

Sufficient Conditions for the Value Function and Optimal Strategy to be Even and Quasi-Convex

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Abstract—Sufficient conditions are identified under which the value function and the optimal strategy of a Markov decision process (MDP) are even and quasi-convex in the state. The key idea behind these conditions is the following. First, sufficient conditions for the value function and optimal strategy to be even are identified. Next, it is shown that if the value function and optimal strategy are even, then one can construct a “folded MDP” defined only on the nonnegative values of the state space. Then, the standard sufficient conditions for the value function and optimal strategy to be monotone are “unfolded” to identify sufficient conditions for the value function and the optimal strategy to be quasi-convex. The results are illustrated by using an example of power allocation in remote estimation.

Index Terms—Markov decision processes (MDPs), stochastic monotonicity, submodularity.

I. INTRODUCTION

A. Motivation

Markov decision theory is often used to identify structural or qualitative properties of optimal strategies. Examples include control limit strategies in machine maintenance [1], [2], threshold-based strategies for executing call options [3], [4], and monotone strategies in queueing systems [5], [6]. In all of these models, the optimal strategy is *monotone* in the state, i.e., if $x > y$, then the action chosen at x is greater (or less) than or equal to the action chosen at y . Motivated by this, general conditions under which the optimal strategy is monotone in scalar-valued states are identified in [7]–[12]. Similar conditions for vector-valued states are identified in [13]–[15]. General conditions under which the value function is increasing and convex are established in [16].

Most of the above results are motivated by queueing models where the state is the queue length that takes nonnegative values. However, for typical applications in systems and control, the state takes both positive and negative values. Often, the system behavior is symmetric for positive and negative values, so one expects the optimal strategy to be even. Thus, for such systems, a natural counterpart of monotone functions is even and quasi-convex (or quasi-concave) functions. In this paper, we identify sufficient conditions under which the value function and optimal strategy are even and quasi-convex.

As a motivating example, consider a remote estimation system in which a sensor observes a Markov process and decides whether to transmit the current state of the Markov process to a remote estimator. There is a cost or constraint associated with transmission. When the transmitter does not transmit or when the transmitted packet is dropped due to interference, the estimator generates an estimate of the state of the Markov process based on the previously received states. The objective is to choose transmission and estimation strategies that minimize

either the expected distortion and cost of communication or minimize expected distortion under the transmission constraint. Variations of such models have been considered in [17]–[23].

In such models, the optimal transmission and estimation strategies are identified in two steps. In the first step, the joint optimization of transmission and estimation strategies is investigated and it is established that there is no loss of optimality in restricting attention to estimation strategies of a specific form. In the second step, estimation strategies are restricted to the form identified in the first step and the structure of the best response transmission strategies is established. In particular, it is shown that the optimal transmission strategies are even and quasi-convex.¹ Currently, in the literature these results are established on a case by case basis. For example, see [18, Th. 1], [20, Th. 3], [24, Th. 1], [21, Th. 1] among others.

In this paper, we identify sufficient conditions for the value functions and optimal strategy of a Markov decision process (MDP) to be even and quasi-convex. We then consider a general model of remote estimation and verify these sufficient conditions.

B. Model and Problem Formulation

Consider a discrete time MDP with state space \mathbb{X} (which is either \mathbb{R} , the real line, or a symmetric subset of the form $[-a, a]$) and action space \mathbb{U} (which is either a subset of integers or a compact subset of reals).

Let $X_t \in \mathbb{X}$ and $U_t \in \mathbb{U}$ denote the state and action at time t , respectively. The initial state X_1 is distributed according to the probability density function μ and the state evolves in a controlled Markov manner, i.e., for any Borel measurable subset A of \mathbb{X}

$$\begin{aligned} \mathbb{P}(X_{t+1} \in A \mid X_{1:t} = x_{1:t}, U_{1:t} = u_{1:t}) \\ = \mathbb{P}(X_{t+1} \in A \mid X_t = x_t, U_t = u_t) \end{aligned}$$

where $x_{1:t}$ is a short-hand notation for (x_1, \dots, x_t) and a similar interpretation holds for $u_{1:t}$. We assume that there exists a (time-homogeneous) controlled transition density $p(y|x; u)$, which is continuous in y for any $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Thus, for any Borel measurable subset A of \mathbb{X}

$$\mathbb{P}(X_{t+1} \in A \mid X_t = x, U_t = u) = \int_A p(y|x; u) dy.$$

We use $p(u)$ to denote transition density corresponding to action $u \in \mathbb{U}$.

The system operates for a finite horizon T . For any time $t \in \{1, \dots, T-1\}$, a measurable and lower semicontinuous² function $c_t : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ denotes the instantaneous cost at time t , and at the terminal time T , a measurable and lower semicontinuous function $c_T : \mathbb{X} \rightarrow \mathbb{R}$ denotes the terminal cost.

¹When the action space is binary—as is the case in most of the models of remote estimation—an even and quasi-convex strategy is equivalent to one in which the action zero is chosen whenever the absolute value of the state is less than a threshold; otherwise, action one is chosen.

²A function is called lower semicontinuous if its lower level sets are closed.

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The actions at time t are chosen according to a Markov strategy g_t , i.e.,

$$U_t = g_t(X_t), \quad t \in \{1, \dots, T-1\}.$$

The objective is to choose a decision strategy $\mathbf{g} := (g_1, \dots, g_{T-1})$ to minimize the expected total cost

$$J_T(\mathbf{g}) := \mathbb{E}^g \left[\sum_{t=1}^{T-1} c_t(X_t, U_t) + c_T(X_T) \right]. \quad (1)$$

We denote such an MDP by $(\mathbb{X}, \mathbb{U}, p, c_t)$.

