On Controllability of Leader-Follower Dynamics over a Directed Graph

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***Abstract*— In this paper, we investigate the problem of determining the structural controllability of leader-follower systems defined over directed graphs. We identify the notions of graph structural controllability and strong graph structural controllability in leader-follower systems on directed graphs. We show that accessibility is a necessary and sufficient condition for graph structural controllability over directed graphs. Next, we identify a sufficient and a necessary condition for strong graph structural controllability. Finally, we derive a sufficient condition for controllability when two graphs are cascaded. We present examples that illustrate the various concepts.**

I. Introduction

A. Motivation and literature overview

Multi-agent systems arise in a variety of applications including mobile robotics, sensor networks, distributed energy systems, and social networks. Such systems are often modeled as a graph, where the vertices correspond to the agents and the system dynamics depend on the graph Laplacian. In recent years, there has been a lot of interest in understanding the relationship between the underlying graph structure and system theoretic properties such as reachability, controllability, and stabilizability of multi-agent systems [1]–[3].

Leader-follower dynamics is a commonly used framework to model multi-agent systems. In this framework, the agents are partitioned into two sets: (i) *leaders*, which are directly influenced by control inputs; and (ii) *followers*, which are only indirectly influenced through the system dynamics. Leader-follower dynamics may be defined over either undirected or directed graphs depending on whether the coupling between the agents is bidirectional or not. Various necessary and sufficient conditions for controllability of leader-follower dynamics over *undirected* graphs are presented in [4]–[10]. Necessary conditions for controllability of leader-follower dynamics over *directed* graphs are presented in [11].

Most of the literature on controllability of leader-follower dynamics assumes that the edge weights (which determine the strength of the interaction) are known a priori. The necessary conditions of [11] for controllability of directed graphs are in terms of *almost equitable partitions (AEPs)*, which depend on the edge weights. Identifying AEPs is a combinatorial optimization problem. Moreover, even a small change in edge weights drastically changes the AEPs, and the controllability condition needs to be checked again.

For certain applications it is useful to be able to provide guarantees similar to structural controllability [12], [13] or strong structural controllability [14] for leader-follower dynamics. Both structural and strong structural controllability are defined for general linear dynamical systems. In this paper, we adapt these definitions for leader-follower dynamics (see Section II-B for precise definitions).

It is worth noting that there is a subtle difference between the standard definitions of structural (and strong structural) controllability for general linear dynamic systems and the definitions presented in this paper for leader-follower dynamics, as explained in Sec. II-B. To avoid any confusion, we refer to the notions defined in this paper as *graph* structural controllability and strong *graph* structural controllability.

In this paper, we focus on leader-follower dynamics defined over a directed graph. Our main contributions are as follows. We prove rigorously that accessibility, in the sense of there existing a directed path connecting every follower to a leader node, is a necessary and sufficient condition for graph structural controllability (Sec. II-E). We show that the existence of a Hamiltonian path, together with an absence of other forward-pointing (with respect to the Hamiltonian path) edges, is sufficent for the dynamics to be strong graph structurally controllable (Sec. II-F). Next, we derive a necessary condition for strong graph structural controllability, based on the existence of a local triangular sub-graph (Sec. II-G). We illustrate the properties derived above for a number of examples, including the role of strong graph structural controllability in cascades of graphs.

B. Preliminaries on directed graphs

A directed graph $\mathcal{G}$ is a tuple $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is the set of vertices and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of ordered edges. An edge (i, j) in $\mathcal{E}$ is considered to be ordered from i to j and i is the *in-neighbour* of j; j is the *out-neighbour* of i. The set of in-neighbours of i, called the *in-neighbourhood of* i, is denoted by $\mathcal{N}_i^-$; the set of out-neighbours of i, called the *out-neighbourhood of* i, is denoted by $\mathcal{N}_i^+$.

A *path* is a sequence of distinct vertices such that each vertex has a directed edge to the next vertex in the sequence, and no edge is repeated. A path is said to be *Hamiltonian* if it visits each vertex of the graph exactly once. A node j is said to be *accessible* from i if there exists a directed path from i to j.

A weighted graph is a graph $\mathcal{G}$ with a weight function $w\colon \mathcal{E} \to \mathbb{R} \setminus \{0\}$.[1] For any node $i \in \mathcal{N}$, $j \in \mathcal{N}_i^-$, w_{ji} denotes the weight of the edge from node j to node i.

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[1]To avoid trivialities, we assume that all weights are non-zero (and a zero weight corresponds to the absence of an edge).

The (weighted) *in-Laplacian* $L^-(\mathcal{G})$ of the graph $\mathcal{G}$ with weight function $w\colon \mathcal{E} \to \mathbb{R}$ is defined by

$$L^-_{ij}(\mathcal{G}) = \begin{cases} -\sum_{k \in \mathcal{N}_i^-} w_{ki}, & \text{if } i = j \\ w_{ji}, & \text{if } j \in \mathcal{N}_i^- \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

II. Controllability of systems with leader-follower dynamics

A. Model for leader-follower dynamics over a directed graph

We consider a continuous-time multi-agent system with leader-follower dynamics defined over directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with a weight function $w\colon \mathcal{E} \to \mathbb{R}\backslash\{0\}$. The nodes $\mathcal{N}$ of the graph correspond to the agents, the edges $\mathcal{E}$ correspond to the dynamical coupling between the agents, and the weight corresponds to the strength of the coupling. A subset $\mathcal{M}$ of $\mathcal{N}$ are *leaders*, i.e., an external control input can be applied to these nodes. Control inputs cannot be applied to other nodes, which are called *followers*.

