Distortion-transmission trade-off in real-time transmission of Gauss-Markov sources

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***Abstract*—The problem of optimal real-time transmission of a Gauss-Markov source under constraints on the expected number of transmissions is considered. This setup is motivated by applications where transmission is sporadic and the cost of switching on the radio and transmitting is significantly more important than the size of the transmitted data packet. The structure of optimal transmission and estimation strategies had been established in the literature. We use these structural results to determine the distortion-transmission function, i.e., the minimum expected distortion that can be achieved when the expected number of transmissions is less than or equal to a particular value. We characterize how the distortion-transmission function scales with the variance of the source and show that it can be computed numerically by solving two Fredholm integral equations of the second kind.**

I. Introduction

In many real-time communication systems such as networked control systems, sensor and surveillance networks, and transportation networks, etc., the transmitter is often a battery-powered device that transmits over a wireless packet-switched network. In such systems, the cost of switching on the radio and transmitting is significantly more important than the size of the data packet. This motivates the study of the fundamental trade-off between the distortion incurred when transmitting a source with real-time (or zero-delay) reconstruction under constraints on the expected number of transmissions. Chakravorty and Mahajan [1] recently investigated such a setup for Markov chains defined over integers and characterized the "distortion-transmission" trade-off. In this paper, we characterize the distortion-transmission function for scalar Gauss-Markov sources.

A. Problem formulation

We investigate the following communication setup. Let $\{X_t\}_{t=0}^\infty$, $X_t \in \mathbb{R}$, be a scalar first-order Gauss-Markov source. The initial state $X_0 = 0$ and for $t > 0$,

$$X_{t+1} = X_t + W_t, \tag{1}$$

where $\{W_t\}_{t=0}^\infty$ are i.i.d. zero-mean Gaussian random variables with variance σ^2.

A transmitter causally observes this source and at each time chooses whether or not to transmit the current source observation. This decision is denoted by $U_t \in \{0,1\}$, where $U_t = 0$ denotes no transmission and $U_t = 1$ denotes transmission. The decision to transmit is made using a *transmission strategy* $f = \{f_t\}_{t=0}^\infty$, where

$$U_t = f_t(X_{0:t}, U_{0:t}). \tag{2}$$

We use the short hand notation $X_{0:t}$ to denote the sequence $\big(X_0, \cdots, X_t\big)$. Similar interpretation holds for $U_{0:t-1}$.

The transmitted symbol, which is denoted by Y_t, is given by

$$Y_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \mathfrak{E}, & \text{if } U_t = 0, \end{cases} \tag{3}$$

where $Y_t = \mathfrak{E}$ denotes no transmission.

The receiver causally observes $\{Y_t\}_{t=0}^\infty$ and generates a source reconstruction $\{\hat{X}_t\}_{t=0}^\infty$ (where $\hat{X}_t \in \mathbb{R}$) in real-time using an *estimation strategy* $g = \{g_t\}_{t=0}^\infty$, i.e.

$$\hat{X}_t = g_t(Y_{0:t}). \tag{4}$$

The fidelity of reconstruction is measured by squared error distortion $(X_t - \hat{X}_t)^2$.

The objective is to choose the transmission and estimation strategies (called the *communication strategy* in short) to minimize the expected distortion under a constraint on the expected number of transmissions. Given a communication strategy, let

$$D(f,g) \coloneqq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T-1}(X_t - \hat{X}_t)^2 \;\Big|\; X_0 = 0\Big]$$

denote the expected long-term average distortion and

$$N(f,g) \coloneqq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T-1}U_t \;\Big|\; X_0 = 0\Big]$$

denote the expected long-term average number of transmissions.

We are interested in the following constrained long-term average cost problem: given $\alpha \in (0,1)$, find a strategy (f^*, g^*) such that

$$D^*(\alpha) \coloneqq D(f^*, g^*) \coloneqq \inf_{(f,g): N(f,g) \le \alpha} D(f,g), \tag{AVG}$$

where the infimum is taken over all history-dependent communication strategies of the form (2) and (4).

