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On the optimal thresholds in remote state estimation with communication costs

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Abstract— **In this paper, we consider a remote sensing system that consists of a sensor and an estimator. A sensor observes a first order Markov source and must communicate it to a remote estimator. Communication is noiseless but expensive. At each time, based on the history of its observations and decisions, the sensor chooses whether to transmit or not. If the sensor does not transmit, then the estimator must estimate the Markov process using its past observations. It was shown by Lipsa and Martins, 2011 and by Nayyar et al, 2013 that the optimal strategy has the following structure. The optimal estimation strategy is Kalman-like and the optimal communication strategy is to communicate when the estimation error is greater than a threshold. We derive closed form expressions for infinite horizon discounted cost version of the problem. Our solution approach is inspired by the idea of calibration used in multi-armed bandits. We identify the value of the communication cost for which one is indifferent between two consecutive threshold based strategies. Using these values, we characterize the optimal thresholds as a function of the communication cost. Lastly, we present an example of birth-death Markov chain to illustrate our results.**

I. INTRODUCTION

A. Motivation

In this paper, we consider a model that captures a fundamental trade-off between communication cost and estimation accuracy. This model is motivated by applications in smart grids and environmental monitoring.

In smart grids, it is envisioned that smart meters will measure the energy consumption in households and communicate these measurements to an aggregator which will use this information for demand response etc. In such a scenario, it is important not to flood the communication network with measurement information by communicating periodically. Instead, one can model the signaling overhead as a cost and optimally trade-off communication cost with estimation accuracy.

In environmental monitoring, a sensor network is used to measure an environmental variable such as rainfall, soil moisture, etc. Energy consumption at the sensor is an important consideration in such systems because it is expensive to replace the sensor battery. Thus, to conserve battery, it is important not to transmit periodically. Instead, one can model the energy consumed while communicating as a cost and optimally trade-off communication cost with estimation accuracy.

Similar scenarios also arise in other applications such as networked control systems. Consider the following model that captures the above scenarios. A sensor observes a first order Markov process and must communicate it to a remote estimator. Communication is noiseless but expensive. At each time, based on the history of its observations and decisions, the sensor chooses whether to transmit or not. If the sensor does not transmit, then the estimator must estimate the Markov process using its past observations. The objective is to minimize a weighted combination of communication cost and estimation error.

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The remote estimation is conceptually difficult due to information decentralization. When the sensor decides not to communicate, its decision is based on the current value of the Markov process. So, even when the estimator does not receive an observation, the 'absence of observation' conveys some information about the Markov process. Such signaling problems are known to be notoriously hard, with the Witsenhausen's counterexample [1] being the most famous example.

The above model has received considerable attention in the literature. The most closely related papers are [2]–[5], which are briefly summarized below. Other related work includes models where the sensor is allowed to sleep for a pre-specified amount of time [6] and models where the estimator decides when the sensor should transmit [7]–[9]. The setup of this paper is also related to the censoring sensors considered in hypothesis testing [10], [11] (where the sensor takes one measurement and decides whether to transmit it or not) and real-time source coding [12], [13] (where the sensor must transmit a quantized version of the observation).

In [2], the authors considered a remote estimation problem where the sensor could communicate a finite number of times. They assumed that the sensor used a threshold strategy to decide when to communicate and determined the optimal estimation strategy and the value of the thresholds. In [3], the authors considered remote estimation of Gauss-Markov processes. They assumed a particular form of the estimator and show that the estimation error is a sufficient statistic for the sensor.

In [4] too, the authors considered remote estimation of Gauss-Markov processes but do not impose any assumption on the communication or estimation strategy. They use ideas from majorization theory to show that the optimal estimation strategy is Kalman-like and the optimal communication strategies are threshold based. In [5], the authors considered remote estimation of countable state Markov processes where the sensor harvests energy to communicate. They show that if the Markov process is symmetric in an appropriate sense, then the results of [4] continue to hold. Both [4] and [5]

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identified dynamic programs to find the optimal thresholds.

Threshold based transmission policies may be viewed as event-based transmission policies: transmission takes place when an event (estimation error greater than a threshold) takes place. Such event-based transmission has also received considerable attention in the literature, a detailed overview of which is given in [14]. In recent years, various event-triggered policies have been proposed and analyzed for different stochastic and deterministic setups, of which a few works are given in [15]–[19].