For any node $i \in \mathcal{N}$, let $x_i(t) \in \mathbb{R}$ denote the state of agent i at time t. For any leader $i \in \mathcal{M}$, let $u_i(t) \in \mathbb{R}$ denote the control input at leader i at time t. The dynamics of the system are given as follows: for a leader i, $i \in \mathcal{M}$, we have

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i^-} w_{ji}(x_j(t) - x_i(t)) + b_i u_i(t), \tag{2a}$$

and for any other agent i, $i \in \mathcal{N} \setminus \mathcal{M}$, we have

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i^-} w_{ji}(x_j(t) - x_i(t)). \tag{2b}$$

The dynamics (2) may be written in vector form as follows. Let $n = |\mathcal{N}|$ and $m = |\mathcal{M}|$. Define $x(t) = \{x_i(t)\}_{i \in \mathcal{N}} \in \mathbb{R}^n$ and $u(t) = \{u_i(t)\}_{i \in \mathcal{M}} \in \mathbb{R}^m$ to denote the state and control input of the system at time t. Then,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$ is the in-Laplacian of $\mathcal{G}$ (defined in (1)) and $B \in \mathbb{R}^{n \times m}$. We refer to the above dynamics as system $(\mathcal{G}, \mathcal{M}, w)$. We use the short hand notation $(\mathcal{G}, \mathcal{M})$ when a certain property is (almost or entirely) independent of the choice of weights, and refer to it as a structural property.

B. (Strong) graph structural controllability

The controllability of system $(\mathcal{G}, \mathcal{M}, w)$ depends on the graph $\mathcal{G}$ as well as the weight w. In certain applications, it is useful to identify controllability properties that depend on the graph structure $(\mathcal{G}, \mathcal{M})$ but not on the weight function w. Inspired by structural controllability [13] and strong structural controllability [14], we define two notions of controllability for leader-follower dynamics.

Definition 1 (Graph structural controllability) *A graph structure $(\mathcal{G}, \mathcal{M})$ is* graph structurally controllable *if there exists a weight function $w\colon \mathcal{E} \to \mathbb{R} \setminus \{0\}$ such that the system $(\mathcal{G}, \mathcal{M}, w)$ is controllable.*

Definition 2 (Strong graph structural controllability) *A graph structure $(\mathcal{G}, \mathcal{M})$ is* strong graph structurally controllable *if for every weight function $w\colon \mathcal{E} \to \mathbb{R} \setminus \{0\}$, the system $(\mathcal{G}, \mathcal{M}, w)$ is controllable.*

The notions of structural controllability [12], [13] and strong structural controllability [14] are different from the graph structural and strong graph structural controllability defined above. Structural and strong structural controllability are defined for any linear dynamical system $\dot{x}(t) = Ax(t) + Bu(t)$.

1) A system (A, B) is structurally controllable if *there exist* matrices (A', B') with the same sparsity pattern (or structure) as (A, B) such that the system (A', B') is controllable.
2) A system (A, B) is strong structurally controllable if *for all* matrices (A', B') with the same sparsity pattern as (A, B), the system (A', B') is controllable.

In a system with leader-follower dynamics, the A matrix in (3) is the in-Laplacian matrix of the graph. Thus, for graph structural and strong graph structural controllability we have to check controllability for matrices (A', B) such that there exists a weight function w for which A' is the in-Laplacian of the weighted graph $(\mathcal{G}, w)$. An equivalent representation is that A' has the same sparsity pattern as A and $A\mathbf{1} = 0$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones. Thus, we have the following.

Graph structural controllability is a stronger property than structural controllability. In particular, a system (A, B) with leader-follower dynamics (3) could be structurally controllable, i.e., there may exist matrices (A', B') with the same structure as (A, B) such that the system (A', B') is controllable. However, A' need not be a graph in-Laplacian and B' need not be same as B. Thus, structural controllability does not imply graph structural controllability. On the other hand, graph structural controllability implies structural controllability.

Strong graph structural controllability is a weaker property than strong structural controllability. In particular, a system (A, B) with leader follower dynamics (3) could be strong structurally controllable, i.e., for all matrices A' with the same sparsity pattern as A such that $A'\mathbf{1} = 0$, the system (A', B) is controllable. However, strong graph structural controllability does not say anything about controllability of system (A', B') where A' has the same sparsity pattern as A but $A'\mathbf{1} \neq 0$. Thus, strong graph structural controllability does not imply strong structural controllability. On the other hand, strong structural controllability implies strong graph structural controllability.

C. Examples

To illustrate the aforementioned concepts, consider the following two examples.

Example 1 Consider the two agent system with leader follower dynamics shown in Fig. 1a. Here, node 1 is the leader and node 2 is a follower. The dynamics corresponding to this

(a) System for Example 1 (b) System for Example 2

Fig. 1: Some examples of multi-agent systems with leader-follower dynamics. The solid circles denote leader nodes and the hollow circles denote the follower nodes.

graph are

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ w_{12} & -w_{12}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$

The controllability matrix of the system is

$$\begin{bmatrix}1 & 0\\ 0 & w_{12}\end{bmatrix}$$

which has full rank for every $w_{12} \neq 0$. Hence, the system is strong graph structurally controllable and, a fortiori, graph structurally controllable.

Example 2 Consider the three agent system with leader follower dynamics shown in Fig. 1b. Here, node 1 is the leader and nodes 2 and 3 are followers. The dynamics corresponding to this graph are

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ w_{12} & -w_{12} & 0\\ w_{13} & 0 & -w_{13}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}u(t)$$

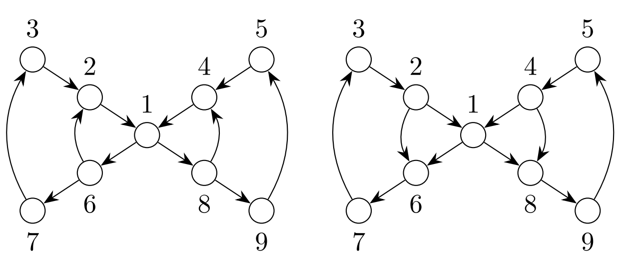
The controllability matrix of this system is

$$\begin{bmatrix}1 & 0 & 0\\ 0 & w_{12} & -w_{12}^2\\ 0 & w_{13} & -w_{13}^2\end{bmatrix}$$

which is rank deficient when $w_{12} = w_{13}$. Hence, the system is graph structurally controllable but not strong graph structurally controllable.