From Markov decision theory [25], we know that an optimal strategy is given by the solution of the following dynamic program. Recursively define value functions $V_t : \mathbb{X} \rightarrow \mathbb{R}$ and value-action functions $Q_t : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ as follows: for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$

$$V_T(x) = c_T(x) \quad (2)$$

and for $t \in \{T-1, \dots, 1\}$

$$\begin{aligned} Q_t(x, u) &= c_t(x, u) + \mathbb{E}[V_{t+1}(X_{t+1}) \mid X_t = x, U_t = u] \\ &= c_t(x, u) + \int_{\mathbb{X}} p(y|x; u) V_{t+1}(y) dy, \end{aligned} \quad (3)$$

$$V_t(x) = \min_{u \in \mathbb{U}} Q_t(x, u). \quad (4)$$

Then, a strategy $\mathbf{g}^* = (g_1^*, \dots, g_{T-1}^*)$ defined as

$$g_t^*(x) \in \arg \min_{u \in \mathbb{U}} Q_t(x, u)$$

is optimal. To avoid ambiguity when the arg min is not unique, we pick

$$g_t^*(x) = \begin{cases} \max \{v \in \arg \min_{u \in \mathbb{U}} Q_t(x, u)\}, & \text{if } x \geq 0 \\ \min \{v \in \arg \min_{u \in \mathbb{U}} Q_t(x, u)\}, & \text{if } x < 0. \end{cases} \quad (5)$$

Let $\mathbb{X}_{\geq 0}$ and $\mathbb{X}_{> 0}$ denote the sets $\{x \in \mathbb{X} : x \geq 0\}$ and $\{x \in \mathbb{X} : x > 0\}$, respectively. We say that a function $f : \mathbb{X} \rightarrow \mathbb{R}$ is *even* and *quasi-convex* if it is even and for $x, x' \in \mathbb{X}_{\geq 0}$ such that $x < x'$, we have that $f(x) \leq f(x')$. The main contribution of this paper is to identify sufficient conditions under which V_t and g_t^* are even and quasi-convex.

C. Main Result

Definition 1 (Even transition density): For a given $u \in \mathbb{U}$, we say that a controlled transition density $p(u)$ on $\mathbb{X} \times \mathbb{X}$ is *even* if for all $x, y \in \mathbb{X}$, $p(y|x; u) = p(-y|-x; u)$.

Our main result is the following.

Theorem 1: Given an MDP $(\mathbb{X}, \mathbb{U}, p, c_t)$, define for $x, y \in \mathbb{X}_{\geq 0}$ and $u \in \mathbb{U}$

$$S(y|x; u) := 1 - \int_{A_y} [p(z|x; u) + p(-z|x; u)] dz \quad (6)$$

where $A_y = \{z \in \mathbb{X} : z < y\}$. Consider the following conditions:

- (C1) $c_T(\cdot)$ is even and quasi-convex and for $t \in \{1, \dots, T-1\}$ and $u \in \mathbb{U}$, $c_t(\cdot, u)$ is even and quasi-convex.
 - (C2) For all $u \in \mathbb{U}$, $p(u)$ is even.
 - (C3) For all $u \in \mathbb{U}$ and $y \in \mathbb{X}_{\geq 0}$, $S(y|x; u)$ is increasing³ for $x \in \mathbb{X}_{\geq 0}$.
 - (C4) For $t \in \{1, \dots, T-1\}$, $c_t(x, u)$ is submodular⁴ in (x, u) on $\mathbb{X}_{\geq 0} \times \mathbb{U}$.
 - (C5) For all $y \in \mathbb{X}_{\geq 0}$, $S(y|x; u)$ is submodular in (x, u) on $\mathbb{X}_{\geq 0} \times \mathbb{U}$.
- Then, under (C1)–(C3), $V_t(\cdot)$ is even and quasi-convex for all $t \in \{1, \dots, T\}$ and under (C1)–(C5), $g_t^*(\cdot)$ is even and quasi-convex for all $t \in \{1, \dots, T-1\}$.

³We use the terms *increasing* and *decreasing* to mean weakly increasing and weakly decreasing respectively.

⁴Submodularity is defined in Section III-B.

The main idea of the proof is as follows. First, we identify conditions under which the value function and optimal strategy of an MDP are even. Next, we show that if we construct an MDP by “folding” the transition density, then the “folded MDP” has the same value function and optimal strategy as the original MDP for nonnegative values of the state. Finally, we show that if we take the sufficient conditions under which the value function and the optimal strategy of the folded MDP are increasing and “unfold” these conditions back to the original model, we get conditions (C1)–(C5) above. The details are given in Sections II and III.

II. EVEN MDPs AND FOLDED REPRESENTATIONS

We say that an MDP is *even* if for every t and every $u \in \mathbb{U}$, $V_t(x)$, $Q_t(x, u)$, and $g_t^*(x)$ are even in x . We start by identifying sufficient conditions for an MDP to be even.

A. Sufficient Conditions for an MDP to be Even

Proposition 1: Suppose an MDP $(\mathbb{X}, \mathbb{U}, p, c_t)$ satisfies the following properties:

- (A1) $c_T(\cdot)$ is even and for every $t \in \{1, \dots, T-1\}$ and $u \in \mathbb{U}$, $c_t(\cdot, u)$ is even.
- (A2) For every $u \in \mathbb{U}$, the transition density $p(u)$ is even.

Then, the MDP is even.

Proof: We proceed by backward induction. $V_T(x) = c_T(x)$, which is even by (A1). This forms the basis of induction. Now, assume that $V_{t+1}(x)$ is even in x . For any $u \in \mathbb{U}$, we show that $Q_t(x, u)$ is even in x . Consider

$$\begin{aligned} Q_t(-x, u) &= c_t(-x, u) + \int_{\mathbb{X}} p(y|-x; u) V_{t+1}(y) dy \\ &\stackrel{(a)}{=} c_t(x, u) + \int_{\mathbb{X}} p(-z|-x; u) V_{t+1}(-z) dz \\ &\stackrel{(b)}{=} c_t(x, u) + \int_{\mathbb{X}} p(z|x; u) V_{t+1}(z) dz = Q_t(x, u) \end{aligned}$$

where (a) follows from (A1), a change of variables $y = -z$, and the fact that \mathbb{X} is a symmetric interval; and (b) follows from (A2) and the induction hypothesis that $V_{t+1}(\cdot)$ is even. Hence, $Q_t(\cdot, u)$ is even.

Since $Q_t(\cdot, u)$ is even, (4) and (5) imply that V_t and g_t^* are also even. Thus, the result is true for time t and, by induction, true for all time t . ■

B. Folding Operator for Distributions

We now show that if the value function is even, we can construct a “folded” MDP with state space $\mathbb{X}_{\geq 0}$ such that the value function and optimal strategy of the folded MDP match that of the original MDP on $\mathbb{X}_{\geq 0}$. For that matter, we first define the following.

Definition 2 (Folding Operator): Given a probability density π on \mathbb{X} , the folding operator $\mathcal{F}\pi$ gives a density $\tilde{\pi}$ on $\mathbb{X}_{\geq 0}$ such that for any $x \in \mathbb{X}_{\geq 0}$, $\tilde{\pi}(x) = \pi(x) + \pi(-x)$.

As an immediate implication, we have the following.