In this paper, we revisit the model of [4] and [5] and look at it from a slightly different point of view. Instead of asking what is the optimal threshold for a particular communication cost, we ask what is the range of communication costs for which a particular threshold is optimal. To find such a range, we use the idea of calibration from multi-armed bandits. We identify the value of the communication cost for which one is indifferent between two consecutive threshold strategies. Using these values, we obtain the range of communication costs for which a particular strategy is optimal.

B. Notation

$\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set of natural numbers. $x_{1:t}$ is a short hand for the vector $(x_1, \dots, x_t)$. For a matrix A, A_{ij} denotes the (i, j)-th element of A and A_i denotes the i-th row of A. Note that unlike the standard notation, in our notation the indices to denote an element of a matrix take both positive and negative values. Furthermore, $A^\intercal$ denotes the transpose of A. I_k denotes the identity matrix of dimension $k \times k$, $k \in \mathbb{N}$. $\mathbf{1}_k$ denotes $k \times 1$ vector of ones. $\langle v, w\rangle$ denotes the inner product between vectors v and w, $\mathbb{P}(\cdot)$ denotes the probability of an event, $\mathbb{E}[\cdot]$ denotes the expectation of a random variable, and $\mathbb{1}\{\cdot\}$ denotes the indicator function of a statement.

C. Problem Formulation

Consider a remote sensing system, which consists of a sensor and an estimator. The sensor observes the state of a first-order Markov process $\{X_t\}_{t=0}^{\infty}$, $X_t \in \mathbb{Z}$[1], with transition matrix P. Assume that the Markov process starts in state x_0 that is known to the sensor and the estimator.

At time t, the sensor decides between two alternatives: either to transmit the current state X_t and incur a cost c or not transmit and incur no cost. The sensor's decision is denoted by $U_t \in \{0, 1\}$, where $U_t = 0$ denotes no transmission and $U_t = 1$ denotes transmitting the current state. The transmitted symbol Y_t is given by

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \epsilon, & \text{if } U_t = 0 \end{cases}$$

where ϵ means no transmission.

The sensor's decision is generated as follows:

$$U_t = f_t(X_{1:t}, U_{1:t-1}, Y_{1:t-1}) \tag{1}$$

[1]The results generalize to $X_t \in \mathbb{R}^n$, but for ease of exposition we restrict our discussion to $X_t \in \mathbb{Z}$

where f_t is called the *communication rule* at time t and the collection $\mathbf{f} = (f_1, f_2, \cdots)$ is called the *communication strategy*.

The estimator observes the transmitted symbols and generates an estimate $\hat{X}_t \in \mathbb{Z}$ as follows:

$$\hat{X}_t = g_t(Y_{1:t}) \tag{2}$$

where g_t is called the *estimation rule* at time t and the collection $\mathbf{g} = (g_1, g_2, \cdots)$ is called the *estimation strategy*.

We are interested in the following optimization problem:

Problem 1: In the above model, given the transition matrix P of the Markov process, the communication cost c and a distortion function $\ell : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$, choose transmission and estimation strategies $(\mathbf{f}, \mathbf{g})$ to minimize the discounted cost

$$J(\mathbf{f}, \mathbf{g}) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \beta^t \big[cU_t + \ell(X_t, \hat{X}_t)\big] \;\Big|\; X_0 = x_0\Big] \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor.

D. Results of [4] and [5]

Slight variations of the above model have been considered in [4], [5] where the objective was to minimize the finite horizon cost

$$\mathbb{E}\Big[\sum_{t=0}^{T} \big[cU_t + \ell(X_t, \hat{X}_t)\big] \;\Big|\; X_0 = x_0\Big]. \tag{4}$$

The model in [4] assumed a Gauss-Markov process with square error distortion while the model in [5] assumed that sensor had limited battery that was replenished using energy harvesting. The structure of optimal communication and estimation strategies was established in [5] under the following assumptions.

(A1) The transition matrix P of the Markov process is a symmetric and banded Toeplitz matrix with decaying terms, i.e.

$$P_{ij} = \begin{cases} p_{|i-j|}, & \text{if } |i-j| \le b \\ 0, & \text{if } |i-j| > b \end{cases}$$

where b is fixed and $p_0 > p_1 > \cdots > p_b$.