The function $D^*(\alpha)$ represents the minimum expected distortion that can be achieved when the number of transmissions is less than α. It is analogous to the distortion-rate function; for that reason we call it the *distortion-transmission* function.

Problem (AVG) is constrained optimization problem. We will first investigate its Lagrange relaxation. For any Lagrange multiplier $\lambda \geq 0$ and any (history dependent) communication strategy (f, g), define $C(f, g; \lambda)$ as

$$\limsup_{T\to\infty} \mathbb{E}^{(f,g)} \frac{1}{T} \Big[\sum_{t=0}^{T-1} \big[(X_t - \hat{X}_t)^2 + \lambda U_t \big] \;\Big|\; X_0 = 0 \Big].$$

The Lagrange relaxation of Problem (AVG) is the following: for any $\lambda \geq 0$, find a strategy (f^*, g^*) such that

$$C(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} C(f, g; \lambda) \tag{LAG}$$

where the infimum is taken over all history-dependent communication strategies of the form (2) and (4).

B. Literature overview

The communication system described above is similar to the classical information theory setup. In particular, it may be viewed as minimizing the average distortion while transmitting over a channel under an average-power constraint. However, unlike the classical information theory setup, the source reconstruction must be done in real-time (i.e. with zero delay). Due to this real-time constraint on source reconstruction, traditional information theoretic approach does not apply.

Two approaches have been used in the literature to investigate real-time or zero-delay communication. The first approach considers coding of individual sequences [2]–[4]; the second approach considers coding of Markov sources [5]–[10]. The model presented above fits with the latter approach. In particular, it may be viewed as real-time transmission over a noiseless channel with input cost. In most of the results in the literature on real-time coding of Markov sources, the focus has been on identifying sufficient statistics (or information states) at the transmitter and the receiver; for some of the models, a dynamic programming decomposition has also been derived. However, very little is known about the solution of these dynamic programs.

The communication system described above is much simpler than the general real-time communication setup due to the following feature: whenever the transmitter transmits, it sends the current realization of the source to the receiver. These transmitted events *reset* the system. Variations of the above communication systems have been considered in [11]–[14]. The authors in [11] assume a specific form of the transmitter and identify the optimal receiver. The authors in [12] assume a specific form of the receiver and identify the optimal transmitter. The authors in [13] and [14] do not make any assumptions on the transmitter or the receiver and show that threshold based strategies are optimal at the transmitter and optimal estimation strategies are Kalman-like. We use these structural results and obtain the optimal thresholds for an arbitrary Lagrange multiplier. We then use the continuity of the optimal value function to find the optimal thresholds for the constrained optimization problem (AVG), and use that to completely characterize the distortion-transmission function.

II. Finite horizon setup and the structure of optimal strategies

To identify the structure of the optimal communication strategy, consider the finite-horizon setup of Problem (LAG). Given a time horizon $T \in \mathbb{Z}_{>0}$, a Lagrange multiplier λ, the performance of a strategy (f, g), where $f = (f_0, \dots, f_T)$ and $g = (g_0, \dots, g_T)$, is given by

$$C_T(f, g; \lambda) \coloneqq \mathbb{E}^{(f,g)} \Big[\sum_{t=0}^{T} \big[(X_t - \hat{X}_t)^2 + \lambda U_t \big] \;\Big|\; X_0 = 0 \Big].$$

The finite-horizon optimization problem is to find a finite-horizon strategy (f^*, g^*) such that

$$C_T(f^*, g^*; \lambda) = \inf_{(f,g)} C_T(f, g; \lambda) \tag{FIN}$$

where the infimum is taken over all history-dependent communication strategies of the form (2) and (4).

