Measure and cost dependent properties of information strucutres

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Is this really true? Can we have two systems with identical information structures that behave differently?

A controller with no memory

- State Equation: $X_{t+1} = f_t(X_t, U_t, W_t)$
- Observation Equation: $Y_t = h_t(X_t, N_t)$
- Controller with no memory: $U_t = g_t(Y_t)$

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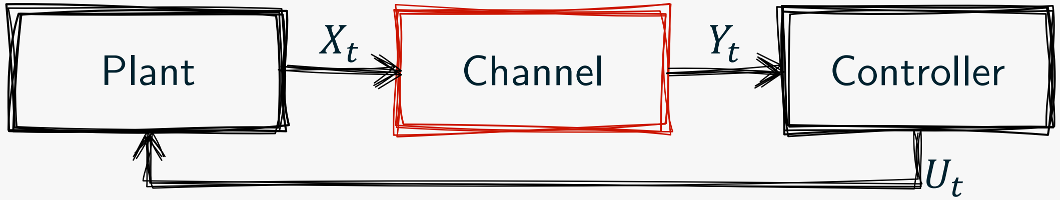

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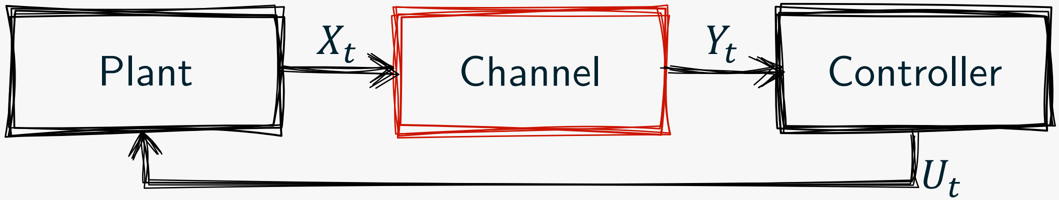

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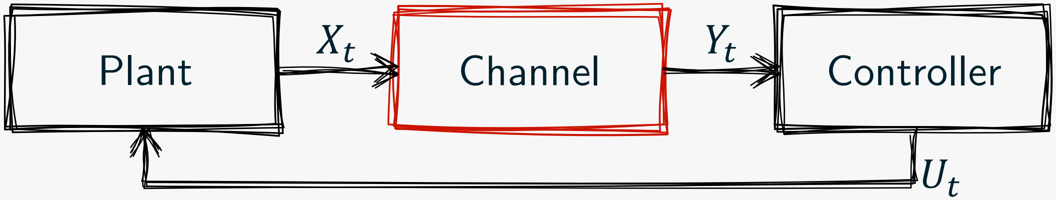

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When the channel is noiseless, the system is an MDP --- a centralized system

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Two systems with identical info structures

Perfect observations ⇒ centralized
Imperfect observations ⇒ decentralized

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We present a generalization of information structures, which we call P-generalization, that captures the usefulness of information.

This generalization depends on the coupling of the cost function and the independence properties of the probability measure

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- Implications: Follow a two step approach
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- Present coupled dynamic programs to find pbpo solution of quasiclassical info structures
  - Works for non-linear systems
  - Need to only solve parametric optimization problem

Outline of the paper

- Model
- Information Structures
- P-generalization of info structures
- Coupled dynamic programs for quasiclassical info structure
- Example

The intrinsic model

Originally proposed by Witsenhausen, 1971 and 1975

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- Objective: Choose $(g_1, \ldots, g_N)$ to minimize expected cost

Salient Features

Agents are coupled in two ways:

- Coupling through dynamics
  - D_n^*: set of agents that can influence the observations of agent n
  - $m \in D_n^* \Rightarrow$ there exist $m = m_0, m_1, \dots, m_\ell = n$ such that

$$m_{i-1} \in D_{m_i}, \quad i = 1, \dots, \ell$$

- Coupling through cost

  C_n^*: agents coupled to agent n through cost

$$C_n^* = \bigcup_{k=1}^{K} C_k \mathbb{1}\{n \in C_k\}$$

Information Structures

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  Collection of information known to each agent

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- **Classification of info structures**

  - Classical info structure
    Each agent knows the data available to all agents that act before it

  - Quasiclassical info structure
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  - Strictly classical info structures
    Each agent . . . data and control actions . . .

  - Strictly quasiclassical info structure
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Expansion of info structures

- Classical expansion of info structure

A new system obtained by

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- Quasiclassical expansion of info structure

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$$Y_n \mapsto (Y_n, Y_{D_n^*}, U_{D_n^*})$$

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- Find optimal policy g for SC (using dynamic programing)

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P-classical info structure:

Let $Q_n := \sum_{k=1}^{K} \rho_k(\omega, U_{C_k}) \mathbb{1}\{\{n \in C_k\} \cup \{\exists m \in C_k : n \in D_m^*\}\}$.