D. Problem Statement

We are interested in understanding properties of the graph structure $(\mathcal{G}, \mathcal{M})$ that characterize whether or not a system is (strong) graph structurally controllable.



(a) (b)

Fig. 2: Two "butterfly graphs"

For example, consider the multi-agent system defined over the "butterfly graphs" shown in Fig. 2. Based on the features of the graph, we want to answer if the system is controllable by a single leader when the leader is at node i (say for $i = 1$ or $i = 2$). Our results show that irrespective of the location of the leader, both graphs are always graph structurally controllable (see Theorem 2). We provide a sufficient condition for strong graph structural controllability (see Theorem 3) that is satisfied for the graph of Fig. 2(b) with leader at node 6 or 8 but not for Fig. 2(a) for any location of the leader.

E. Main results for graph structural controllability

Similar to [13], we show that a leader-follower system is graph structurally controllable if and only if it is controllable for almost all choices of the weight functions.

Theorem 1 *The system $(\mathcal{G}, \mathcal{M})$ is graph structurally controllable if and only if for any weight function w_0 and any $\epsilon \in \mathbb{R}_{>0}$, there exists a weight function w_1 satisfying $\|w_1 - w_0\| < \epsilon$ such that the system $(\mathcal{G}, \mathcal{M}, w_1)$ is controllable.*

As argued earlier, graph structural controllability is a stronger notion than structural controllability. Therefore, the above result does not follow from [13]. However, the proof is similar in spirit. See Appendix I.

Our main result for graph structural controllability is the following.

Theorem 2 *A leader-follower system is graph structurally controllable if and only if every node is accessible from some leader node.*

See Appendix II for proof.

F. Main result for strong graph structural controllability

Theorem 3 *Consider a leader-follower system with a single leader. Suppose the following condition holds:*

(C) *There exists a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, \ldots, n)$ such that*
  a) *i_1 is the leader node*
  b) *$(i_1, i_2, \ldots, i_n)$ is Hamiltonian path*
  c) *For any $k, \ell \in \{1, \ldots, n\}$ such that $k < \ell + 1$, the graph $\mathcal{G}$ does not have an edge from node i_k to node i_ℓ.*

Then,

1) *The system is strong graph structurally controllable.*
2) *For any weight function $w \colon \mathcal{E} \to \mathbb{R} \setminus \{0\}$, the eigenvalues $\{\lambda_k\}_{k=1}^n$ of the controllability matrix of the system $(\mathcal{G}, \mathcal{M}, w)$ are given by: $\lambda_1 = 1$ and for $k \in \{2, \ldots, n\}$*

$$\lambda_k = \prod_{\ell=1}^{k-1} w_{i_\ell, i_{\ell+1}}. \tag{4}$$

See Appendix III for proof. The above result is similar in spirit to [14, Theorem 1], but is richer for having identified the eigenvalues of the controllability matrix. This allows us to measure how far the system is from being uncontrollable.

A system with a Hamiltonian path and no forward-pointing edges can be thought of as a hierarchical system [12]. General hierarchical systems were shown to be structurally controllable in [12]. In comparison, the Laplacian structure allows us to strengthen the result and include *all* non-zero edge weights.

Although we haven't identified a necessary condition for strong graph structural controllability, we provide an example to show that of a leader-follower network that does not satisfy condition (C) of Theorem 3 and is not strong graph structurally controllable.

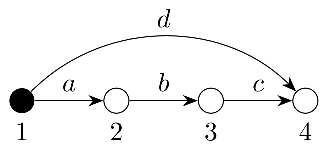


Fig. 3: An example of a leader-follower system that doesn't satisfy condition (C) of Theorem 3 and is not strong graph structurally controllable.

Consider the four node system shown in Fig. 3. Note that this system does not satify condition (C) due to the forward edge $(1, 4)$. For this system

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & -a & 0 & 0 \\ 0 & b & -b & 0 \\ d & 0 & c & -(c+d) \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The controllability matrix of this system is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & -a^2 & a^3 \\ 0 & 0 & ab & -ab(a+b) \\ 0 & d & -d(c+d) & d(c+d)^2 + abc \end{bmatrix}$$

and

$$\det(C) = a^2 b(c+d)(ab - ad - bd + cd + d^2)$$

which is zero for $c = -d$. Thus, the system is not strong graph structurally controllable.

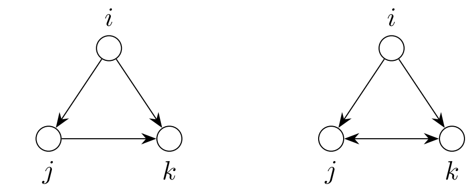
G. Necessary condition for classical controllability

Although we have identified necessary and sufficient conditions for graph structural controllability as well as sufficient condition for strong graph structural controllability, there are situations where one is interested in controllability of a graph with a pre-specified weight function. In [11], necessary conditions for controllability for leader-follower dynamics over directed graphs are provided in terms of *almost equitable partitions (AEPs)*. Here, we present an alternative and easier to verify necessary condition for controllability of $(\mathcal{G}, \mathcal{M}, w)$.