Lemma 1: If $f : \mathbb{X} \rightarrow \mathbb{R}$ is even, then for any probability distribution π on \mathbb{X} and $\tilde{\pi} = \mathcal{F}\pi$, we have

$$\int_{\mathbb{X}} f(x) \pi(x) dx = \int_{\mathbb{X}_{\geq 0}} f(x) \tilde{\pi}(x) dx.$$

Now, we generalize the folding operator to transition densities.

Definition 3: Given a transition density p on $\mathbb{X} \times \mathbb{X}$, the folding operator $\mathcal{F}p$ gives a transition density \tilde{p} on $\mathbb{X}_{\geq 0} \times \mathbb{X}_{\geq 0}$ such that for any $x, y \in \mathbb{X}_{\geq 0}$, $\tilde{p}(y|x) = p(y|x) + p(-y|x)$.

Definition 4 (Folded MDP): Given an MDP $(\mathbb{X}, \mathbb{U}, p, c_t)$, define the *folded MDP* as $(\mathbb{X}_{\geq 0}, \mathbb{U}, \tilde{p}, c_t)$, where for all $u \in \mathbb{U}$, $\tilde{p}(u) = \mathcal{F}p(u)$.

Let \tilde{Q}_t , \tilde{V}_t , and \tilde{g}_t^* denote respectively the value-action function, the value function, and the optimal strategy of the folded MDP. Then, we have the following.

Proposition 2: If the MDP $(\mathbb{X}, \mathbb{U}, p, c_t)$ is even, then for any $x \in \mathbb{X}$ and $u \in \mathbb{U}$

$$Q_t(x, u) = \tilde{Q}_t(|x|, u), \quad V_t(x) = \tilde{V}_t(|x|), \quad g_t^*(x) = \tilde{g}_t^*(|x|). \quad (7)$$

Proof: We proceed by backward induction. For $x \in \mathbb{X}$ and $\tilde{x} \in \mathbb{X}_{\geq 0}$, $V_T(x) = c_T(x)$ and $\tilde{V}_T(\tilde{x}) = c_T(\tilde{x})$, respectively. Since $V_T(\cdot)$ is even, $V_T(x) = V_T(|x|) = \tilde{V}_T(|x|)$. This is the basis of induction. Now, assume that for all $x \in \mathbb{X}$, $V_{t+1}(x) = \tilde{V}_{t+1}(|x|)$. Consider $x \in \mathbb{X}_{\geq 0}$ and $u \in \mathbb{U}$. Then, we have

$$\begin{aligned} Q_t(x, u) &= c_t(x, u) + \int_{\mathbb{X}} p(y|x; u) V_{t+1}(y) dy \\ &\stackrel{(a)}{=} c_t(x, u) + \int_{\mathbb{X}_{\geq 0}} \tilde{p}(y|x; u) V_{t+1}(y) dy \\ &\stackrel{(b)}{=} c_t(x, u) + \int_{\mathbb{X}_{\geq 0}} \tilde{p}(y|x; u) \tilde{V}_{t+1}(y) dy = \tilde{Q}_t(x, u) \end{aligned}$$

where (a) uses Lemma 1 and that V_{t+1} is even and (b) uses the induction hypothesis.

Since the Q -functions match for $x \in \mathbb{X}_{\geq 0}$, (4) and (5) imply that the value functions and the optimal strategies also match on $\mathbb{X}_{\geq 0}$, i.e., for $x \in \mathbb{X}_{\geq 0}$

$$V_t(x) = \tilde{V}_t(x) \quad \text{and} \quad g_t^*(x) = \tilde{g}_t^*(x).$$

Since V_t and g_t^* are even, we get that (7) is true at time t . Hence, by the principle of induction, it is true for all t . ■

III. MONOTONICITY OF THE VALUE FUNCTION AND THE OPTIMAL STRATEGY

We have shown that under (A1) and (A2) the original MDP is equivalent to a folded MDP with state space $\mathbb{X}_{\geq 0}$. Thus, we can use standard conditions to determine when the value function and the optimal strategy of the folded MDP are monotone. Translating these conditions back to the original model, we get the sufficient conditions for the original model.

A. Monotonicity of the Value Function

The results on monotonicity of value functions rely on the notion of stochastic monotonicity.

Given a transition density p defined on \mathbb{X} , the cumulative transition distribution function P is defined as

$$P(y|x) = \int_{A_y} p(z|x) dz, \quad \text{where } A_y = \{z \in \mathbb{X} : z < y\}.$$

Definition 5 (Stochastic Monotonicity): A transition density p on \mathbb{X} is said to be *stochastically monotone increasing* if for every $y \in \mathbb{X}$, the cumulative distribution function $P(y|x)$ corresponding to p is decreasing in x .

Proposition 3: Suppose the folded MDP $(\mathbb{X}_{\geq 0}, \mathbb{U}, \tilde{p}, c_t)$ satisfies the following:

(B1) $c_T(x)$ is increasing in x for $x \in \mathbb{X}_{\geq 0}$; for any $t \in \{1, \dots, T-1\}$ and $u \in \mathbb{U}$, $c_t(x, u)$ is increasing in x for $x \in \mathbb{X}_{\geq 0}$.

(B2) For any $u \in \mathbb{U}$, $\tilde{p}(u)$ is stochastically monotone increasing.

Then, for any $t \in \{1, \dots, T\}$, $\tilde{V}_t(x)$ is increasing in x for $x \in \mathbb{X}_{\geq 0}$.

A version of this proposition when \mathbb{X} is a subset of integers is given in [8, Th. 4.7.3]. The same proof argument also works when \mathbb{X} is a subset of reals.

Recall the definition of S given in (6). (B2) is equivalent to the following:

(B2') For every $u \in \mathbb{U}$ and $x, y \in \mathbb{X}_{\geq 0}$, $S(y|x; u)$ is increasing in x .

An immediate consequence of Propositions 1–3 is the following.

Corollary 1: Under (A1), (A2), (B1), and (B2) [or (B2')], the value functions $V_t(\cdot)$ is even and quasi-convex.

Remark 1: Note that (A1) and (B1) are equivalent to (C1), (A2) is the same as (C2), and (B2) [or equivalently, (B2')] is equivalent to (C3). Thus, Corollary 1 proves the first part of Theorem 1.

B. Monotonicity of the Optimal Strategy

Now, we state sufficient conditions under which the optimal strategy is increasing. These results rely on the notion of submodularity.