(A2) The distortion function is given by $\ell(X_t, \hat{X}_t)$, where ℓ is monotone increasing and even function given by $\ell(X_t, \hat{X}_t) = \mathbb{1}\{X_t \neq \hat{X}_t\}$ or $\ell(X_t, \hat{X}_t) = |X_t - \hat{X}_t|^d$ for some $d > 0$. With a slight abuse of notation, we use $\ell(X_t - \hat{X}_t)$ to denote the distortion function.

Remark 1: Instead of assumption (A1), it was assumed in [5] that the Markov process evolves according to

$$X_{t+1} = X_t + N_t$$

where N_t has an even and a.s.u. (almost symmetric and unimodal) distribution. That assumption was motivated by the model in [4] where the (real-valued) Markov process evolves according to

$$X_{t+1} = aX_t + N_t$$

where N_t is a zero-mean Gaussian process. Our assumption (A1) is equivalent to the assumption of [5] and, in our opinion, a more natural characterization of a Markov process. In addition, [5] had assumed that the distribution of X_0 is even and a.s.u. We do not need that assumption because we assume that x_0 is fixed.

Definition 1: Let Z_t denote the most recent transmitted value of the Markov process. The process $\{Z_t\}_{t=0}^{\infty}$ evolves in a controlled Markov manner as follows:

$$Z_0 = x_0$$

and

$$Z_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ Z_{t-1}, & \text{if } U_t = 0. \end{cases}$$

Note that since U_t can be inferred from the transmitted symbol Y_t, the estimator can also keep track of Z_t as follows:

$$Z_0 = x_0$$

and

$$Z_t = \begin{cases} Y_t, & \text{if } Y_t \neq \epsilon \\ Z_{t-1}, & \text{if } Y_t = \epsilon. \end{cases}$$

Theorem 1: *[5, Theorem 2], [4, Proposition 1] Consider the finite horizon version of Problem 1 under assumptions (A1) and (A2). The process $\{Z_t\}_{t=0}^{T}$ is a sufficient statistic at the estimator and an optimal estimation strategy is given by:*

$$\hat{X}_t = g_t(Z_t) = Z_t. \tag{5}$$

In general, the optimal estimation strategy depends on the choice of the communication strategy and vice-versa. Theorem 1 shows that when the Markov process and the distortion function satisfy appropriate symmetry assumptions, the optimal estimation strategy can be specified in closed form. Consequently, we can fix the estimator to be of the above form, and consider the centralized problem of identifying the best communication strategy.

Definition 2: Let $E_t = X_t - Z_{t-1}$. The process $\{E_t\}_{t=0}^{\infty}$ evolves in a controlled Markov manner as follows:

$$\mathbb{P}(E_{t+1} = n \mid E_t = e, U_t = u) = \begin{cases} P_{0n} & \text{if } u = 1 \\ P_{en} & \text{if } u = 0. \end{cases}$$

Theorem 2: *[5, Theorem 3], [4, Theorem 1] Consider the finite horizon version of Problem 1 under assumptions (A1) and (A2). The process $\{E_t\}_{t=1}^{T}$ is a sufficient statistic at the sensor and an optimal communication strategy is characterized by a sequence of thresholds $\{k_t\}_{t=1}^{T}$ i.e.,*

$$U_t = f_t(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k_t \\ 0, & \text{if } |E_t| < k_t. \end{cases}$$

The optimal strategy mentioned above is given by the solution of the following dynamic programming

$$V_{T+1}(\cdot) = 0 \tag{6}$$

and for $t = T, \dots, 0$

$$V_t(e) = \min\Big\{c + \sum_{n=-\infty}^{\infty} P_{0n} V_{t+1}(n), \quad \ell(e) + \sum_{n=-\infty}^{\infty} P_{en} V_{t+1}(n)\Big\}. \tag{7}$$

E. Contributions of this paper

Theorem 2 shows that the optimal communication strategy is threshold based and, hence, easy to implement. However, we still need to solve an appropriate dynamic program to identify the thresholds. In this paper, we consider the infinite horizon discounted cost problem. From Theorem 1, the optimal estimation strategy is time-invariant. Using standard results for countable state Markov decision processes, we show that under appropriate conditions the optimal communication strategy is time-invariant threshold strategy and given by the fixed point of a dynamic program.