Then, an info structure is P-classical if

$$\mathbb{E}\{Q_n \mid Y_n, U_n\} = \mathbb{E}\{Q_n \mid Y_{[n]}, U_{[n]}\}$$

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The main idea (2)

We ask a similar question for quasiclassical info structures.

- What is the most relaxed info structure that we can start with such that
  - if we take its quasiclassical expansion
  - find the optimal policy for the quasiclassical expansion
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- Difficulty: No appropriate solution technique for quasiclassical systems
  - Solutions for LQG quasiclassical systems rely convexity of static LQG teams. These results do not extend to non-LQG systems.
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P-quasiclassical info structure:

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Proof outline

- The proof for both cases is constructive
  - Take expanded info structure
  - Find an optimal (or pbpo) policy
  - Construct a corresponding policy that is implementable in original system
- The details of each step conceptually simple, but notationally cumbersome due to generality of the model

Coupled Dynamic programs for quasiclassical info structure

Any quasiclassical system can be broken into a collection of coupled systems where each subsystem has a classical info structure

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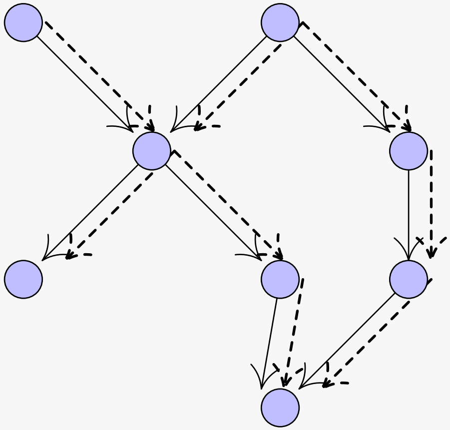

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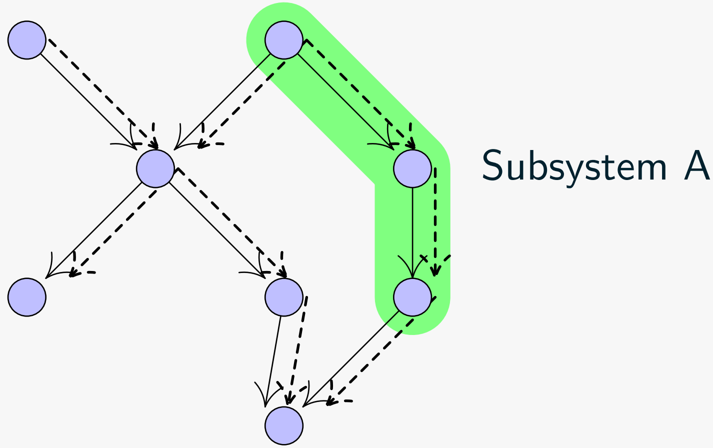

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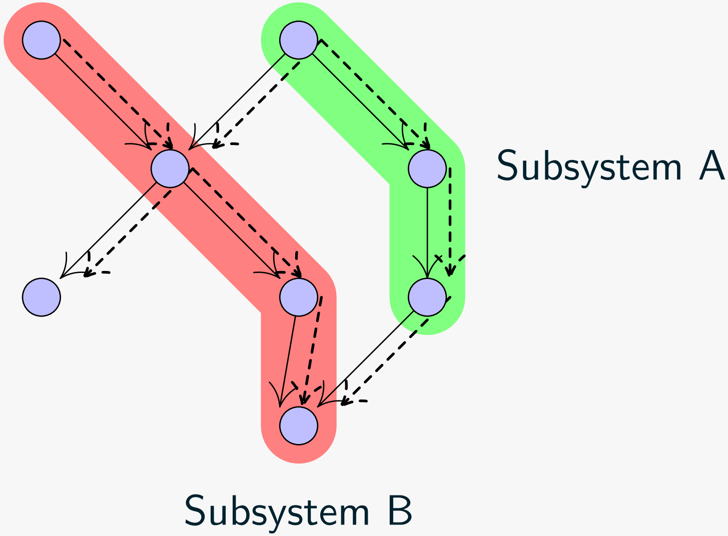

