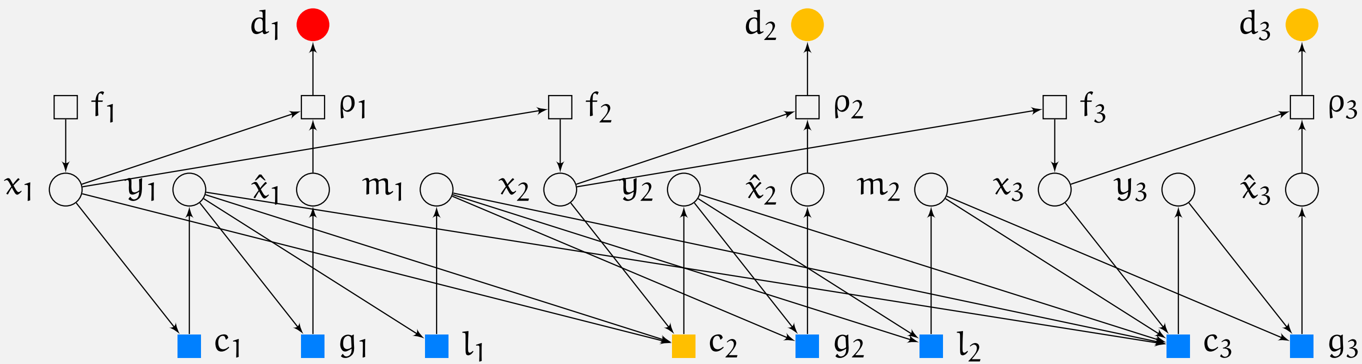
Structural Results for Dec MDP – Step 3



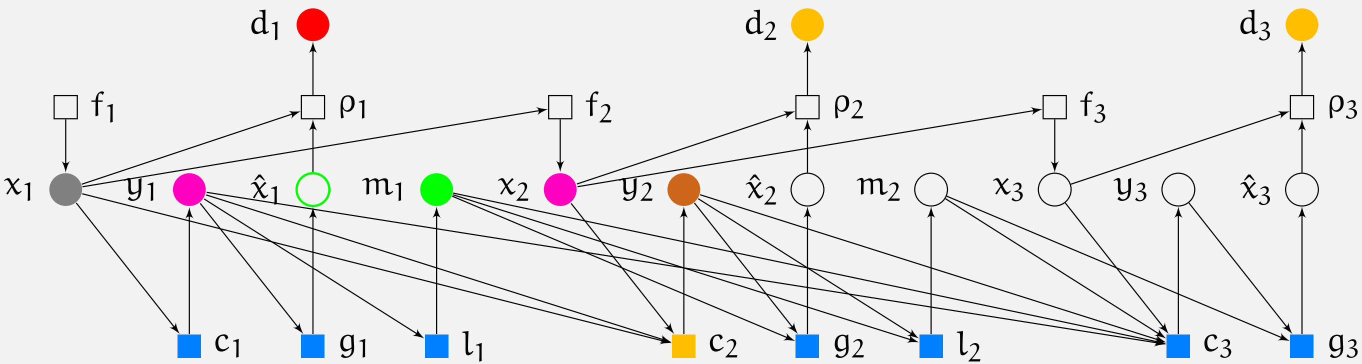
Structural Results for Dec MDP – Step 4



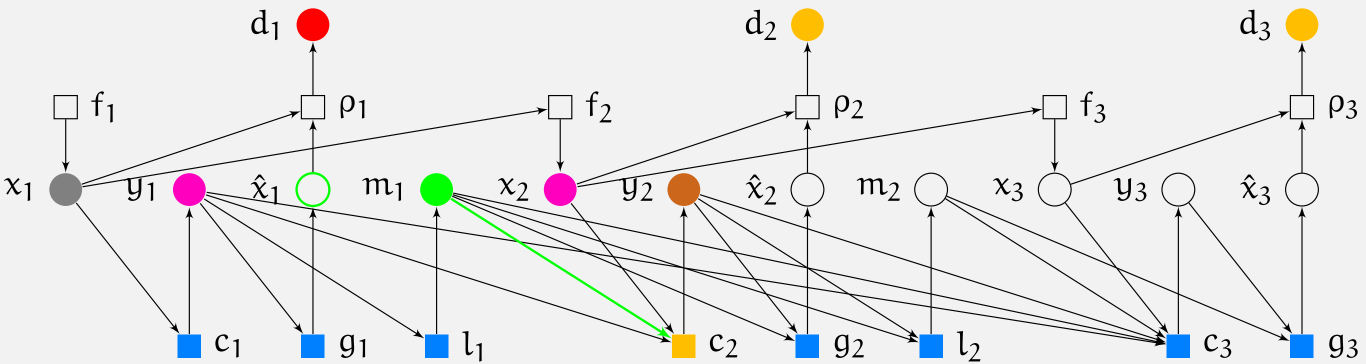
