
# Remote-state estimation with packet drop

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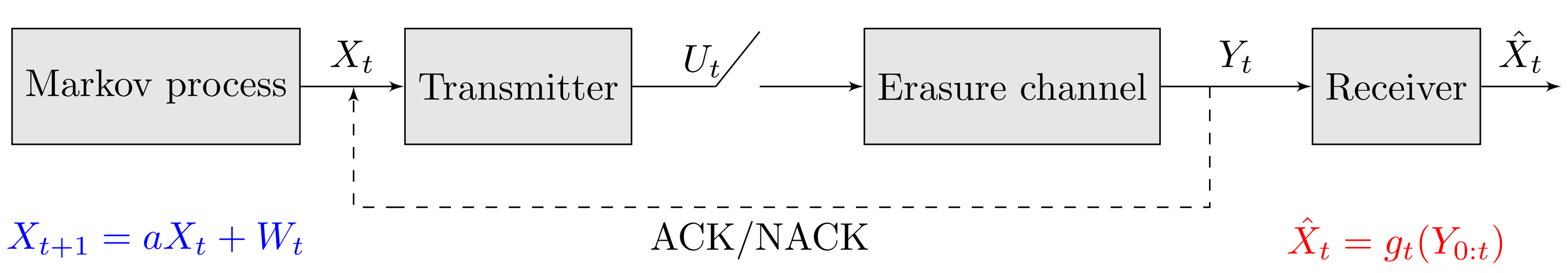
## Block diagram

$U_t = f_t(X_{0:t}, U_{0:t-1}), \in \{0,1\}$ $H_t \in \{\text{ON}(1\text{-}\varepsilon), \text{OFF}(\varepsilon)\}$




## Difficulty

Decentralized optimization problem; non-classical information structure

## Salient features

- **Real-time** transmission; **size** of data-packet **not important**; **sensoring is cheap** compared to transmission
- **Known prior** of packet-drop, **TCP**-like protocol

## Model

- $a, X_t, W_t \in \mathbb{Z}$; $W_t \sim$ unimodal and symmetric pmf
- $Y_t = U_t H_t X_t + (1 - U_t H_t)\mathfrak{E}$
- Per-step distortion: $d(X_t - \hat{X}_t)$; $d(0) = 0$, $d(x) = d(-x)$, $d(x) \le d(x+1)$ for any $x \in \mathbb{Z}_{\ge 0}$

## Performance metrics

$$D_\beta(f,g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty}\beta^t d(X_t - \hat{X}_t) \;\Big|\; X_0 = 0\Big]$$

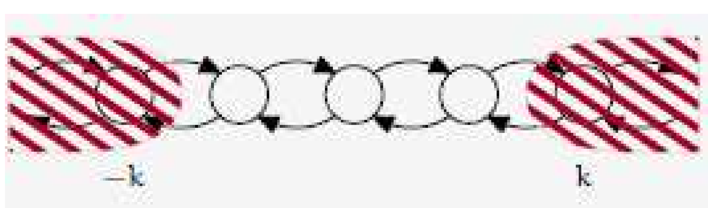
$$N_\beta(f,g) := (1-\beta)\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{\infty}\beta^t U_t \;\Big|\; X_0 = 0\Big].$$

## Optimization problems

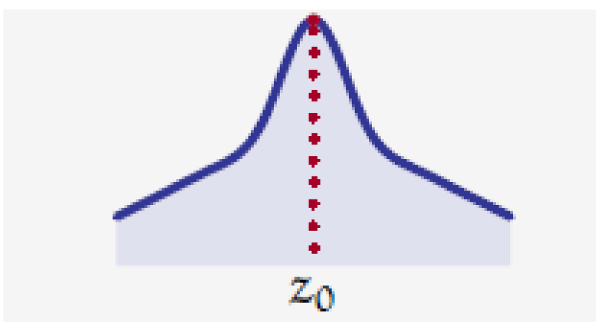
- **Costly communication**: $C^*_\beta(\lambda) := \arg\inf_{(f,g)} C_\beta(f,g;\lambda) = \arg\inf_{(f,g)} D_\beta(f,g) + \lambda N_\beta(f,g)$
- **Constrained communication**: $\alpha \in (0,1)$; $D^*_\beta(\alpha) := \arg\inf_{(f,g):N_\beta(f,g)\le\alpha} D_\beta(f,g)$