Definition 6 (Submodular function): A function $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ is called submodular if for any $x, y \in \mathbb{X}$ and $u, v \in \mathbb{U}$ such that $x \geq y$ and $u \geq v$, we have

$$f(x, u) + f(y, v) \leq f(x, v) + f(y, u).$$

An equivalent characterization of submodularity is that

$$\begin{aligned} f(y, u) - f(y, v) &\geq f(x, u) - f(x, v) \\ \iff f(x, v) - f(y, v) &\geq f(x, u) - f(y, u) \end{aligned}$$

which implies that the differences in one variable are decreasing in the other variable.

Proposition 4: Suppose that in addition to (B1) and (B2) [or (B2')], the folded MDP $(\mathbb{X}_{\geq 0}, \mathbb{U}, \tilde{p}, c_t)$ satisfies the following:

(B3) For all $t \in \{1, \dots, T-1\}$, $c_t(x, u)$ is submodular in (x, u) on $\mathbb{X}_{\geq 0} \times \mathbb{U}$.

(B4) For all $y \in \mathbb{X}_{\geq 0}$, $S(y|x; u)$ is submodular in (x, u) on $\mathbb{X}_{\geq 0} \times \mathbb{U}$, where $S(y|x; u)$ is defined in (6).

Then, for every $t \in \{1, \dots, T-1\}$, the optimal strategy $\tilde{g}_t^*(x)$ is increasing in x for $x \in \mathbb{X}_{\geq 0}$.

A version of this proposition when \mathbb{X} is a subset of integers is given in [8, Th. 4.7.4]. The same proof argument also works when \mathbb{X} is a subset of reals.

An immediate consequence of Propositions 1–4 is the following.

Corollary 2: Under (A1), (A2), (B1), (B2) [or (B2')], (B3), and (B4), the optimal strategy $g_t^*(\cdot)$ is even and quasi-convex.

Remark 2: As argued in Remark 1, (A1), (A2), (B1), and (B2) [or (B2')] are equivalent to (C1)–(C3). Note that (B3) and (B4) are the same as (C4) and (C5). Thus, Corollary 2 proves the second part of Theorem 1.

IV. REMARK ON INFINITE HORIZON SETUP

Although we restricted attention to finite horizon models, the results extend immediately to infinite horizon discounted cost setup. In particular, suppose the per-step cost is time-homogeneous and given by $c : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ and future is discounted by a discount factor $\beta \in (0, 1)$. Define the following Bellman operators: for any $g : \mathbb{X} \rightarrow \mathbb{U}$, and $V : \mathbb{X} \rightarrow \mathbb{R}$

$$[\mathcal{B}_g V](x) = c(x, g(x)) + \beta \int_{\mathbb{X}} p(y|x; g(x)) V(y) dy$$

and

$$\mathcal{B}^* V = \min_{g: \mathbb{X} \rightarrow \mathbb{U}} \mathcal{B}_g V.$$

Suppose the model satisfies standard conditions (see [25, Ch. 4]) so that \mathcal{B}^* is a contraction and has a unique fixed point (which we denote by V_β) and there exists a strategy $g_\beta : \mathbb{X} \rightarrow \mathbb{U}$ such that $V_\beta = \mathcal{B}_{g_\beta} V_\beta$. Then, the result of Theorem 1 is also true for V_β and g_β . In particular, we have the following.

Corollary 3: Given an MDP $(\mathbb{X}, \mathbb{U}, p, c)$ and a discount factor $\beta \in (0, 1)$, consider the following conditions:

(C1') For $u \in \mathbb{U}$, $c(\cdot, u)$ is even and quasi-convex.

(C4') $c(x, u)$ is submodular in (x, u) on $\mathbb{X}_{\geq 0} \times \mathbb{U}$.

Then, under (C1'), (C2), and (C3), $V_\beta(\cdot)$ is even and quasi-convex, and under (C1'), (C2), (C3), (C4'), and (C5), $g_\beta^*(\cdot)$ is even and quasi-convex.

Proof: Note that the equivalence to folded MDP continues to hold for infinite horizon setup. Therefore, the result follows from extension of Propositions 3 and 4 to infinite horizon setup. For example, see [8, Sec. 6.11]. ■

V. REMARKS ABOUT DISCRETE \mathbb{X}

So far, we assumed that \mathbb{X} was a subset of the real line. Now, suppose \mathbb{X} is discrete (either the set \mathbb{Z} of integers or a symmetric subset of the form $\{-a, \dots, a\}$). With a slight abuse of notation, let $p(y|x; u)$ denote $\mathbb{P}(X_{t+1} = y | X_t = x, U_t = u)$.

Theorem 2: The result of Theorem 1 is true for discrete \mathbb{X} with S defined as

$$S(y|x; u) = 1 - \sum_{z \in A_y} [p(z|x; u) + p(-z|x; u)]$$

where $A_y = \{x \in \mathbb{X} : x < y\}$.

The proof proceeds along the same lines as the proof of Theorem 1. In particular,

- 1) Proposition 1 is also true for discrete \mathbb{X} .
- 2) Given a probability mass function π on \mathbb{X} , define the folding operator \mathcal{F} as follows: $\tilde{\pi} = \mathcal{F}\pi$ means that $\tilde{\pi}(0) = \pi(0)$ and for any $x \in \mathbb{X}_{>0}$, $\tilde{\pi}(x) = \pi(x) + \pi(-x)$.
- 3) Use this definition of the folding operator to define the folded MDP, as in Definition 4. Proposition 2 remains true with this modified definition.
- 4) A discrete state Markov chain with transition function p is stochastically monotone increasing if for every $y \in \mathbb{X}$

$$P(y|x) = \sum_{z \in A_y} p(z|x), \quad \text{where } A_y = \{z \in \mathbb{X} : z < y\},$$

is decreasing in x .

- 5) Propositions 3 and 4 are also true for discrete \mathbb{X} .
- 6) The result of Theorem 2 follows from Corollaries 1 and 2.

A. Monotone Dynamic Programming

Under (C1)–(C5), the even and quasi-convex property of the optimal strategy can be used to simplify the dynamic program given by (2)–(4). For conciseness, assume that the state space \mathbb{X} is a set of integers of the form $\{-a, -a+1, \dots, a-1, a\}$ and the action space \mathbb{U} is a set of integers of the form $\{\underline{u}, \underline{u}+1, \dots, \bar{u}-1, \bar{u}\}$.