Problem (FIN) is a dynamic team (or a decentralized control) problem with two decision-makers—the transmitter and the receiver—who have to coordinate to achieve a common objective. A slight variation of Problem (FIN) was investigated in [13] and [14], where the authors identified the structure of optimal strategies. To explain these results, we define the following processes.

Definition 1 Let Z_t denote the most recently transmitted value of the Markov source. The process $\{Z_t\}_{t=0}^{\infty}$ evolves in a controlled Markov manner as follows: $Z_0 = 0$ and

$$Z_t = \begin{cases} X_t, & \text{if } U_t = 1; \\ Z_{t-1}, & \text{if } U_t = 0. \end{cases}$$

□

Definition 2 (Error process) Let $E_t = X_t - Z_{t-1}$. The process $\{E_t\}_{t=0}^{\infty}$ evolves in a controlled Gauss-Markov manner as follows: $E_0 = 0$ and

$$E_{t+1} = \begin{cases} E_t + W_t, & \text{if } U_t = 0 \\ W_t, & \text{if } U_t = 1. \end{cases} \tag{5}$$

. □

The following structure was proved in [13]) and [14].

Theorem 1 *In Problem* (FIN)*,* $\{Z_t\}_{t=0}^{T}$ *and* $\{E_t\}_{t=0}^{T}$ *are sufficient statistics for the receiver and the transmitter respectively. In particular, there is no loss of optimality in considering transmission strategies of the form*

$$U_t = f_t(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k_t; \\ 0, & \text{if } |E_t| < k_t, \end{cases} \tag{6}$$

and estimation strategies of the form

$$\hat{X}_t = g_t^*(Z_t) = Z_t. \tag{7}$$

□

III. Main results

A. Performance of a threshold based strategy

The structural results of Theorem 1 extend to the infinite horizon setup as well. The optimal estimation strategy is completely specified by Theorem 1. Note that the optimal estimation strategy does not depend on the choice of the transmission strategy. Therefore, we can fix the estimation strategy and seek to find a transmission strategy that is the *best response* to this estimation strategy. Identifying such a best response strategy is a centralized stochastic control problem. Since the optimal estimation strategy is time-homogeneous, it can be shown that the best response optimal transmission strategy (i.e., the choice of the optimal thresholds $\{k_t\}_{t=0}^{\infty}$) is time-homogeneous as well [1]. Thus, we can search within the class of all time-homogeneous transmission strategies (denoted by $\mathcal{F}$) to solve Problem (LAG).

Let $f^{(k)} \in \mathcal{F}$ denote the strategy with threshold k, $k \in \mathbb{R}_{\geq 0}$, i.e.,

$$f^{(k)}(e) := \begin{cases} 1, & \text{if } |e| \geq k; \\ 0, & \text{if } |e| < k. \end{cases}$$

Let $D^{(k)}(e)$ and $N^{(k)}(e)$ denote the expected long-term average distortion and the expected long-term average number of transmissions under strategy $f^{(k)}$ when the system starts in state e. Note that,

$$D^{(k)}(0) = D(f^{(k)}, g^*) \quad \text{and} \quad N^{(k)}(0) = N(f^{(k)}, g^*),$$

where g^* is the strategy given by (7).

Similarly, let $C^{(k)}(e; \lambda)$ denote the long-term average performance of strategy $f^{(k)}$ for the Lagrange relaxation with Lagrange multiplier $\lambda \geq 0$ when the system starts in state e. Then,

$$C^{(k)}(e; \lambda) = D^{(k)}(e) + \lambda N^{(k)}(e), \tag{8}$$

and

$$C^{(k)}(0; \lambda) = C(f^{(k)}, g^*; \lambda). \tag{9}$$

Let $\tau^{(k)}$ denote the stopping time when the Gauss-Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{R} : |e| \geq k\}$. Note that $\tau^{(0)} = 1$ and $\tau^{(\infty)} = \infty$.