The dynamic program for the infinite horizon discounted cost problem is similar to the dynamic programs that arises in multi-armed bandits [20]. We exploit this relationship and use the idea of calibration from multi-armed bandits. We first derive a closed form expression for an arbitrary threshold based strategy. We use this expression to obtain a closed form expression of the communication index, i.e. the value of communication cost for which one is indifferent between two consecutive threshold strategies. We show that the communication index provides the complete characterization of the optimal communication strategy for all values of the communication cost.

II. Main results

A. Dynamic program for infinite horizon setup

In contrast to [4], [5], we consider infinite horizon discounted cost problem. The structural result of Theorems 1 and 2 extend to the infinite horizon setup. To show that the corresponding optimal strategy is time-homogeneous, we assume the following:

(A3) There exists positive and finite constant μ_1 and μ_2 and a function $w\colon \mathbb{Z} \to \mathbb{R}$ such that for all $e \in \mathbb{Z}$

$$\max\{c, \ell(e)\} \leq \mu_1 w(e);$$

and

$$\max\Big\{\sum_{n=-\infty}^{\infty} P_{en} w(n), \sum_{n=-\infty}^{\infty} P_{0n} w(n)\Big\} \leq \mu_2 w(e).$$

Theorem 3: *Consider Problem 1 under assumptions (A1), (A2) and (A3). The process $\{E_t\}_{t=1}^{\infty}$ is a sufficient statistic at the sensor and an optimal communication strategy is characterized by a time-invariant threshold k, i.e.,*

$$U_t = f(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k \\ 0, & \text{if } |E_t| < k. \end{cases} \tag{8}$$

The optimal strategy mentioned above is given by the unique fixed point of the following dynamic programming

$$V(e;c) = \min\Big\{c + \beta \sum_{n=-\infty}^{\infty} P_{0n} V(n;c), \quad \ell(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} V(n;c)\Big\}. \tag{9}$$

We are interested in the sensitivity of the optimal strategy to a change in the communication cost c. For that reason, we parametrize value function with the communication cost c.

Proof: The above result is the natural extension of the result of Theorem 2 to infinite horizon setup. The results on uniqueness of the fixed point rely on Banach fixed point theorem. To apply Banach fixed point theorem, the value function must be bounded with respect to a sup-norm. In our case, the loss function is unbounded, so we cannot assume that the value function is uniformly bounded. Therefore, we need to work with a weighted sup-norm. As shown in [21, Assumptions 6.10.1, 6.10.2], if a weighting function $w(\cdot)$ defined in (A3) exists, then the Bellman operator has a unique fixed point and the result of the theorem follows from [21, Proposition 6.10.3]. ■

Assumption (A3) is fairly mild. In Section III, we show that this assumption is always satisfied by a birth-death Markov chain.

B. Performance of a threshold strategy

The result of Theorem 3 shows that we can restrict attention to time-homogeneous threshold based communication strategies. In this section we obtain a closed form expression for the performance of an arbitrary strategy of this form.

Let $\mathcal{F}$ denote the class of all time-homogeneous threshold-based strategies of the type (8). Let $f_k \in \mathcal{F}$ denote the strategy with threshold k, $k \in \mathbb{N}$, i.e.

$$f_k(e) = \begin{cases} 1, & \text{if } |e| \geq k \\ 0, & \text{if } |e| < k. \end{cases}$$

Let $W_k(e;c)$ denote the performance of strategy f_k when the system starts in state e and has a communication cost c. From standard results in Markov decision processes, W_k is the unique fixed point of the following equation:

$$W_k(e;c) = \begin{cases} c + \beta \sum_{n=-\infty}^{\infty} P_{0n} W_k(n;c), & \text{if } |e| \geq k, \\ \ell(e) + \beta \sum_{n=-\infty}^{\infty} P_{en} W_k(n;c), & \text{if } |e| < k \end{cases} \tag{10}$$