Initialize $V_T(x)$ as in (2). Now, suppose $V_{t+1}(\cdot)$ has been calculated. Instead of computing $Q_t(x, u)$ and $V_t(x)$ according to (appropriately modified versions of) (3) and (4), we proceed as follows:

- 1) Set $x = 0$ and $w_x = \underline{u}$.
- 2) For all $u \in [w_x, \bar{u}]$, compute $Q_t(x, u)$ according to (3).
- 3) Instead of (4), compute

$$V_t(x) = \min_{u \in [w_x, \bar{u}]} Q_t(x, u), \quad \text{and set}$$

$$g_t(x) = \max\{v \in [w_x, \bar{u}] \text{ s.t. } V_t(x) = Q_t(x, v)\}.$$

- 4) Set $V_t(-x) = V_t(x)$ and $g_t(-x) = g_t(x)$.
- 5) If $x = a$, then stop. Otherwise, set $w_{x+1} = g_t(x)$ and $x = x + 1$. Go to step 2.

B. Remark on Randomized Actions

Suppose \mathbb{U} is a discrete set of the form $\{\underline{u}, \underline{u}+1, \dots, \bar{u}\}$. In constrained optimization problems, it is often useful to consider the action space $\mathbb{W} = [\underline{u}, \bar{u}]$, where for $u, u+1 \in \mathbb{U}$, an action $w \in (u, u+1)$ corresponds to a randomization between the “pure” actions u and $u+1$. More precisely, let transition probability \check{p} corresponding to \mathbb{W} be given as follows: for any $x, y \in \mathbb{X}$ and $w \in (u, u+1)$

$$\check{p}(y|x; w) = (1 - \theta(w))p(y|x; u) + \theta(w)p(y|x; u+1)$$

where $\theta : \mathbb{W} \rightarrow [0, 1]$ is such that for any $u \in \mathbb{U}$

$$\lim_{w \downarrow u} \theta(w) = 0, \quad \text{and} \quad \lim_{w \uparrow u+1} \theta(w) = 1, \quad (8)$$

and for any $v, w \in (u, u+1)$ such that $v \leq w$, $\theta(v) \leq \theta(w)$. Thus, $\check{p}(w)$ is continuous at all $u \in \mathbb{U}$.

Theorem 3: If $p(u)$ satisfies (C2), (C3), and (C5), then so does $\check{p}(w)$.

Proof: Since $\check{p}(w)$ is linear in $p(u)$ and $p(u+1)$, both of which satisfy (C2) and (C3), so does $\check{p}(w)$.

To prove that $\check{p}(w)$ satisfies (C5), note that

$$\check{S}(y|x; w) = S(y|x; u) + \theta(w)[S(y|x; u+1) - S(y|x; u)].$$

So, for $v, w \in (u, u+1)$ such that $v > w$, we have that

$$\check{S}(y|x; v) - \check{S}(y|x; w) = (\theta(v) - \theta(w)) [S(y|x; u+1) - S(y|x; u)].$$

Since $\theta(\cdot)$ is increasing in $(u, u+1)$, $\theta(v) - \theta(w) \geq 0$. Moreover, since $S(y|x; u)$ is submodular in (x, u) , $S(y|x; u+1) - S(y|x; u)$ is decreasing in x , and, therefore, so is $\check{S}(y|x; v) - \check{S}(y|x; w)$. Hence, $\check{S}(y|x; w)$ is submodular in (x, w) on $\mathbb{X} \times (u, u+1)$. Due to (8), $\check{S}(y|x; w)$ is continuous in w . Hence, $\check{S}(y|x; w)$ is submodular in (x, w) on $\mathbb{X} \times [u, u+1]$. By piecing intervals of the form $[u, u+1]$ together, we get that $\check{S}(y|x; w)$ is submodular on $\mathbb{X} \times \mathbb{W}$. ■

VI. EXAMPLE: OPTIMAL POWER ALLOCATION STRATEGIES IN REMOTE ESTIMATION

Consider a remote estimation system that consists of a sensor and an estimator. The sensor observes a first-order autoregressive process $\{X_t\}_{t \geq 1}$, $X_t \in \mathbb{X}$, where \mathbb{X} is either \mathbb{R} or \mathbb{Z} . The system starts with $X_1 = 0$ and for $t > 1$

$$X_{t+1} = aX_t + W_t$$

where $a \in \mathbb{X}$ is a constant and $\{W_t\}_{t \geq 1}$, $W_t \in \mathbb{X}$ is an independent identically distributed (i.i.d.) noise process with probability density/mass function φ .

At each time step, the sensor uses power U_t to send a packet containing X_t to the remote estimator. U_t takes values in $[0, u_{\max}]$, where $U_t = 0$ denotes that no packet is sent. The packet is received with probability $q(U_t)$, where q is an increasing function with $q(0) = 0$ and $q(u_{\max}) \leq 1$.

Let Y_t denote the received symbol. $Y_t = X_t$ if the packet is received, and $Y_t = \mathfrak{E}$ if the packet is not received. Packet reception is acknowledged, so the sensor knows Y_t with one unit delay. At each stage, the receiver generates an estimate \hat{X}_t as follows. \hat{X}_0 is 0 and for $t > 0$

$$\hat{X}_t = \begin{cases} a\hat{X}_{t-1}, & \text{if } Y_t = \mathfrak{E} \\ Y_t, & \text{if } Y_t \neq \mathfrak{E}. \end{cases} \quad (9)$$

Under some conditions, such an estimation rule is known to be optimal [18], [20], [22], [23], [26].⁵

There are two types of costs: 1) a communication cost $\lambda(U_t)$, where λ is an increasing function with $\lambda(0) = 0$; and 2) an estimation cost $d(X_t - \hat{X}_t)$, where d is an even and quasi-convex function with $d(0) = 0$.

⁵The model presented above appears as an intermediate step in the analysis of remote estimation problem. One typically starts with a model where the transmission strategy is of the form $U_t = g_t(X_{1:t}, Y_{1:t-1}, U_{1:t-1})$ and the estimation strategy is of the form $\hat{X}_t = h_t(Y_{1:t})$. This is a decentralized control problem. After a series of simplifications, it is shown that there is no loss of optimality to restrict attention to estimation strategies of the form (9) (see [18, Fact B.3], [20, Th. 3], [23, Th. 1] among others). Once the attention is restricted to estimation strategies of the form (9), the next step is to simplify the structure of the optimal transmission strategy (see [18, Fact A.4], [20, Th. 3], [24, Th. 1], [23, Th. 1] among others). The model presented above corresponds to this step.