Let $L^{(k)}(e)$ and $M^{(k)}(e)$ respectively denote the expected distortion incurred until stopping and expected stopping time under $f^{(k)}$. Then, using balance equations, we can show the following:

Lemma 1 *$L^{(k)}$ and $M^{(k)}$ satisfy the following:*

$$L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w - e) L^{(k)}(w) dw; \tag{10}$$

$$M^{(k)}(e) = 1 + \int_{-k}^{k} \phi(w - e) M^{(k)}(w) dw, \tag{11}$$

where $\phi(w)$ is the pdf of a zero-mean normal variable w with variance σ^2. □

Equations (10) and (11) are Fredholm integral equations of second kind [15]. We discuss the numerical solution of these equations in Section IV.

[1] The general idea behind the proof is as follows. The model satisfies [18, Assumptions 4.2.1, 4.2.2]. Therefore, by [18, Theorem 4.2.3], the structural results extend to the infinite horizon discounted cost setup. It can be shown that the discounted model satisfies [18, Assumptions 4.2.1, 5.4.1] and therefore the structural results extend to the long-term average setup due to [18, Theorem 5.4.3].

Let $\mathcal{C}^{(k)}$ denote the space of bounded functions from $[-k, k]$ to $\mathbb{R}$. Define the operator $\mathcal{B}^{(k)} : \mathcal{C}^{(k)} \to \mathcal{C}^{(k)}$ as follows. For any $v \in \mathcal{C}^{(k)}$,

$$\big[\mathcal{B}^{(k)} v\big](e) = \int_{-k}^{k} \phi(w - e) v(w) dw. \tag{12}$$

Then (10) and (11) can be written as

$$\big[L^{(k)} - \mathcal{B}^{(k)} L^{(k)}\big](e) = e^2, \tag{13}$$

$$\big[M^{(k)} - \mathcal{B}^{(k)} M^{(k)}\big](e) = 1. \tag{14}$$

Let $\|\cdot\|_\infty$ denote the sup-norm, i.e., for any $v \in \mathcal{C}^{(k)}$, $\|v\|_\infty = \sup_{e \in [-k,k]} v(e)$.

Lemma 2 *The operator $\mathcal{B}^{(k)}$ is a contraction, i.e., for any $v \in \mathcal{C}^{(k)}$, $\|\mathcal{B}^{(k)} v\|_\infty < \|v\|_\infty$. Therefore, for any $h \in \mathcal{C}^{(k)}$, the equation $\big[v - \mathcal{B}^{(k)} v\big] = h$ has a unique bounded solution v.* □

Proof: By the definition of sup-norm, we have that

$$\begin{aligned} \|\mathcal{B}^{(k)} v\|_\infty &= \sup_{e \in [-k,k]} \int_{-k}^{k} \phi(w - e) v(w) dw \\ &\leq \sup_{e \in [-k,k]} \|v\|_\infty \int_{-k}^{k} \phi(w - e) dw \\ &< \|v\|_\infty, \quad \text{(since } \phi \text{ is a pdf).} \end{aligned}$$

Now, consider the operator $\mathcal{B}'$ given as: $\mathcal{B}' v = h + \mathcal{B} v$. Then we have,

$$\|\mathcal{B}'(v_1 - v_2)\|_\infty = \|\mathcal{B}(v_1 - v_2)\|_\infty < \|v_1 - v_2\|_\infty.$$

Hence, by Banach fixed point theorem, $\mathcal{B}'$ has a unique fixed point. ■

Lemma 3 *The solution $L^{(k)}$ and $M^{(k)}$ of* (10) *and* (11) *are continuous, differentiable and monotonically increasing in k.* □

Proof: Consider $k, l \in \mathbb{R}_{\geq 0}$ such that $k < l$ and the set $S^{(k)} := (-k, k)$. A sample path starting from $e \in S^{(k)}$ must escape $S^{(k)}$ before it escapes $S^{(l)}$. Thus

$$L^{(l)}(e) \geq L^{(k)}(e).$$

In addition, the above inequality is strict because W_t has a unimodal distribution. The differentiability can be proved from elementary algebra using the definition and the fact that the operator $\mathcal{B}^{(k)}$ is a contraction. Continuity follows from differentiabilty. The detailed proof is omitted here due to lack of space and can be found in [19]. ■