For ease of notation, denote

$$\hat{W}_k(0;c) = \sum_{n=-\infty}^{\infty} P_{0n} W_k(n;c).$$

Note that $W_k(0;c) = \beta \hat{W}_k(0;c)$ and an equivalent characterization of $W_k(0;c)$ is given by

$$W_k(0;c) = \mathbb{E}\Big[\sum_{t=0}^{\tau_k - 1} \beta^t \ell(E_t) + \beta^{\tau_k}(c + \beta \hat{W}_k(0;c)) \,\Big|\, E_0 = 0\Big] \tag{11}$$

where τ_k denotes the stopping time when the Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{Z} : |e| \geq k\}$. Define

$$L_k = \mathbb{E}\Big[\sum_{t=0}^{\tau_k - 1} \beta^t \ell(E_t) \,\Big|\, E_0 = 0\Big]$$

and

$$T_k = \frac{1 - \mathbb{E}[\beta^{\tau_k}]}{1 - \beta}.$$

Then, substituting $W_k(0;c)$ and L_k in (11), we get that

$$W_k(0;c) = \beta \hat{W}_k(0;c) = \frac{1}{(1-\beta)T_k}(L_k + c(1 - (1-\beta)T_k)). \tag{12}$$

Next, we seek to find a closed form expression for L_k and T_k. First, let us define square matrices $P^{(k)}$ and $Q^{(k)}$ and a column vector $\ell^{(k)}$ indexed by $I^{(k)} = \{-k+1, \cdots, k-1\}$ as follows:

$$P^{(k)}_{ij} = P_{ij}, \quad i, j \in I^{(k)};$$
$$Q^{(k)} = [I_{2k-1} - \beta P^{(k)}]^{-1};$$
$$\ell^{(k)} = [\ell(-k+1), \ell(-k+2), \cdots, \ell(k-2), \ell(k-1)]^{\intercal}.$$

We can now state the following theorem which gives an easily computable expression for L_k and T_k.

Lemma 4: *L_k and T_k defined above can be expressed in a closed form as follows:*

$$L_k = \Big\langle Q_0^{(k)}, \ell^{(k)} \Big\rangle; \tag{13}$$

$$T_k = \Big\langle Q_0^{(k)}, \mathbf{1}_{2k-1} \Big\rangle, \tag{14}$$

where $Q_0^{(k)}$ denotes the row with index 0 in $Q^{(k)}$. Substituting these in (12), we get a closed form expression of $W_k(0;c)$.

Proof: Recall that for a matrix A, A_0 denotes the row with index 0 (and our index set includes negative values as well). $P^{(k)}$ is a substochastic matrix that captures the probability of Markov chain not leaving the set $I^{(k)}$. Thus

$$\begin{aligned} L_k &= \mathbb{E}\Big[\sum_{t=0}^{\tau_k-1} \beta^t \ell(E_t) \,\Big|\, E_0 = 0\Big] = \sum_{t=0}^{\infty} \beta^t \Big[\sum_{e \in I^{(k-1)}} (P^{(k)}_{0e})^t \ell(e)\Big] \\ &= \sum_{t=0}^{\infty} \Big\langle \Big[(\beta P^{(k)})^t\Big]_0, \ell^{(k)} \Big\rangle = \Big\langle \sum_{t=0}^{\infty} \Big[(\beta P^{(k)})^t\Big]_0, \ell^{(k)} \Big\rangle \\ &= \Big\langle \Big[\sum_{t=0}^{\infty} (\beta P^{(k)})^t\Big]_0, \ell^{(k)} \Big\rangle = \Big\langle Q_0^{(k)}, \ell^{(k)} \Big\rangle \end{aligned} \tag{15}$$

(14) can be proved by a similar argument. ■

C. Characterization of optimal strategy

Our approach to characterize the optimal strategy is inspired by the idea of *calibration* in multi-armed bandits [20]. Let c_k be the value of the communication cost for which, starting from state 0, one is indifferent between communication strategy f_k and f_{k+1}, i.e., c_k is such that

$$W_k(0; c_k) = W_{k+1}(0; c_k). \tag{16}$$

For reasons that will become apparent later, we call the sequence $\{c_k\}_{k=1}^{\infty}$ as the *communication indices*. We show that such a c_k exists under the following assumption.

(A4) L_k/T_k is increasing in k.

Lemma 5: *$T_k < T_{k+1}$ and $L_k < L_{k+1}$.*

Due to dearth of availability of space, we omit the proof of the above lemma.