Define the error process $\{E_t\}_{t \geq 0}$ as $E_t = X_t - a\hat{X}_{t-1}$. The error process $\{E_t\}_{t \geq 0}$ evolves in a controlled Markov manner as follows:

$$E_{t+1} = \begin{cases} aE_t + W_t, & \text{if } Y_t = \mathfrak{e} \\ W_t, & \text{if } Y_t \neq \mathfrak{e}. \end{cases} \quad (10)$$

Due to packet acknowledgments, E_t is measurable at the sensor at time t . If a packet is received, then $\hat{X}_t = X_t$ and the estimation cost is 0. If the packet is dropped, $X_t - \hat{X}_t = E_t$ and an estimation cost of $d(E_t)$ is incurred.

The objective is to choose a transmission strategy $\mathbf{g} = (g_1, \dots, g_T)$ of the form $U_t = g_t(E_t)$ to minimize⁶

$$\mathbb{E} \left[\sum_{t=1}^T [\lambda(U_t) + (1 - q(U_t))d(E_t)] \right].$$

The above model is an MDP with state $E_t \in \mathbb{X}$, control action $U_t \in [0, u_{\max}]$, per-step cost

$$c(e, u) = \lambda(u) + (1 - q(u))d(e) \quad (11)$$

and transition density/mass function

$$p(e_+ | e; u) = q(u)\varphi(e_+) + (1 - q(u))\varphi(e_+ - ae). \quad (12)$$

For ease of reference, we restate the assumptions imposed on the cost:

- (M0) $q(0) = 0$ and $q(u_{\max}) \leq 1$.
- (M1) $\lambda(\cdot)$ is increasing with $\lambda(0) = 0$.
- (M2) $q(\cdot)$ is increasing.
- (M3) $d(\cdot)$ is even and quasi-convex with $d(0) = 0$.

In addition, we impose the following assumptions on the probability density/mass function of the i.i.d. process $\{W_t\}_{t \geq 1}$:

- (M4) $\varphi(\cdot)$ is even.
- (M5) $\varphi(\cdot)$ is unimodal (i.e., quasi-concave).

Claim 1: We have the following:

- 1) Under assumptions (M0) and (M3), the per-step cost function given by (11) satisfies (C1).
- 2) Under assumptions (M0), (M2), and (M3), the per-step cost function given by (11) satisfies (C4).
- 3) Under assumption (M4), the transition density $p(u)$ given by (12) satisfies (C2).
- 4) Under assumptions (M0), (M2), (M4), and (M5), the transition density $p(u)$ satisfies (C3) and (C5).

The proof is given in Appendix A.

An immediate consequence of Theorem 1 and Claim 1 is the following.

Theorem 4: Under assumptions (M0) and (M2)–(M5), the value function and the optimal strategy for the remote estimation model are even and quasi-convex.

Remark 3: Although Theorem 4 is derived for continuous action space, it is also true when the action space is a discrete set. In particular, if we take the action space to be $\{0, 1\}$ and $q(1) = 1$, we get the results of [18, Th. 1], [17, Proposition 1], [20, Th. 3], and [21, Th. 1]; if we take the action space to be $\{0, 1\}$ and $q(1) = 1 - \varepsilon$, we get the result of [22, Th. 1] and [23, Th. 2].

To illustrate the above result, consider the case when $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \{0, 1\}$, $a = 1$, $W_t \sim \mathcal{N}(0, 1)$, $d(e) = e^2$, $\lambda = 1$, $q(0) = 0$, $q(1) = 0.9$, and $T = 4$. We discretize the state space with a uniform grid of width 0.01 and numerically solve the resulting dynamic program (2)–(4). The value functions across time are shown in Fig. 1. The optimal strategy is of the form

$$g_t(e) = \begin{cases} 1, & \text{if } |e| > k_t \\ 0, & \text{if } |e| \leq k_t \end{cases}$$

⁶It can be shown that $\{E_t\}_{t \geq 0}$ is a controlled Markov process. Thus, restricting attention to strategies of the form $U_t = g_t(E_t)$ is without loss of optimality.

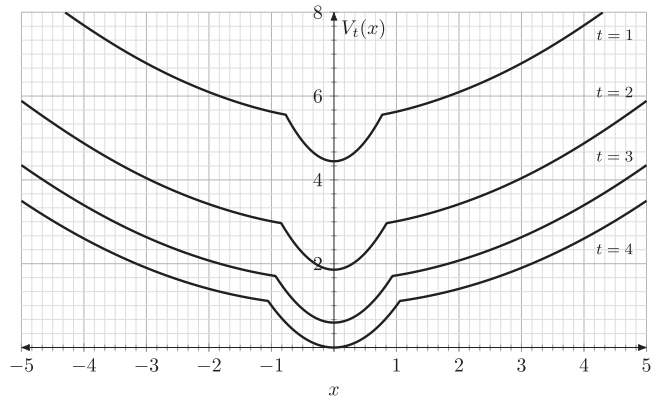


Fig. 1. Value function for the remote estimation problem of horizon $T = 4$ with state space $\mathbb{X} = \mathbb{R}$, action space $\mathbb{U} = \{0, 1\}$, $a = 1$, $W_t \sim \mathcal{N}(0, 1)$, $d(e) = e^2$, $\lambda = 1$, $q(0) = 0$, and $q(1) = 0.9$. The kink in the value function corresponds to the point where the optimal action changes.

where $k_1 = 0.77$, $k_2 = 0.84$, $k_3 = 0.93$, and $k_4 = 1.05$. The code for the calculations is available in [27]. Note that, as expected, both the value function and the optimal policy are even and quasi-convex; therefore, the value functions of the folded MDP are identical to the value functions above when restricted to the domain $\mathbb{R}_{\geq 0}$.

A. Some Comments on the Conditions

The result does not depend on (M1) for the following reason. Suppose there are two power levels $u_1, u_2 \in \mathbb{U}$ such that $u_1 < u_2$ but $\lambda(u_1) > \lambda(u_2)$. Define an alternative action space $\mathbb{U}' = \{0, u'_1, u'_2\}$ where $u'_1 < u'_2$ and a bijection $\sigma: \mathbb{U} \rightarrow \mathbb{U}'$ such that $\sigma(0) = 0$, $\sigma(u_1) = u'_2$ and $\sigma(u_2) = u'_1$. Now, consider a remote estimation system with communication cost $\lambda' = \lambda \circ \sigma^{-1}$ and success probabilities $q' = q \circ \sigma^{-1}$. By construction, q' satisfies (M0) and (M2).⁷ If $d(\cdot)$ and $\varphi(\cdot)$ are chosen to satisfy (M3)–(M5), then by Theorem 4, the optimal strategy $g': \mathbb{X} \rightarrow \mathbb{U}'$ is even and quasi-convex. In particular, we can pick λ , d , and φ such that $g'(0) = 0$, $g'(\pm 1) = u'_1$, and $g'(\pm 2) = u'_2$ and $g'(x) = g(x)$ for $x \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$. However, this means that with the original labels, the optimal strategy would have been $g = g' \circ \sigma^{-1}$, which means $g(0) = 0$, $g(\pm 1) = u_2$ and $g(\pm 2) = u_1$, and, hence, the optimal strategy is not quasi-convex.