Theorem 4 *Consider a leader-follower system with a given weight function w. Suppose the following condition holds:*

(D) *There exist nodes i, j, k such that*

a) *nodes j and k are not leaders;*
b) $N_j^- = \{i\}$ *or* $N_j^- = \{i, k\}$*;*
c) $N_k^- = \{i, j\}$*;*
d) $w_{ij} = w_{ik}$ *or* $w_{ij}w_{ik} + w_{ik}w_{kj} + w_{ij}w_{jk} = 0$



(a) $N_j^- = \{i\}$ and $N_k^- = \{i, j\}$.   (b) $N_j^- = \{i, k\}$ and $N_k^- = \{i, j\}$.

Fig. 4: Two possible sub-graphs that satisfy condition (D.b) and (D.c) of Theorem 4

Then, the system is not controllable.
It can be shown that under condition (D), the partition consisting of nodes $\{j, k\}$ in one cell and all other nodes in their own unique cell is an equitable partition [10]. We present a more direct proof in Appendix IV.

Remark 1 When conditions (D.a)–(D.c) of Theorem 4 are satisfied, the system is not strong graph structurally controllable. Thus, the lack of such a sub-graph is a necessary condition for strong graph structural controllability.

Remark 2 In the special case when all edge weights are 1, the first part of (D.d) is always satisfied (while the second part cannot be satisfied).

III. Controllability under Graph Cascades

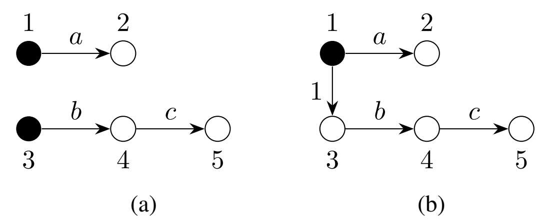


Fig. 5: An example illustrating that the cascade of two strong graph structurally controllable systems need not be controllable.

Given two controllable systems $(\mathcal{G}_1, \mathcal{M}_1, w_1)$ and $(\mathcal{G}_2, \mathcal{M}_2, w_2)$ (where $|\mathcal{M}_2| = 1$), suppose we connect one of the nodes in $\mathcal{G}_1$ to the leader node in $\mathcal{G}_2$ with a weighted edge and make the leader node of $\mathcal{G}_2$ a follower. Is the cascaded system controllable? Such a question arises when we have a controllable system (e.g., a UAV network) and we want to control it as part of a bigger network (e.g., from a network of mobile ground robots). In general, such a cascade may not result in a controllable system *even if* $(\mathcal{G}_1, \mathcal{M}_1)$ *and* $(\mathcal{G}_2, \mathcal{M}_2)$ *are strong graph structurally controllable*. To see this, suppose we have the two networks shown in Fig. 5(a) both of which are strongly controllable and we connect node 1 to node 3 to obtain the network shown in Fig. 5(b). It can be shown that the system of Fig. 5 is not strong graph structurally controllable. Specifically, it loses controllability if $a = b$ or $a = c$. Below we identify sufficient conditions under which the cascade of two graphs is controllable.

Theorem 5 *Consider two systems* $(\mathcal{G}_1, \mathcal{M}_1, w_1)$ *and* $(\mathcal{G}_2, \mathcal{M}_2, w_2)$ *where* $|\mathcal{M}_1| = |\mathcal{M}_2| = 1$*. Suppose the following conditions are satisfied:*

1) $\mathcal{G}_1$ *satisfies condition (C) of Theorem 3.*
2) $(\mathcal{G}_2, \mathcal{M}_2, w_2)$ *is controllable.*

Then, the cascaded system formed by connecting the terminal node (i.e., node with label i_{n_1} *in (C)) of* $\mathcal{G}_1$ *to the leader of* $\mathcal{G}_2$ ***with any weight*** *(and making the leader of* $\mathcal{G}_2$ *a follower) is controllable.*

See Appendix V for proof. The above theorem shows that the cascaded system formed by connecting node 2 to node 3 in Fig. 5(a) is controllable.

IV. Conclusions

In this paper, we identified the notions of graph structural controllability and strong graph structural controllability in linear leader-follower systems on directed graphs. Their relation to the prevalent notions of structural controllability and strong structural controllability were pointed out. We proved that accessibility is a necessary and sufficient condition for graph structural controllability. We also proved that the existence of a Hamiltonian path and an absence of other forward-pointing edges (with respect to the Hamiltonian path) is sufficient for strong graph structural controllability. The effects of adding a forward edge were demonstrated through a necessary condition for loss of strong graph structural controllability, involving a triangle subgraph. Finally, we demonstrated the role played by strong graph structural controllability in preserving controllability when two dynamical systems are cascaded.

Appendix I
Proof of Theorem 1

To prove sufficiency, note that if the system $(\mathcal{G}, \mathcal{M}, w_1)$ is controllable, then by definition, $(\mathcal{G}, \mathcal{M})$ is structurally controllable.

To prove necessity, suppose $(\mathcal{G}, \mathcal{M})$ is structurally controllable. If the system $(\mathcal{G}, \mathcal{M}, w_0)$ is controllable, then we are done. If not, then let w_2 be a weight function for which the system $(\mathcal{G}, \mathcal{M}, w_2)$ is controllable. Define, $w(\lambda) = (1-\lambda)w_0 + \lambda w_2$. Note that $\|w(\lambda) - w_0\| = \lambda \|w_2 - w_0\|$. Thus, there exists a $\lambda_0 \in (0,1)$ such that for all $\lambda \in (0, \lambda_0)$, $\|w(\lambda) - w_0\| < \epsilon$.

Let $C(\lambda)$ denote the controllability matrix of the system $(\mathcal{G}, \mathcal{M}, w(\lambda))$. By assumption, the system $(\mathcal{G}, \mathcal{M}, w(1))$ is controllable. Thus $\operatorname{rank} C(1) = n$ (where $n = |\mathcal{N}|$), which means that there exists an $n \times n$ submatrix $D(1)$ of $C(1)$ such that $\det(D(1)) \neq 0$. Let $D(\lambda)$ denote the same submatrix for a general λ.