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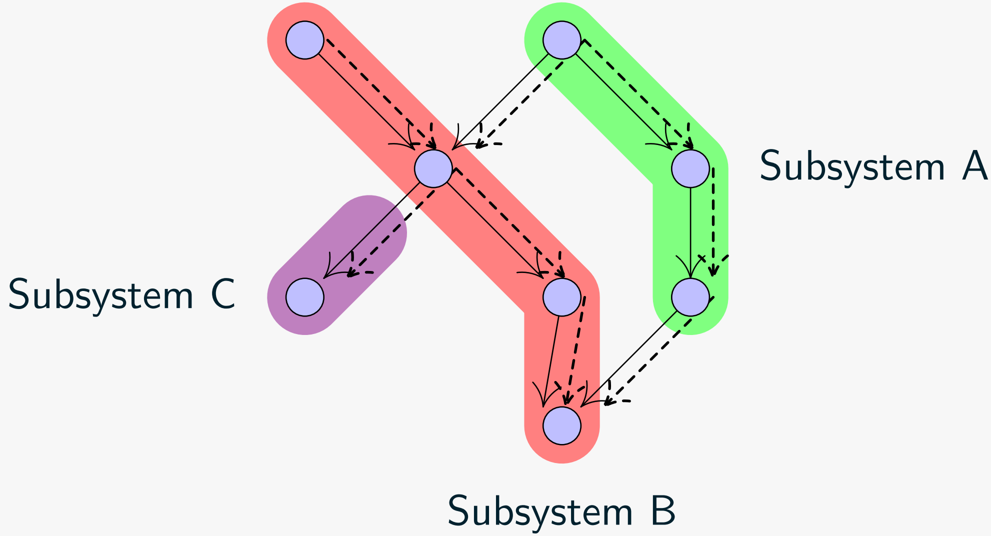

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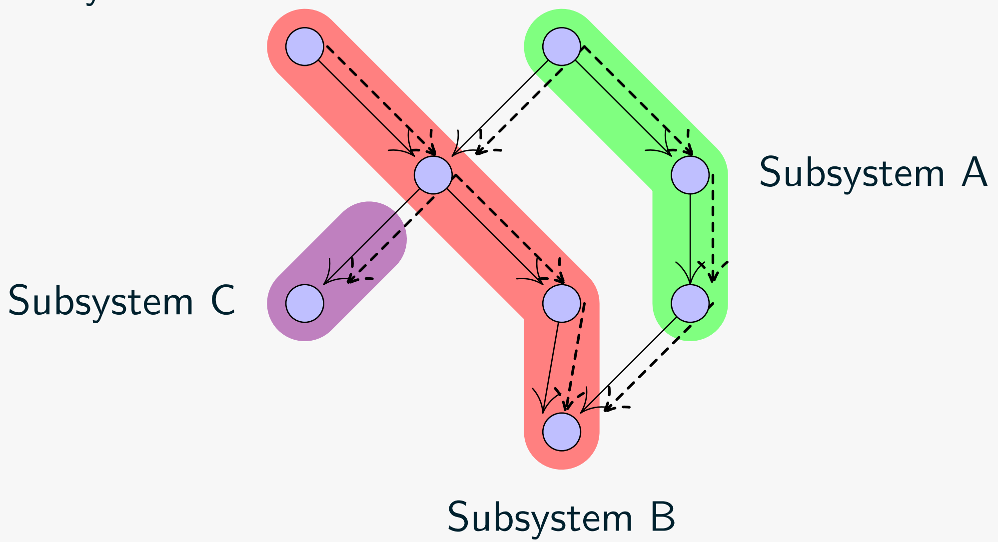

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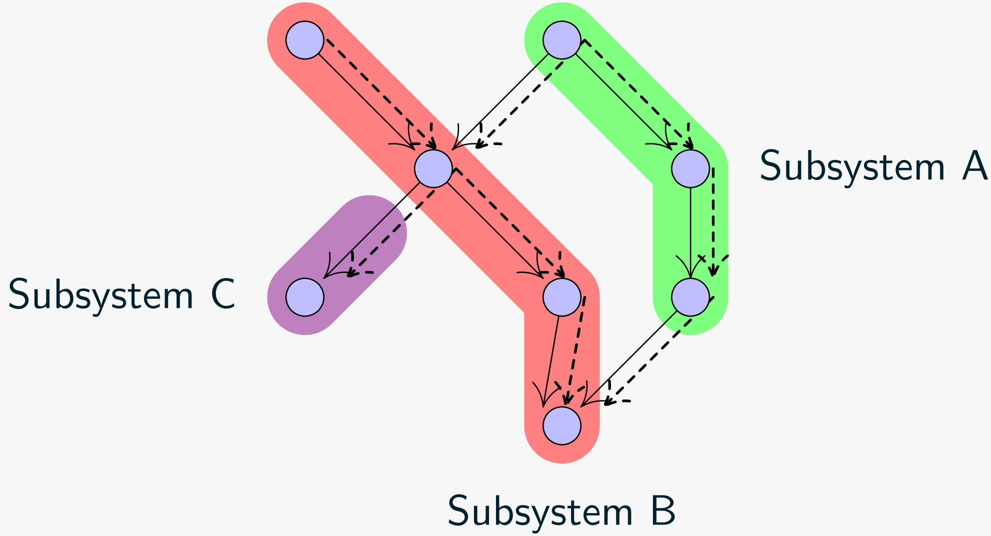
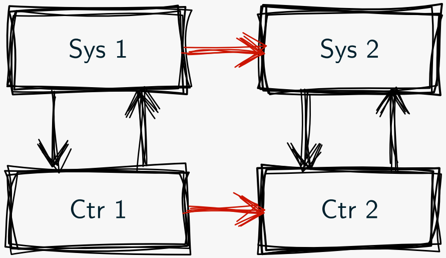