The process $\{E_t\}$ is a regenerative process that *resets* at each transmission. Therefore, by standard results in renewal theory, we can show the following:

Lemma 4 *$D^{(k)}(0)$ and $N^{(k)}(0)$ can be expressed in terms of $L^{(k)}(0)$ and $M^{(k)}(0)$ as follows:*

$$D^{(k)}(0) = \frac{L^{(k)}(0)}{M^{(k)}(0)} \quad \textit{and} \quad N^{(k)}(0) = \frac{1}{M^{(k)}(0)}. \tag{15}$$

□

The proof is omitted due to lack of space. An immediate consequence of Lemma 3 and 4 is the following.

Corollary 1 *$D^{(k)}(0)$ and $N^{(k)}(0)$ are continuous and differentiable in k. Furthermore, $N^{(k)}(0)$ is strictly decreasing in k.* □

Lemma 5 *$D^{(k)}(0)$ is increasing in k.* □

The proof follows in the following manner. The strict monotonicity in k is proved for the discounted cost setup. The result then follows by taking limit $\beta \uparrow 1$. See [19] for a detailed proof.

B. *Solution of Problem* (LAG)

We use $L_k^{(k)}$, $M_k^{(k)}$, $D_k^{(k)}$, $N_k^{(k)}$ and $C_k^{(k)}$ to denote the derivative of $L^{(k)}$, $M^{(k)}$, $D^{(k)}$, $N^{(k)}$ and $C^{(k)}$ with respect to k.

Theorem 2 *If the pair (λ, k) satisfy the following*

$$\lambda = -\frac{D_k^{(k)}(0)}{N_k^{(k)}(0)}, \tag{16}$$

then the strategy $(f^{(k)}, g^*)$ is λ-optimal for Problem (LAG). *Furthermore, for any $k > 0$, there exists a $\lambda \geq 0$ that satisfies* (16). □

Proof: The choice of λ implies that $C_k^{(k)}(0; \lambda) = 0$. Hence strategy $(f^{(k)}, g^*)$ is λ-optimal.

Note that, (16) can also be written as $\lambda = \big((M^{(k)}(0))^2 D_k^{(k)}(0)\big)/M_k^{(k)}(0)$. By Lemma 3, $M_k^{(k)}(0) > 0$ and by Lemma 5, $D_k^{(k)}(0) \geq 0$. Hence, for any $k > 0$, λ given by (16) is positive. This completes the proof. ■

C. *Solution of Problem* (AVG)

By definition, the distortion-transmission function $D^*(\alpha)$ is continuous and decreasing function of α. It can be completely characterized as follows:

Theorem 3 *For any $\alpha \in (0, 1)$, there exists a $k^*(\alpha)$ such that*

$$N^{(k^*(\alpha))}(0) = \alpha. \tag{17}$$

The strategy $(f^{(k^*(\alpha))}, g^*)$ is optimal for Problem (AVG). *Moreover, the distortion-transmission function $D^*(\alpha)$ is given by*

$$D^*(\alpha) = D^{(k^*(\alpha))}(0). \tag{18}$$

□

Proof: A strategy (f°, g°) is optimal for a constrained optimization problem if the following conditions hold [16]:

- (C1) $N(f^\circ, g^\circ) = \alpha$,
- (C2) There exists a Lagrange multiplier $\lambda^\circ \geq 0$ such that (f°, g°) is optimal for $C(f, g; \lambda^\circ)$.

We will show that for a given α, there exists a $k^*(\alpha) \in \mathbb{R}_{\geq 0}$ such that $(f^{(k^*(\alpha))}, g^*)$ satisfy conditions (C1) and (C2).