Note that each term in $C(\lambda)$ and thus $D(\lambda)$ is a polynomial in λ. Hence, $\det(D(\lambda))$ is also a polynomial in λ. We know that $\det(D(1)) \neq 0$, hence the polynomial is not identically zero and it has only a finite number of real roots. If we pick a $\lambda \in (0, \lambda_0)$ other than these roots, then $\det(D(\lambda)) \neq 0$, which implies that $\operatorname{rank} C(\lambda) = n$. Thus, there exists a weight function $w_1 = w(\lambda)$ such that $\|w_1 - w_0\| < \epsilon$ and the system $(\mathcal{G}, \mathcal{M}, w_1)$ is controllable.

Appendix II
Proof of Theorem 2

A. Proof of necessity

If there exists a set consisting of nodes that are inaccessible from the leaders, we can find a permutation of indices so that the matrix A is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

while $B = [B_1 \ 0]^\mathsf{T}$. This is the well-known Kalman decomposition, with the second set of states (corresponding to A_{22}) being uncontrollable.

B. Preliminaries lemmas for the proof of sufficiency

We first prove two preliminary lemmas that are used as part of the proof of sufficiency.

Lemma 1 *Given a graph* $\mathcal{G}$*, let* $\mathcal{G}'$ *denote a graph formed by adding an additional edge in* $\mathcal{G}$*. If the system* $(\mathcal{G}, \mathcal{M})$ *is graph structurally controllable, then so is* $(\mathcal{G}', \mathcal{M})$*.*

Proof: Since $(\mathcal{G}, \mathcal{M})$ is graph structurally controllable, there exists a weight function w such that $(\mathcal{G}, \mathcal{M}, w)$ is controllable. Let $w'(\lambda)$ be a weight function for $(\mathcal{G}', \mathcal{M})$ which assigns the same weight as w to the edges in $\mathcal{G}$ and assigns a weight λ to the new edge. Let $C(\lambda)$ denote the controllability matrices of $(\mathcal{G}', \mathcal{M}, w'(\lambda))$. Note that the system $(\mathcal{G}', \mathcal{M}, w'(0))$ is same as the system $(\mathcal{G}, \mathcal{M}, w)$, which is controllable. Thus, $\operatorname{rank} C(0) = n$ (where $n = |\mathcal{N}|$), which means that there exists an $n \times n$ submatrix $D(0)$ of $C(0)$ such that $\det(D(0)) \neq 0$. Let $D(\lambda)$ denote the same submatrix for general λ.

Note that each term in $C(\lambda)$ and thus $D(\lambda)$ is a polynomial in λ. Hence, $\det(D(\lambda))$ is also a polynomial in λ. We know that $\det(D(0)) \neq 0$, hence the polynomial is not identically zero and it has only a finite number of real roots. If we pick a λ that is not a root, $\det(D(\lambda)) \neq 0$, which implies that $\operatorname{rank} C(\lambda) = n$. Thus, there exists a weight function $w_1 = w'(\lambda)$ such that the system $(\mathcal{G}', \mathcal{M}, w_1)$ is controllable. ∎

Lemma 2 *Given a graph* $\mathcal{G}$ *with a single leader (i.e.,* $|\mathcal{M}| = 1$*), let* $\mathcal{G}'$ *denote a graph formed by adding a new node and an incoming edge to it from a node in* $\mathcal{G}$*. If the system* $(\mathcal{G}, \mathcal{M})$ *is graph structurally controllable, then so is* $(\mathcal{G}', \mathcal{M})$*.*

Proof: Let (A, B) denote the system matrices of the system $(\mathcal{G}, \mathcal{M})$ and let C be the controllability matrix of (A, B). Let A', B', and C' denote the corresponding terms of the system $(\mathcal{G}', \mathcal{M})$. Let the number of nodes in $\mathcal{G}$ be n. Then number of nodes in $\mathcal{G}'$ is $(n+1)$. We assume that the index of the leader node is 1; thus, B and B' are n and $(n+1)$ dimensional column vectors with only their first entries being 1 and the rest being 0.

Let e denote a standard n-dimensional unit vector with the position of 1 corresponding to the node in $\mathcal{G}$ which has an outgoing edge to the new node and let λ denote the weight of that edge. Then

$$A' = \begin{bmatrix} A & 0 \\ \lambda e^\mathsf{T} & -\lambda \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Furthermore, we have that

$$C' = \begin{bmatrix} B & AB & \dots & A^{n-1}B & A^nB \\ c_0 & c_1 & \dots & c_{n-1} & c_n \end{bmatrix},$$

where $c_0 = 0$ and for $k \geq 0$, $c_{k+1} = \lambda e^\mathsf{T} A^k B - \lambda c_k$. By recursive substitution, we get that

$$c_k = \sum_{j=0}^{k-1} (-1)^j \lambda^{j+1} a_{k-j-1}, \tag{5}$$

where we have used $a_k = e^\mathsf{T} A^k B$.

Since $(\mathcal{G}, \mathcal{M})$ is graph structurally controllable, C is full rank and, hence, the columns of C are linearly independent. Since the columns of C are sub-vectors of the first n columns of C', the latter are also linearly independent. Thus, to show that C' is full rank, we need to verify that there exists a weight λ such that the $n+1$ column of C' is linearly independent of the first n.