Structural Results for Dec MDP – Step 5



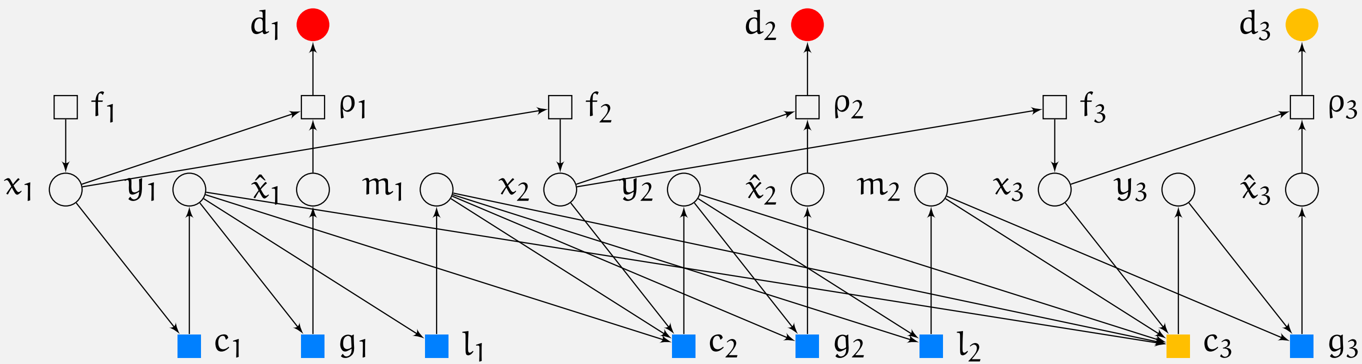
Structural Results for Dec MDP – Step 6



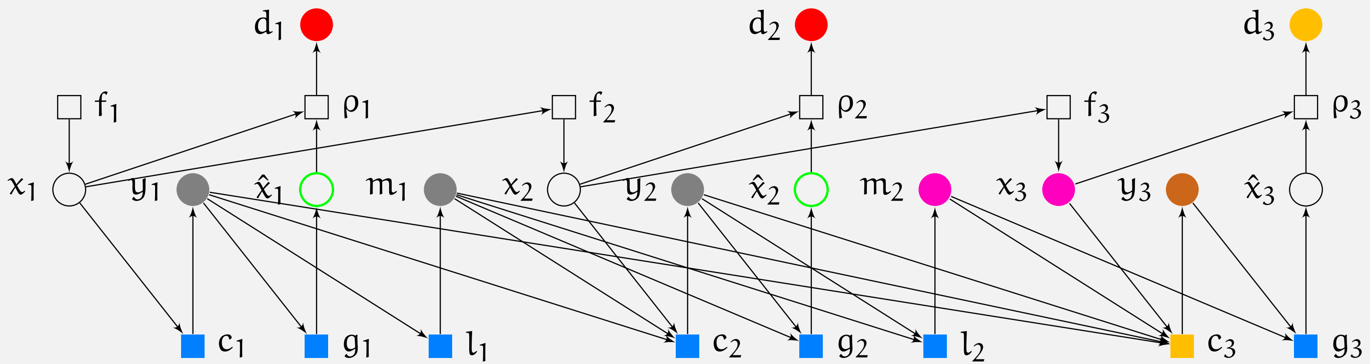
Structural Results for Dec MDP – Step 7



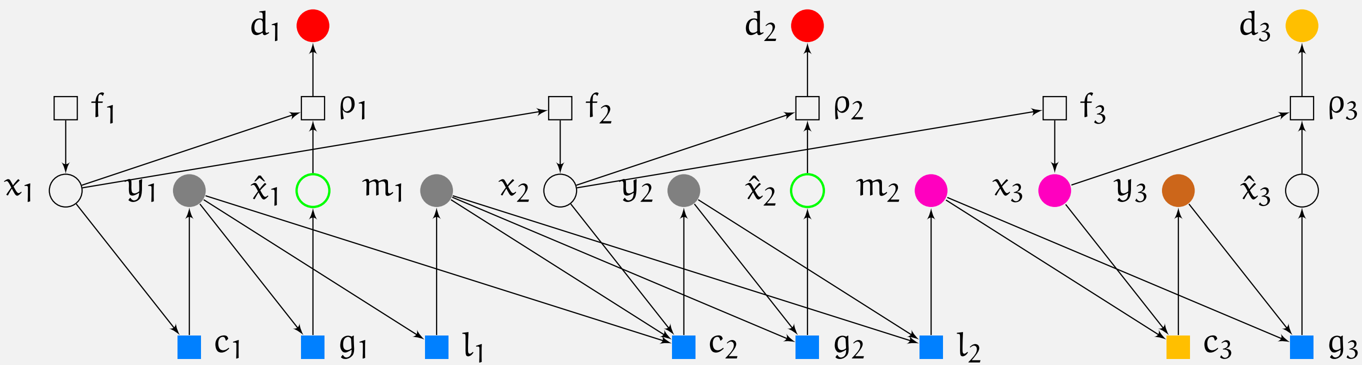