The other conditions, (M0) and (M2)–(M5), in addition to being sufficient are also necessary for the reasons given below.

The necessity of (M2) is illustrated with the following example. Suppose $\mathbb{X} = \mathbb{Z}$ and $\mathbb{U} = \{0, u_1, u_2\}$ such that $u_1 < u_2$ but $q(u_1) > q(u_2)$. Define an alternative action space $\mathbb{U}' = \{0, u'_1, u'_2\}$ where $u'_1 < u'_2$ and a bijection $\sigma: \mathbb{U} \rightarrow \mathbb{U}'$ such that $\sigma(0) = 0$, $\sigma(u_1) = u'_2$ and $\sigma(u_2) = u'_1$. Now, consider a remote estimation system with communication cost $\lambda' = \lambda \circ \sigma^{-1}$ and success probabilities $q' = q \circ \sigma^{-1}$. By construction, q' satisfies (M0) and (M2).⁷ If $d(\cdot)$ and $\varphi(\cdot)$ are chosen to satisfy (M3)–(M5), then by Theorem 4, the optimal strategy $g': \mathbb{X} \rightarrow \mathbb{U}'$ is even and quasi-convex. In particular, we can pick λ , d , and φ such that $g'(0) = 0$, $g'(\pm 1) = u'_1$, and $g'(\pm 2) = u'_2$ and $g'(x) = g(x)$ for $x \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$. However, this means that with the original labels, the optimal strategy would have been $g = g' \circ \sigma^{-1}$, which means $g(0) = 0$, $g(\pm 1) = u_2$ and $g(\pm 2) = u_1$, and, hence, the optimal strategy is not quasi-convex.

Conditions (M3) and (M4) are necessary. If they are not satisfied, then it is easy to construct examples where the value function is not even.

The necessity of (M5) is illustrated by the following example. Consider an example where $\mathbb{X} = \mathbb{Z}$. In particular, let $a = 1$ and φ have support $\{-1, 0, 1\}$, where $\varphi(0) = 1 - 2p$ and $\varphi(\pm 1) = p$. Suppose $p > 1/3$, so that (M5) is not satisfied. Furthermore, suppose $T = 2$, $\mathbb{U} = \{0, 1\}$ and consider the following functions: $\lambda(0) = 0$, $\lambda(1) = K$; $q(0) = 0$ and $q(1) = 1$; and $d(0) = 0$, $d(\pm 1) = 1$, and for any $e \notin \{-1, 0, 1\}$, $d(e) = 1 + k$, where k is a positive constant. Note that $q(\cdot)$ satisfies (M0) and (M2); $d(\cdot)$ satisfies (M3); and $\varphi(\cdot)$ satisfies (M4) but not (M5). Suppose $K > 2(1 + k)$, so that action 1 is not optimal at any time. Thus, $V_2(e) = d(e)$ and $V_1(0) = 2p$ and $V_1(\pm 1) = p(1 + k) + (1 - 2p) = pk + 1 - p$. Now, if $k < (3p - 1)/p$, then $V_1(-1) < V_1(0) > V_1(1)$, and hence

⁷ λ' does not satisfy (M1), but (M1) is not needed for Theorem 4.

the value function is not quasi-convex. Hence, condition (M5) is necessary.

VII. CONCLUSION

We identify sufficient conditions under which the value function and the optimal strategy of an MDP are even and quasi-convex. The proof relies on a folded representation of the MDP and uses stochastic monotonicity and submodularity. We present an example of optimal power allocation in remote estimation and show that the sufficient conditions are easily verified.

Establishing that the value function and optimal strategy are even and quasi-convex has two benefits. First, such structured strategies are easier to implement. Second, the structure of the value function and optimal strategy may be exploited to efficiently solve the dynamic program.

For example, when the action space is discrete, say $|\mathbb{U}| = m$, then even and quasi-convex strategy is characterized by $m - 1$ thresholds. Such a threshold-based strategy is simpler to implement than an arbitrary strategy. Furthermore, the threshold structure also simplifies the search of the optimal strategy. For discrete state spaces, see the monotone dynamic programming presented in Section V-A; for continuous state spaces, see [28], where a simulation based algorithm is presented to compute the optimal thresholds in remote estimation over a packet drop channel.

Even for continuous action spaces, it is easier to search within the class of even and quasi-convex strategies. Typically, some form of approximation is needed to search for an optimal strategy. Two commonly used approximation schemes are discretizing the action space or projecting the strategy on to a parametric family of function. If the action state is discretized, then the search methods for discrete action spaces may be used. If the strategy is projected on to a parametric family of function, then the structure may help in reducing the size of the parameter space. For example, when approximating an even and quasi-convex strategy as a finite-order polynomial, one can restrict attention to polynomials where the coefficients of even powers are positive and the coefficients of odd powers are zero.

In this paper, we assume that the state space \mathbb{X} is a subset of reals. It will be useful to generalize these results to higher dimensions.

APPENDIX A PROOF OF CLAIM 1

We first prove some intermediate results.

Lemma 2: Under (M4) and (M5), for any $x, y \in \mathbb{X}_{\geq 0}$, we have that

$$\varphi(y - x) \geq \varphi(y + x).$$

Proof: We consider two cases: $y \geq x$ and $y < x$.

- 1) If $y \geq x$, then $y + x \geq y - x \geq 0$. Thus, (M5) implies that $\varphi(y + x) \geq \varphi(y - x)$.
- 2) If $y < x$, then $y + x \geq x - y$. Thus, (M5) implies that $\varphi(y + x) \geq \varphi(x - y) = \varphi(y - x)$, where the last equality follows from (M4). \blacksquare

Some immediate implications of Lemma 2 are the following.