We prove the independence by contradiction. Suppose that is not the case, then there must exist weights $\{\beta_k\}_{k=0}^n$, $\beta_n = 1$, such that for all weights $\lambda \neq 0$,

$$\sum_{k=0}^n \beta_k A^k B = 0 \quad \text{and} \quad \sum_{k=0}^n \beta_k c_k = 0. \tag{6}$$

Substituting the values of c_k from (5) in the second equation in (6), we get a polynomial in λ, which we denote by $p(\lambda)$. The coefficient of λ^{j+1} in $p(\lambda)$ is given by

$$(-1)^j \left[\sum_{k=j+1}^n \beta_k a_{k-j-1} \right].$$

Thus, for the second equation in (6) to be satisfied for all values of λ, $p(\lambda) = 0$ for all λ, which means that all the coefficients of λ must be identically zero. We argue below that there is a term in $p(\lambda)$ that is not identically zero.

Note that all $\{a_k\}_{k=0}^n$ cannot be identically zero because that will imply that there is a row of C that is identically zero. Let ℓ denote the smallest k such that $a_k \neq 0$. Then, the coefficient of $\lambda^{n-\ell}$ in the second equation of (6) is $(-1)^{n-\ell-1}\beta_n a_\ell$ which is non-zero. Thus, $p(\lambda)$ cannot be identically zero for all λ. Hence (6) cannot be satisfied for all λ. So, there exists a weight λ such that all columns of C' are linearly independent. ■

C. Proof of sufficiency

We separately prove the result for single and multiple leaders.

1) Part 1: The case with single leader: Consider any system $(\mathcal{G}, \mathcal{M})$ where $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $|\mathcal{M}| = 1$, and every node is accessible from the leader node. Let $\mathcal{G}_0$ denote the graph $(\mathcal{M}, \emptyset)$ consisting of just the leader node. Clearly $\mathcal{G}_0$ is structurally controllable. Since all nodes of $\mathcal{G}$ are accessible from the leader node, it follows that we can build $\mathcal{G}$ starting from $\mathcal{G}_0$ by applying a sequence of operations, where each operation falls into one of two categories:

1) Add a node and an edge from the existing graph to the new node; from Lemma 2, the resulting graph is graph structurally controllable.
2) Add an edge connecting the existing nodes; from Lemma 1, the resulting graph is graph structurally controllable.

After each operation, the system remains graph structurally controllable. Hence, the final system $(\mathcal{G}, \mathcal{M})$ is also graph structurally controllable. This completes the proof of Theorem 2 for a single leader. ■

2) Part 2: The case with multiple leaders: Consider any system $(\mathcal{G}, \mathcal{M})$ where $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, $|\mathcal{M}| = m > 1$, and every node is accessbile from one of the leader nodes. The set of nodes can be partitioned into disjoint subsets $\mathcal{N}_1, \dots, \mathcal{N}_m$ such that (i) each leader node belongs to exactly one $\mathcal{N}_i$; (ii) no $\mathcal{N}_i$ has more than one leader node, and (iii) each node in $\mathcal{N}_i$ is accessible from its leader, with the edges from $\mathcal{E}$.

For every $i \in \{1, \dots, m\}$, we define the induced subgraph $\mathcal{G}_i = (\mathcal{N}_i, \mathcal{E}_i)$, where $\mathcal{E}_i = \{(i,j) \in \mathcal{E} : i, j \in \mathcal{N}_i\}$. It follows from Part 1 above that $\mathcal{G}_i$, $i \in \{1, \dots, m\}$, is graph structurally controllable. Thus, the system $(\mathcal{G}', \mathcal{M})$, where

$$\mathcal{G}' = \bigcup_{i=1}^m G_i = \left(\mathcal{N}, \bigcup_{i=1}^m \mathcal{E}_i \right)$$

is also graph structurally controllable. Graph $\mathcal{G}$ may obtained from $\mathcal{G}'$ by adding the mising edges one-by-one. By Lemma 1, $(\mathcal{G}, \mathcal{M})$ is graph structurally controllable.

Appendix III
Proof of Theorem 3

We use the notation $a = \bullet$ to denote that the variable a may take any value and use $a = *$ to denote that a is non-zero.

For ease of exposition, we assume that the agents have been labeled such that the permutation $(i_1, i_2, \dots, i_n)$ is $(1, 2, \dots, n)$. Since there is no edge of the form (k, ℓ) for $k < \ell + 1$ in the graph $\mathcal{G}$, the matrix A is of the form:

$$A_{ij} = \begin{cases} \bullet, & \text{if } i < j+1 \\ *, & \text{if } i = j+1 \\ 0, & \text{if } i > j+1 \end{cases} \tag{7}$$

where $A_{j+1,j} = w_{j,j+1}$.

Using induction, we can show that the form A^k is similar. In particular, we have the following.

Lemma 3 *For any $k \in \{1, \dots, n-1\}$, A^k is a $n \times n$ matrix of the form*

$$A^k_{ij} = \begin{cases} \bullet, & \textit{if } i < j+k \\ *, & \textit{if } i = j+k \\ 0, & \textit{if } i > j+k \end{cases} \tag{8}$$

where

$$A^k_{j+k,j} = w_{j,j+1} w_{j+1,j+2} \cdots w_{j+k-1,j+k}. \tag{9}$$

Proof: We prove the result by induction. For $k = 1$, the form of A given by (7) is the same as (8). This forms the basis of induction. Now assume that (8) is true for some

$k \in \{1, \ldots, n-2\}$, and consider A^{k+1}. Note that

$$A_{ij}^{k+1} = \sum_{\ell=1}^{n} A_{i\ell}^{k} A_{\ell j} \stackrel{(a)}{=} \sum_{\ell=i-k}^{j+1} A_{i\ell}^{k} A_{\ell j}$$
$$= \begin{cases} 0, & \text{if } i-k > j+1 \\ A_{j+1+k,j+1}^{k} A_{j+1,j}, & \text{if } i-k = j+1 \\ \bullet, & \text{if } i-k < j+1 \end{cases} \quad (10)$$

where (a) uses the fact that $A_{i\ell}^{k} = 0$ for $\ell < i-k$ (Eq. (8), the induction hypothesis) and that $A_{\ell j} = 0$ for $\ell > j+1$ (Eq. (7)). From the induction hypothesis, we also have $A_{j+1+k,j+1}^{k} = *$. Thus, (10) is of the same form as (8). Hence, by the principle of induction, Eq. (8) holds for all k.