Lemma 6: *Under (A4) such a c_k always exists, is positive, and is given by*

$$c_k = \left(\frac{L_{k+1}}{T_{k+1}} - \frac{L_k}{T_k}\right) \Bigg/ \left(\frac{1}{T_k} - \frac{1}{T_{k+1}}\right). \tag{17}$$

Proof: By definition of c_k, $W_k(0; c_k) = W_{k+1}(0; c_k)$. Using (12), we get that

$$\frac{L_k + c_k(1-(1-\beta)T_k)}{(1-\beta)T_k} = \frac{L_{k+1} + c_k(1-(1-\beta)T_{k+1})}{(1-\beta)T_{k+1}}.$$

Under (A4), the above equation always has a solution for c_k. By rearranging the terms, we get c_k as given by (17), which is positive. ■

Theorem 7: *Suppose (A4) holds and $\{c_k\}_{k=1}^{\infty}$ is an increasing sequence. Then, for all $c \in (c_k, c_{k+1}]$ such that $c_k \neq c_{k+1}$, the strategy f_{k+1} is discounted cost optimal.*

Proof: Using (12), we get that for any c,

$$W_k(0; c) = W_k(0; c_k) + \frac{(1-T_k)}{T_k}(c - c_k).$$

Using the above equation with (16), we get

$$W_k(0; c) - W_{k+1}(0; c) = \left[\frac{1}{T_k} - \frac{1}{T_{k+1}}\right](c - c_k)$$

Since by Lemma 5 we have $T_k < T_{k+1}$, the first term in the brackets is positive and the sign of $W_k(0; c) - W_{k+1}(0; c)$ is the same as that of $(c - c_k)$.

Now, consider a $c \in (c_k, c_{k+1}]$. Since $c > c_k \geq c_{k-1} \geq \cdots \geq c_1$, we have that

$$W_{k+1}(0; c) < W_k(0; c) \leq W_{k-1}(0; c) \leq \cdots \leq W_1(0; c).$$

Moreover, since $c \leq c_{k+1} \leq c_{k+2} \leq \cdots$ we have that

$$W_{k+1}(0; c) \leq W_{k+2}(0; c) \leq W_{k+3}(0; c) \leq \cdots$$

Hence, f_{k+1} is optimal among all threshold strategies. From Theorem 3, we get that f_{k+1} is globally optimum. ■

Lemma 8: *The value function $V(e; \cdot)$, as given in (9), is a piece-wise linear, continuous, concave and increasing function of c for all $e \in \mathbb{Z}$.*

The proof of the above lemma is omitted here due to limitation of space.

D. Discussion of the result

The result of Theorem 7 identifies the range of values for the communication cost for which an arbitrary strategy f_k is optimal. For a particular Markov process and loss function, we can compute the communication indices $\{c_k\}_{k=1}^{\infty}$ by substituting the values of L_k and T_k from Lemma 4 into (17). These communication indices determine the optimal communication strategy for all values of the communication cost. For a particular communication cost c, we find the smallest threshold c_{k+1} that is larger than c, and use the strategy f_{k+1}. This is in contrast to the result of [4] and [5] where a separate dynamic program needs to be solved for each value of c.

III. An example: aperiodic, symmetric birth-death Markov chain

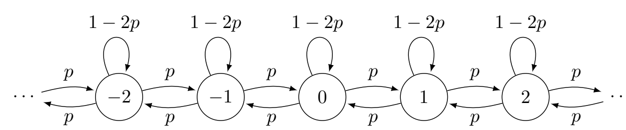

Fig. 1. A birth-death Markov chain

Consider a birth-death Markov chain, shown in Fig. 1, with transition probabilities as follows:

$$P_{ij} = \begin{cases} p, & \text{if } j = i+1, i-1 \\ 1-2p, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}$$

where $p \in (0, \frac{1}{2})$. Let the distortion function $\ell(\cdot)$ to be $\ell(e) = |e|$. Note that P and ℓ satisfy (A1) and (A2).