By Corollary 1, $N^{(k)}(0)$ is continuous and strictly decreasing in k. It is easy to see that $\lim_{k\to 0} N^{(k)}(0) = 1$ and $\lim_{k\to\infty} N^{(k)}(0) = 0$. Hence, for a given $\alpha \in (0, 1)$, there exists a $k^*(\alpha)$ such that $N^{(k^*(\alpha))}(0) = N(f^{(k^*(\alpha))}, g^*) = \alpha$. Thus, $(f^{(k^*(\alpha))}, g^*)$ satisfies (C1).

Now, for $k^*(\alpha)$, we can find a λ satisfying (16) and hence we have by Theorem 2 that $(f^{(k^*(\alpha))}, g^*)$ is optimal for $C(f, g; \lambda)$. Thus, $(f^{(k^*(\alpha))}, g^*)$ satisfies (C2). Hence, $(f^{(k^*(\alpha))}, g^*)$ is optimal for Problem (AVG).

Lastly, the optimal distortion, namely the *distortion-transmission function*, which is function of α, is given by $D^*(\alpha) \coloneqq D(f^{(k^*(\alpha))}, g^*) = D^{(k^*(\alpha))}(0)$. This completes the proof. ■

IV. Scaling and computation

A. *Scaling with variance*

In this section, we investigate the scaling of the distortion-transmission function with the variance σ^2 of the increments W_t. So, we parameterize $L^{(k)}$, $M^{(k)}$, $D^{(k)}$, $N^{(k)}$, $\mathcal{B}^{(k)}$, k^*, and D^* by subscript σ to show the dependence on σ.

Lemma 6 *Let $L_\sigma^{(k)}$ and $M_\sigma^{(k)}$ be the solutions of* (13) *and* (14) *respectively, when the variance of W_t is σ^2. Then*

$$L_\sigma^{(k)}(e) = \sigma^2 L_1^{(k/\sigma)}\Big(\frac{e}{\sigma}\Big), \quad M_\sigma^{(k)}(e) = M_1^{(k/\sigma)}\Big(\frac{e}{\sigma}\Big), \tag{19}$$

$$D_\sigma^{(k)}(e) = \sigma^2 D_1^{(k/\sigma)}\Big(\frac{e}{\sigma}\Big), \quad N_\sigma^{(k)}(e) = N_1^{(k/\sigma)}\Big(\frac{e}{\sigma}\Big). \tag{20}$$

□

Proof: $L_\sigma^{(k)}$ is the solution of the following equation

$$\big[L_\sigma^{(k)} - \mathcal{B}_\sigma^{(k)} L_\sigma^{(k)}\big](e) = e^2.$$

Define $\hat{L}_\sigma^{(k)}(e) \coloneqq \sigma^2 L_1^{(k/\sigma)}(e/\sigma)$. Then, it can be shown that

$$\big[\mathcal{B}_\sigma^{(k)} \hat{L}_\sigma^{(k)}\big](e) = \sigma^2 \big[\mathcal{B}_1^{(k/\sigma)} L_1^{(k/\sigma)}\big]\Big(\frac{e}{\sigma}\Big). \tag{21}$$

Therefore

$$\big[\hat{L}_\sigma^{(k)} - \mathcal{B}_\sigma^{(k)} \hat{L}_\sigma^{(k)}\big](e) = \sigma^2 \big[L_1^{(k/\sigma)} - \mathcal{B}_1^{(k/\sigma)} L_1^{(k/\sigma)}\big]\Big(\frac{e}{\sigma}\Big) = e^2.$$

This proves the scaling of $L_\sigma^{(k)}$. The scaling of $M_\sigma^{(k)}$ can be proved similarly. The scaling of $D_\sigma^{(k)}$ and $N_\sigma^{(k)}$ follow from Lemma 4. This completes the proof. ■