Furthermore, Eq (10) implies that the $A_{j+k,j}^{k}$ elements satisfy the following recursion: $A_{j+1,j} = w_{j,j+1}$, and for any $k \in \{1, \ldots, n-2\}$,

$$A_{j+1+k,j}^{k+1} = A_{j+1+k,j+1}^{k} A_{j+1,j}.$$

By recursively expanding this expression, we get

$$A_{j+k,j}^{k} = A_{j+k,j+k-1} A_{j+k-1,j+k-2} \cdots A_{j+1,j},$$

which is the same as (9). ■

Lemma 4 *Let $C = [B \; AB \; \cdots A^{n-1}B]$ denote the controllability matrix of the system. Then, C is an upper triangular matrix of the form*

$$C_{ij} = \begin{cases} \bullet, & \text{if } j < i \\ A_{i,1}^{i-1}, & \text{if } j = i \\ 0, & \text{if } j > i \end{cases} \quad (11)$$

where $A_{11}^{0} = 1$ and for $i \in \{1, \ldots, n-1\}$,

$$A_{i,1}^{i-1} = w_{12} w_{23} \ldots w_{i-1,i}. \quad (12)$$

Therefore, we have the following:

1) *C is full rank.*
2) $\det(C) = w_{12}^{n-1} w_{23}^{n-2} \cdots w_{n-2,n-1}^{2} w_{n-1,n}$.
3) *The eigenvalues of C are given by $\{A_{i,1}^{i-1}\}_{i=1}^{n}$.*

Proof: Recall that B is a $n \times 1$ matrix of the form $[1 \; 0 \; \cdots \; 0]^{\intercal}$. Thus, the first column of C is of the form (11) (observe that $A^0 = I$). Moreover, $A^{i-1}B$ (which is the i-th column of C) is equal to the first column of A^{i-1}; from (8), we get that the first column of A^{i-1} is of the form (11). This establishes the upper triangular structure of C.

From (9), we get that the form of A_{i1}^{i-1} given in (12). Thus, C is an upper triangular matrix with non-zero diagonal terms. Hence, it is full rank.

The determinant of an upper triangular matrix is equal to the product of its diagonal terms. Thus,

$$\det(C) = A_{11}^{0} A_{21}^{1} A_{31}^{2} \cdots A_{n1}^{n-1}. \quad (13)$$

Substituting (12) in (13), we get the determinant of C.

Since C is a square matrix, its eigenvalues are given by the roots of the characteristic polynomial $\det(\lambda I - C)$. $\lambda I - C$ is also upper triangular, so its determinant is given by the product of the diagonal terms. Hence, the $\{A_{i1}^{i-1}\}_{i=1}^{n}$ are the roots of $\det(\lambda I - C)$. ■

The proof of Theorem 3 follows immediately from Lemma 4.

APPENDIX IV
PROOF OF THEOREM 4

Without loss of generality, we permute the nodes such that nodes (i, j, k) are the first three nodes and write the A matrix as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} = \begin{bmatrix} w_{ii} & w_{ji} & w_{ki} \\ w_{ij} & w_{jj} & w_{kj} \\ w_{ik} & w_{jk} & w_{kk} \end{bmatrix}.$$

Conditions (D.b) and (D.c) imply that the corresponding rows of A_{12} are zero. Thus, the weights satisfy

$$w_{ij} + w_{jj} + w_{kj} = 0 \quad \text{and} \quad w_{ik} + w_{jk} + w_{kk} = 0. \quad (14)$$

Now let a and b be non-zero real numbers such that $w_{ik}/a = w_{ij}/b$ and let $v = \begin{bmatrix} 0 & a & -b & 0 & \cdots & 0 \end{bmatrix}$. Consider

$$vA = \begin{bmatrix} \begin{bmatrix} 0 & a & -b \end{bmatrix} A_{11} & 0 & \cdots & 0 \end{bmatrix} \quad (15)$$

where we have used the fact the rows corresponding to nodes j and k in A_{12} are zero. Now consider the first component of (15).

$$\begin{bmatrix} 0 & a & -b \end{bmatrix} A_{11}$$
$$= \begin{bmatrix} aw_{ij} - bw_{ik} & aw_{jj} - bw_{jk} & aw_{kj} - bw_{kk} \end{bmatrix}$$
$$\stackrel{(a)}{=} \begin{bmatrix} 0 & -(aw_{ij} + aw_{kj} + bw_{jk}) & aw_{kj} + bw_{ik} + bw_{jk} \end{bmatrix} \quad (16)$$

where (a) uses (14) and the relation between a and b. Now suppose

$$\frac{aw_{ij} + aw_{kj} + bw_{jk}}{a} = \frac{aw_{kj} + bw_{ik} + bw_{jk}}{b}. \quad (17)$$

Substituting this in (16), we get that

$$\begin{bmatrix} 0 & a & -b \end{bmatrix} A_{11} = \frac{-(aw_{ij} + aw_{kj} + bw_{jk})}{a} \begin{bmatrix} 0 & a & -b \end{bmatrix}.$$

Substituting in (15), we get that v is a left eigenvector of A. Since nodes j and k are not the leader nodes, $vB = 0$, so v is a left null vector of B. Hence, by the PBH test, (A, B) is not controllable.