Lemma 3: Under (M4) and (M5), for any $a \in \mathbb{X}$ and $x, y \in \mathbb{X}_{\geq 0}$, we have that

$$a[\varphi(y - ax) - \varphi(y + ax)] \geq 0.$$

Proof: For $a \in \mathbb{X}_{\geq 0}$, from Lemma 2, we get that $\varphi(y - ax) \geq \varphi(y + ax)$. For $a \in \mathbb{X}_{< 0}$, from Lemma 2, we get that $\varphi(y + ax) \geq \varphi(y - ax)$. \blacksquare

Lemma 4: Under (M4) and (M5), for any $a, b, x, y \in \mathbb{X}_{\geq 0}$, we have that

$$\varphi(y - ax - b) \geq \varphi(y + ax + b) \geq \varphi(y + ax + b + 1).$$

Proof: By taking $y = y - b$ and $x = ax$ in Lemma 2, we get

$$\varphi(y - b - ax) \geq \varphi(y - b + ax).$$

Now, by taking $y = y + ax$ and $x = b$ in Lemma 2, we get

$$\varphi(y + ax - b) \geq \varphi(y + ax + b).$$

By combining these two inequalities, we get

$$\varphi(y - ax - b) \geq \varphi(y + ax + b),$$

which proves the first inequality in the result. The second inequality in the result follows from (M5). \blacksquare

Lemma 5: Under (M4) and (M5), for $a \in \mathbb{Z}$ and $x, y \in \mathbb{Z}_{\geq 0}$

$$\Phi(y + ax) + \Phi(y - ax) \geq \Phi(y + ax + a) + \Phi(y - ax - a)$$

where Φ is the cdf (cumulative distribution function) of φ .

Proof: The statement holds trivially for $a = 0$. Furthermore, the statement does not depend on the sign of a . So, without loss of generality, we assume that $a > 0$.

Now, consider the following series of inequalities (which follow from Lemma 4):

$$\begin{aligned} \varphi(y - ax) &\geq \varphi(y + ax + 1) \\ \varphi(y - ax - 1) &\geq \varphi(y + ax + 2) \\ &\dots \geq \dots \\ \varphi(y - ax - a + 1) &\geq \varphi(y + ax + a). \end{aligned}$$

Adding these inequalities, we get

$$\Phi(y - ax) - \Phi(y - ax - a) \geq \Phi(y + ax + a) - \Phi(y + ax)$$

which proves the result. \blacksquare

Proof of Claim 1: First, we assume that $\mathbb{X} = \mathbb{R}$ and prove each part separately.

- 1) Fix $u \in [0, u_{\max}]$. $c(\cdot, u)$ is even because $d(\cdot)$ is even [from (M3)]. $c(\cdot, u)$ is quasi-convex because $1 - q(u) \geq 0$ [from (M0)] and $d(\cdot)$ is quasi-convex [from (M3)].
- 2) Consider $e_1, e_2 \in \mathbb{R}_{\geq 0}$ and $u_1, u_2 \in [0, u_{\max}]$ such that $e_1 \geq e_2$ and $u_1 \geq u_2$. The per-step cost is submodular on $\mathbb{R}_{\geq 0} \times [0, u_{\max}]$ because

$$\begin{aligned} c(e_1, u_2) - c(e_2, u_2) &= (1 - q(u_2))(d(e_1) - d(e_2)) \\ &\stackrel{(a)}{\geq} (1 - q(u_1))(d(e_1) - d(e_2)) \\ &= c(e_1, u_1) - c(e_2, u_1) \end{aligned}$$

where (a) is true because $d(e_1) - d(e_2) \geq 0$ [from (M3)] and $1 - q(u_2) \geq 1 - q(u_1) \geq 0$ [from (M0) and (M2)].

- 3) Fix $u \in [0, u_{\max}]$ and consider $e, e_+ \in \mathbb{R}$. Then, $p(u)$ is even because

$$\begin{aligned} p(-e_+ | -e; u) &= q(u)\varphi(-e_+) + (1 - q(u))\varphi(-e_+ + ae) \\ &\stackrel{(b)}{=} q(u)\varphi(e_+) + (1 - q(u))\varphi(e_+ - ae) \\ &= p(e_+ | e; u) \end{aligned}$$

where (b) is true because φ is even [from (M4)].

4) First note that

$$\begin{aligned} S(y|x; u) &= 1 - \int_{-\infty}^y [p(z|x; u) + p(-z|x; u)] dz \\ &= 1 - \int_{-\infty}^y q(u) [\varphi(z) + \varphi(-z)] dz \\ &\quad - \int_{-\infty}^y (1 - q(u)) [\varphi(z - ax) + \varphi(-z - ax)] dz \\ &\stackrel{(c)}{=} 1 - 2q(u)\Phi(y) \\ &\quad - (1 - q(u)) [\Phi(y - ax) + \Phi(y + ax)] \end{aligned}$$

where Φ is the cumulative distribution function of φ and (c) uses the fact that φ is even [condition (M4)].

Let $S_x(y|x; u)$ denote $\partial S/\partial x$. Then,

$$S_x(y|x; u) = (1 - q(u))a[\varphi(y - ax) - \varphi(y + ax)].$$

From (M0) and Lemma 3, we get that $S_x(y|x; u) \geq 0$ for any $x, y \in \mathbb{R}_{\geq 0}$ and $u \in [0, u_{\max}]$. Thus, $S(y|x; u)$ is increasing in x . Furthermore, from (M2), $S_x(y|x; u)$ is decreasing in u . Thus, $S(y|x; u)$ is submodular in (x, u) on $\mathbb{R}_{\geq 0} \times [0, u_{\max}]$.

Now, let us assume that $\mathbb{X} = \mathbb{Z}$. The proof of the first three parts remains the same. Now, in part 4), it is still the case that

$$\begin{aligned} S(y|x; u) &= 1 - 2q(u)\Phi(y) \\ &\quad - (1 - q(u)) [\Phi(y - ax) + \Phi(y + ax)]. \end{aligned}$$

However, since \mathbb{X} is discrete, we cannot take the partial derivative with respect to x . Nonetheless, following the same intuition, for any $x, y \in \mathbb{Z}_{\geq 0}$, consider

$$\begin{aligned} S(y|x+1; u) - S(y|x; u) &= (1 - q(u)) [\Phi(y + ax) \\ &\quad - \Phi(y + ax + a) + \Phi(y - ax) - \Phi(y - ax - a)]. \end{aligned} \quad (13)$$

Now, by Lemma 5, the term in the square bracket is positive, and hence $S(y|x; u)$ is increasing in x . Moreover, since $(1 - q(u))$ is decreasing in u , so is $S(y|x+1; u) - S(y|x; u)$. Hence, $S(y|x; u)$ is submodular in $\mathbb{Z}_{\geq 0} \times [0, u_{\max}]$.

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