- Subsystems A, B, and C are classical
- Write a DP for each subsystem and solve them iteratively

Idea originally proposed in Teneketzis and Ho, 1987

An Example



$$x^1_{t+1} = f^1(x^1_t, u^1_t, w^1_t) \qquad x^2_{t+1} = f^2(x^1_t, x^2_t, u^2_t, w^2_t)$$

$$y^1_t = h^1(x^1_t, n^1_t) \qquad y^2_t = h^2(x^2_t, n^2_t)$$

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Choose $G^1 := (g^1_1, \dots, g^1_T)$ and $G^2 := (g^2_1, \dots, g^2_T)$ to minimize

$$\mathbb{E}\left\{\sum_{t=1}^{T} \rho(x^1_t, x^2_t, u^1_t, u^2_t)\right\}$$

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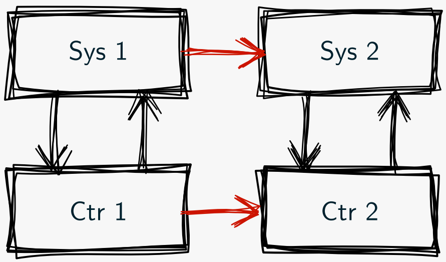

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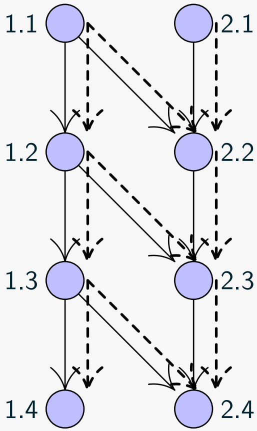
- Quasiclassical info structure
- Non-linear dynamics
- Noisy observations

An Example



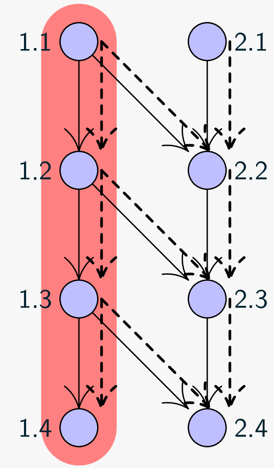
An Example

- Subsystem 1

Fix policy G^2 and solve for G^1

$$V_T^1(y_{[T]}^1, u_{[T-1]}^1) = \mathbb{E}\Big\{\rho(x_T^1, x_T^2, u_T^1, u_T^2)\Big| y_{[T]}^1, u_{[T-1]}^1\Big\}$$

$$V_t^2(y_{[t]}^1, u_{[t-1]}^1) = \mathbb{E}\Big\{\rho(x_t^1, x_t^2, u_t^1, u_t^2) + V_{t+1}^1(y_{[t+1]}^1, u_{[t]}^1)\Big| y_{[t]}^1, u_{[t-1]}^1\Big\}$$



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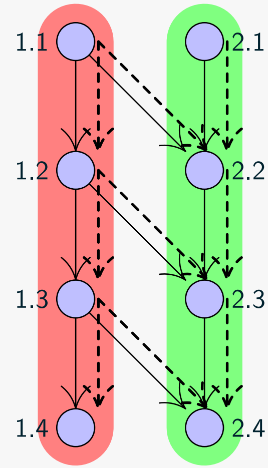

- **Subsystem 2**

  Fix policy G^1 and solve for G^2

$$V_T^2(y_{[T]}^1, y_{[T]}^2, u_{[T-1]}^1 u_{[T-1]}^2) = \mathbb{E}\Big\{\rho(x_T^2, x_T^2, u_T^2, u_T^2)\Big| y_{[T]}^1, y_{[T]}^2, u_{[T-1]}^1, u_{[T-1]}^2\Big\}$$

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Conclusion

- Defined a P-generalization of info structure

> The solution technique for any info structure
> is also applicable to its P-generalization

- Present coupled dynamic programs to find person by person optimal solution of quasiclassical info structures

Thank you