Lemma 9: *(A3) is always satisfied for the above model. The values of the function $w(\cdot)$ and the parameters μ_1 and μ_2 are given by:*

$$w(e) = \max\{c, \ell(e)\}, \mu_1 = 1, \mu_2 = \max\{1 - 2p + 2p/c, \ell(2)\}$$

Proof: The result may be verified separately for $\ell(e) = \mathbb{1}\{e \neq 0\}$ and $\ell(e) = |e|^d$ by substitution. ■

Since (A3) is satisfied, the optimal communication strategy is a time-homogeneous threshold strategy. Hence, the framework of Sections II-B–II-C is applicable. The communication indices $\{c_k\}_{k=1}^{\infty}$ depend on the values of L_k and T_k, which in turn depend on the matrix $Q^{(k)}$. Since, $Q^{(k)}$ is the inverse of a tridiagonal symmetric Toeplitz matrix, an explicit formula for its elements is available [22].

Lemma 10: *Define*

$$D = -2 - (1-\beta)/(\beta p) \quad \textit{and} \quad \lambda = \cosh^{-1}(-D/2).$$

Then $Q^{(k)}$ is given by

$$Q_{ij}^{(k)} = \frac{a_{ij}^{(k)}}{b^{(k)}}, \quad i, i \in I^{(k-1)} \tag{18}$$

TABLE I

VALUES OF c_k FOR A BIRTH-DEATH MARKOV CHAIN WITH $p = 0.3$.

k	1	2	3	4	5	6	7	8	9	10
c_k	1.0811	3.4764	6.7826	10.5126	14.4118	18.3748	22.3612	26.3563	30.3545	34.3538

where

$$a_{ij}^{(k)} = \cosh((2k - |j-i|)\lambda) - \cosh((i+j)\lambda),$$
$$b^{(k)} = 2\beta p \sinh(\lambda)\sinh(2k\lambda).$$

In particular $Q_{0j}^{(k)}$ *is given by:*

$$Q_{0j}^{(k)} = \frac{1}{\beta p}\frac{\cosh((2k-|j|)\lambda) - \cosh(j\lambda)}{2\sinh(\lambda)\sinh(2k\lambda)} \tag{19}$$

Proof: The matrix $I_{2k-1} - \beta P^{(k)} \in \mathbb{R}^{(2k-1)\times(2k-1)}$ is a symmetric tridiagonal matrix given by

$$I_{2k-1} - \beta P^{(k)} = -\beta p \begin{bmatrix} D & 1 & 0 & \cdots & \cdots & 0 \\ 1 & D & 1 & \cdots & \cdots & 0 \\ 0 & 1 & D & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & D & 1 \\ 0 & \cdots & \cdots & 0 & 1 & D \end{bmatrix}$$

$Q^{(k)}$ is the inverse of the above matrix. The inverse of tridiagonal Toeplitz matrix in the above form are computed in closed form in [22]. The result of the Lemma corresponds to the $D \leq -2$ case in [22]. ■

Using the expression for $Q_0^{(k)}$, we can obtain L_k and T_k in closed form. In particular:

Lemma 11: *For the above birth-death Markov chain with* $\ell(e) = |e|$,

$$L_k = -\frac{k\sinh(\lambda) - \sinh(k\lambda)}{4\beta p \sinh^2(\lambda/2)\sinh(\lambda)\cosh(k\lambda)} \tag{20a}$$

$$T_k = \frac{(1-\beta)\sinh^2(k\lambda/2)}{2\beta p \sinh^2(\lambda/2)\cosh(k\lambda)} \tag{20b}$$

Proof: The result follows by substituting $Q_{0j}^{(k)}$ given by (19) into the expressions for L_k and T_k derived in Lemma 4, and simplifying the expressions. ■

Using Lemmas 4 and 10, we can easily verify if assumption (A4) holds and find the values of c_k. To illustrate this, consider $\ell(e) = |e|$ and pick $p = 0.3$ and $\beta = 0.75$. We verified numerically that when (A4) is satisfied for different values of k, the corresponding values of c_k are non-decreasing and the initial 10 values are shown in Table I.

IV. CONCLUSION

In this paper, we study a remote sensing problem with communication cost and generalize the results of [4] and [5] to infinite horizon. We obtain an explicit characterization of the communication indices that represent the value of communication cost for which one is indifferent between two consecutive threshold strategies. We provide closed form expressions of these communication thresholds and use them to completely characterize the optimal communication strategy for all values of the communication cost.

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