Theorem 4 *$D_\sigma^*(\alpha) = \sigma^2 D_1^*(\alpha)$.* □

Proof: By definition of $k^*(\alpha)$ in Theorem 3 and the scaling properties shown in Lemma 6, we have that $k_\sigma^*(\alpha) = \sigma k_1^*(\alpha)$. Therefore,

$$\begin{aligned} D_\sigma^*(\alpha) &= D_\sigma^{(k_\sigma^*(\alpha))}(0) = D_\sigma^{(\sigma k_1^*(\alpha))}(0) \\ &\overset{(a)}{=} \sigma^2 D_1^{(k_1^*(\alpha))}(0) = \sigma^2 D_1^*(\alpha), \end{aligned}$$

where equality (a) is obtained by using (20). ■

An implication of the above theorem is that we only need to numerically compute $D_1^*(\alpha)$. The distortion-transmission function for any other value of σ^2 can be obtained by simply scaling $D_1^*(\alpha)$.

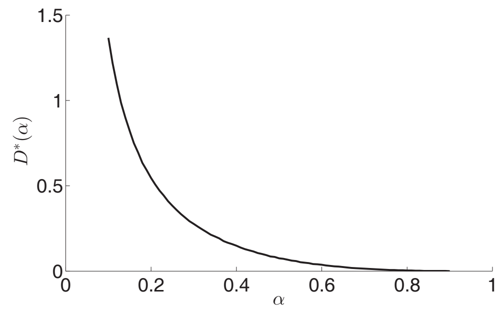

Fig. 1: The distortion-transmission function $D^*(\alpha)$ for $\sigma^2 = 1$.

B. Numerical computation of $D_1^*(\alpha)$

The computation of $D_1^*(\alpha)$ depends on the fixed point solutions $L^{(k)}$ and $M^{(k)}$ as given by the Fredholm equations (10) and (11). There are various methods to solve Fredholm equation of second kind. A standard approach is to use the resolvent kernel of the integral equation; such a solution is known as the Liouville-Neumann series [15]. Another method is to use quadrature method to discretize the integral. The authors of [17] describe Matlab implementation, which uses Simpsons rule to discretize the integral equation with smooth kernels and the natural interpolant of Nystrōm to get an approximate solution. We use their implementation to obtain $D^*(\alpha)$. Since $N^{(k^*(\alpha))}(0)$ is decreasing and continuous in k, we use the bisection method to find the value of $k^*(\alpha)$ for which (17) holds. This requires solving (11) for a different value of k at each iteration. Once $k^*(\alpha)$ is obtained, $D^*(\alpha)$ is computed using (18), which requires solving (10) for $k = k^*(\alpha)$. Fig. 1 shows the plot of $D^*(\alpha)$ for different values of α with variance $\sigma^2 = 1$.

V. Conclusion

In this paper, we characterize the distortion-transmission function of a Gauss-Markov source. The problem is set-up as a constrained decentralized control problem. Previous results [13] and [14] had characterized the structure of optimal transmission and estimation strategies. These results show that the optimal transmission strategy is threshold-based and the optimal estimation strategy is Kalman-like. An interesting feature of these results is that the estimation strategy does not depend on the thresholds used by the transmission strategy. This allows us to investigate the choice of optimal thresholds as a centralized constrained stochastic control problem. We characterize the Lagrange performance of an arbitrary threshold-based strategy and use that to find the optimal threshold for any value of the Lagrange multiplier. By showing that the optimal performance is continuous in the thresholds, we obtain a solution to the constrained optimization problem as well.

We establish the scaling property of the distortion-transmission function, which enables one to compute the value of the distortion-transmission function for a given variance, using the knowledge of the *standard* distortion-transmission function, i.e., the distortion-transmission function with variance 1.

The results are derived under an idealized system model. In particular, we assume that when the transmitter does transmit, it sends the complete state of the source; the channel is noiseless and does not introduce any delay. Relaxing these assumptions to analyze the effects of quantization, channel noise and delay are important future directions.

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