Structural Results for Dec MDP – Step 8



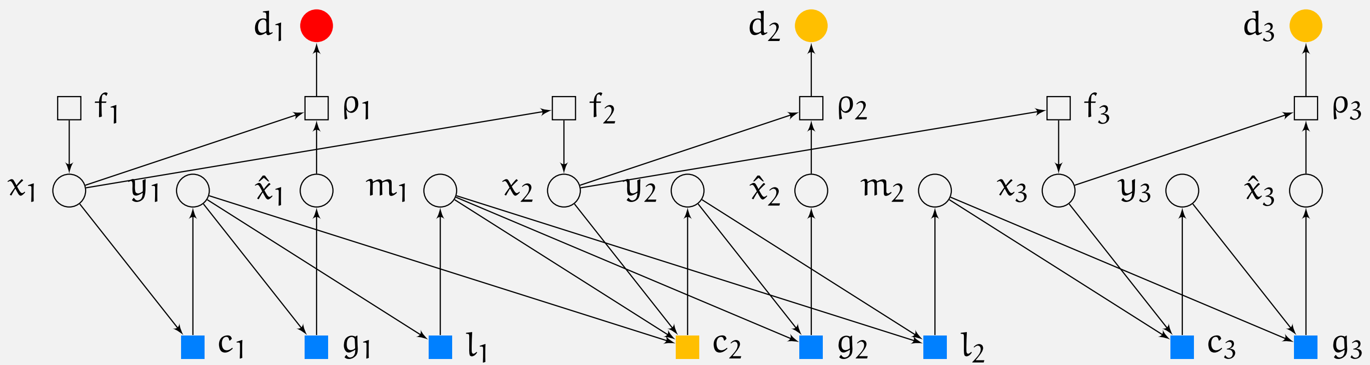
Structural Results for Dec MDP – Step 9



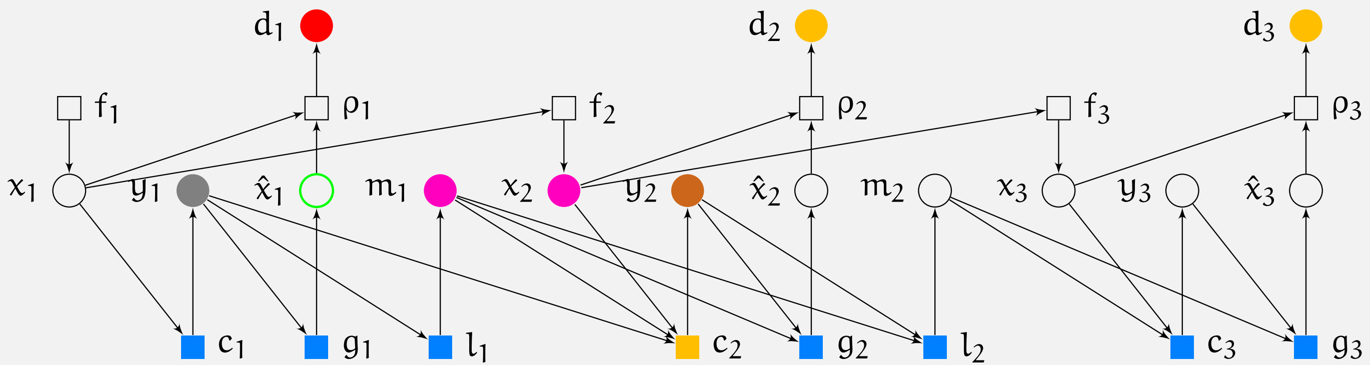
Structural Results for Dec MDP – Step 10