Now we argue that (17) is equivalent to (D.d). Since $w_{ik}/a = w_{ij}/b$, we can write (17) as

$$\frac{w_{ik}w_{ij} + w_{ik}w_{kj} + w_{ij}w_{jk}}{w_{ik}} = \frac{w_{ik}w_{kj} + w_{ij}w_{ik} + w_{ij}w_{jk}}{w_{ij}}.$$

which is equivalent to

$$\left(\frac{1}{w_{ik}} - \frac{1}{w_{jk}} \right) (w_{ik}w_{ij} + w_{ik}w_{kj} + w_{ij}w_{jk}) = 0,$$

which, in turn, implies (D.d). ■

APPENDIX V
PROOF OF THEOREM 5

Let n_i denote the number of nodes of graph $\mathcal{G}_i$, $i \in \{1, 2\}$. Let $(A_i, B_\cdot)$ denote the system matrices for system $(\mathcal{G}_i, \mathcal{M}_i)$. We assume that the leaders of both graphs are their first node and the terminal node in $\mathcal{G}_1$ is indexed by n_1. Then the system matrices of the cascaded system are given by

$$A' = \begin{bmatrix} A_1 & 0 \\ \lambda\Delta_1 & A_2 - \lambda\Delta_2 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{18}$$

where $\lambda \neq 0$ is the weight of the connecting edge, while $\Delta_1 \in \mathbb{R}^{n_2 \times n_1}$ and $\Delta_2 \in \mathbb{R}^{n_2 \times n_2}$ are given by

$$\Delta_{1,ij} = \begin{cases} 1, & i = 1, j = n_1 \\ 0 & \text{otherwise} \end{cases}, \quad \Delta_{2,ij} = \begin{cases} 1, & i = 1, j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that

$$\begin{aligned} &\Delta_2^2 = \Delta_2, \quad \Delta_2\Delta_1 = \Delta_1, \quad \Delta_1 B_1 = 0 \\ &\Delta_1 x = x(n_1)B_2, \ \Delta_2 x = x(1)B_2, \ \text{where } x \in \mathbb{R}^{n_1} \end{aligned} \tag{19}$$

Lemma 5 *If $\mathcal{G}_1$ has a Hamiltonian path and no other forward-pointing edges, we have that*

$$\Delta_1 A_1^i B_1 = \begin{cases} 0, & i < n_1 - 1 \\ \alpha B_2, & i = n_1 - 1 \end{cases} \tag{20}$$

where $\alpha \neq 0$ is a constant.

Proof: This lemma is a straight-forward consequence of Lemma 4. ■

Lemma 6 *Let (A_2, B_2) be controllable. Then,*

$$A'^k B' = \begin{cases} \begin{bmatrix} A_1^k B_1 \\ 0 \end{bmatrix}, & k < n_1 \\[2ex] \begin{bmatrix} A_1^k B_1 \\ q_k \end{bmatrix}, & k \geq n_1 \end{cases} \tag{21}$$

where $q_{n_1+j} = \sum_{i=0}^{j} \alpha_{j,i} A_2^j B_2$ for $0 \leq j \leq n_2 - 1$. Furthermore, $\alpha_{j,j} = \lambda\alpha$ for all $j \leq n_2 - 1$.

Proof: We start by noting that

$$A'B' = \begin{bmatrix} A_1 B_1 \\ 0 \end{bmatrix}, \ A'^2 B' = \begin{bmatrix} A_1^2 B_1 \\ \lambda\Delta_1 A_1 B_1 \end{bmatrix} = \begin{bmatrix} A_1^2 B_1 \\ 0 \end{bmatrix}$$

from Lemma 5. Continuing along these lines, we get

$$A'^k B' = \begin{bmatrix} A_1^k B_1 \\ 0 \end{bmatrix}, \ k \leq n_1 - 1$$

For $k = n_1$, we deduce from Lemma 5 and (21) (where q_k was introduced) that $q_{n_1} = \Delta_1 A_1^{n_1-1} B_1 = \lambda\alpha B_2$. We make the following induction hypothesis: for some $j \in (0, n_2 - 2)$,

$$q_{n_1+j} = \sum_{i=0}^{j} \alpha_{j,i} A_2^i B_2, \ \alpha_{j,j} = \lambda\alpha$$

Note that $\alpha_{0,0} = \lambda\alpha$. Then, from (21), we get

$$\begin{aligned} q_{n_1+j+1} &= \lambda\Delta_1 A_1^{n_1+j} B_1 + (A_2 - \lambda\Delta_2) \underbrace{\sum_{i=0}^{j} \alpha_{j,i} A_2^i B_2}_{q_k} \\ &= \gamma_1 B_2 + \sum_{i=1}^{j+1} \alpha_{j,i} A_2^i B_2 = \sum_{i=0}^{j+1} \alpha_{j+1,i} A_2^i B_2 \end{aligned} \tag{22}$$

where the existence of the constant γ_1 follows from (19), and $\alpha_{j+1,i}$ are also constants. Since $A_2^i B_2$ are independent for $i \leq n_2 - 1$, we obtain $\alpha_{j+1,j+1} = \alpha_{j,j} = \lambda\alpha$. ■

To prove Theorem 5, we start by noting, from Lemma 6, that the controllability matrix of (A', B') is of the form

$$C' = \begin{bmatrix} C_1' & C_3' \\ 0_{n_2 \times n_1} & C_2' \end{bmatrix}$$

where $C_1' = C_1$ and $C_2' = C_2 U$; C_1 and C_2 are the controllabilty matrices corresponding to (A_1, B_1) and (A_2, B_2), respectively, and U is an upper triangular matrix with each diagonal entry being $\lambda\alpha \neq 0$ (from Lemma 6). We recall that C_1 and C_2 are full-ranked. Furthermore, $\det(C_2 U) = \det(C_2)\det(U) \neq 0$, and $C_2 U$ is full-ranked as well. Thus, (A', B') is controllable for any $\lambda \neq 0$.■

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