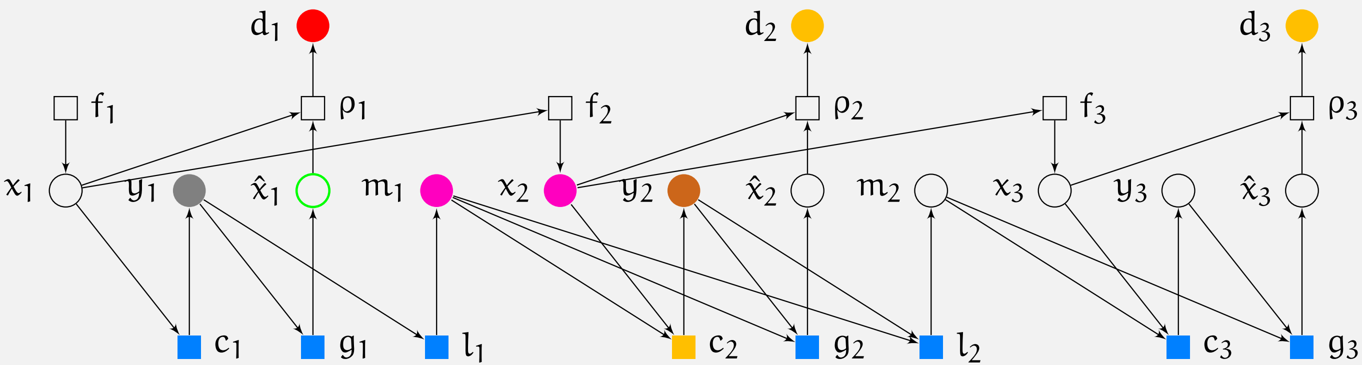
Structural Results for Dec MDP – Step 11



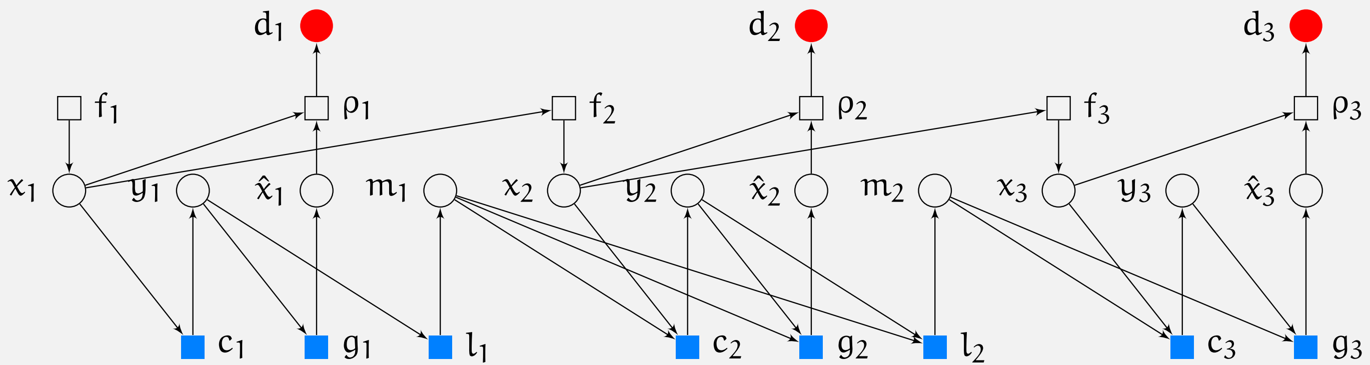
Structural Results for Dec MDP – Step 12



Structural Results for Dec MDP – Step 13

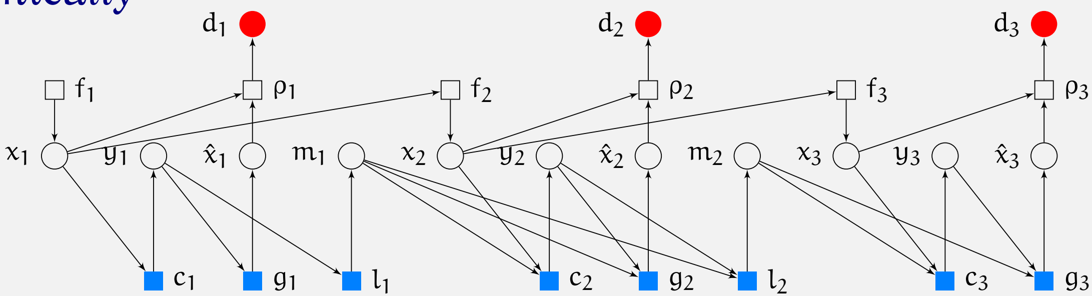


Structural Results for Dec MDP — Step 14



Structural Results for real-time communication

- Graphically



- Mathematically

  - Original Encoder: $y_t = c_t(x_1, \ldots, x_t, y_1, \ldots, y_{t-1})$
  - New encoder: $y_t = c_t(x_t, m_{t-1})$

Automated Structural results

- Simplify Once
  - For each control node
    - Find irrelevant nodes and functionally determined nodes.
    - Remove edges from irrelevant nodes, add edges from functionally determined nodes.
- Find fixed point
  - Keep on simplifying until the graph does not change
- Software Implementation
  - A EDSL to find structural results

    http://pantheon.yale.edu/~am894/code/teams/

Conclusion

Conclusion

An automated method to derive
structural results for sequential teams

- Future Directions
  - Belief States
  - Sequential decomposition

Thank you