## Structure of optimal strategies

- $Z_t := \begin{cases} X_t, & \text{if } Y_t \ne \mathfrak{E}, \\ aZ_{t-1}, & \text{if } Y_t = \mathfrak{E} \end{cases}$, $Z_0 = 0$.
- **Estimation strategy**: $\hat{X}_t = g^*_t(Z_t) = \begin{cases} X_t, & \text{if } Y_t \ne \mathfrak{E} \\ aZ_{t-1}, & \text{if } Y_t = \mathfrak{E} \end{cases}$
- **Transmission strategy**: $E_t := X_t - a\hat{X}_{t-1}$. $U_t = f^*_t(E_t) = \begin{cases} 1, & \text{if } |E_t| \ge k \\ 0, & \text{if } |E_t| < k \end{cases}$




- **Majorization, ASU Schur concavity**
- Belief states, POMDP
- $\{E_t\}_{t\ge0}$ is a **regenerative process**



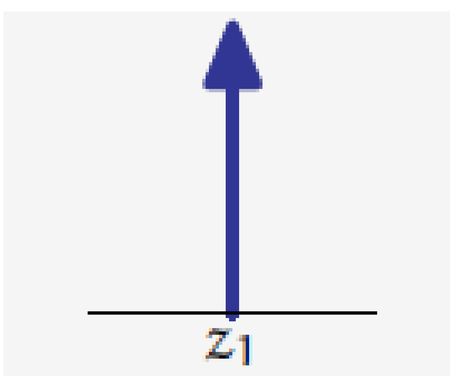




Figure: (a) $Y_1 \ne \mathfrak{E}$

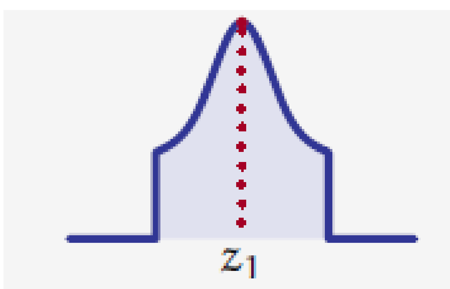



Figure: (b) $Y_1 = \mathfrak{E}$

## Performance of threshold-based strategy

- $L^{(k)}_\beta(e) := \begin{cases} \varepsilon\Big[d(e) + \beta\sum_{n\in\mathbb{Z}} p_{n-ae}L^{(k)}_\beta(n)\Big], & \text{if } |e| \ge k \\ d(e) + \beta\sum_{n\in\mathbb{Z}} p_{n-ae}L^{(k)}_\beta(n), & \text{if } |e| < k. \end{cases}$
- $M^{(k)}_\beta(e) := \begin{cases} \varepsilon\Big[1 + \beta\sum_{n\in\mathbb{Z}} p_{n-ae}M^{(k)}_\beta(n)\Big], & \text{if } |e| \ge k \\ 1 + \beta\sum_{n\in\mathbb{Z}} p_{n-ae}M^{(k)}_\beta(n), & \text{if } |e| < k. \end{cases}$

## Matrix formula

$$\begin{bmatrix} \vdots \\ L^{(2)}_\beta(-2) \\ L^{(2)}_\beta(-1) \\ L^{(2)}_\beta(0) \\ L^{(2)}_\beta(1) \\ L^{(2)}_\beta(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \varepsilon d(-2) \\ d(-1) \\ d(0) \\ d(1) \\ \varepsilon d(2) \\ \vdots \end{bmatrix} + \beta \begin{bmatrix} \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \cdots & \varepsilon p_1 & \varepsilon p_2 & \varepsilon p_3 & \varepsilon p_4 & \varepsilon p_5 & \cdots \\ \cdots & p_2 & p_1 & p_2 & p_3 & p_4 & \cdots \\ \cdots & p_3 & p_2 & p_1 & p_2 & p_3 & \cdots \\ \cdots & p_4 & p_3 & p_2 & p_1 & p_2 & \cdots \\ \cdots & \varepsilon p_5 & \varepsilon p_4 & \varepsilon p_3 & \varepsilon p_2 & \varepsilon p_1 & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ L^{(2)}_\beta(-2) \\ L^{(2)}_\beta(-1) \\ L^{(2)}_\beta(0) \\ L^{(2)}_\beta(1) \\ L^{(2)}_\beta(2) \\ \vdots \end{bmatrix}$$

- $h^{(k)} := \Big[\cdots\ \varepsilon\ \varepsilon\ \underbrace{1\ldots1}_{2k-1}\ \varepsilon\ \varepsilon\ \cdots\Big]^\intercal$; $h^{(k)} \odot P$ is substochastic
- $L^{(k)}_\beta = [I - \beta h^{(k)} \odot P]^{-1} h^{(k)} \odot d$, $\quad M^{(k)}_\beta = [I - \beta h^{(k)} \odot P]^{-1} h^{(k)}$

## Renewal relationships

- For $k \in \mathbb{Z}_{>0}$, $\beta \in (0,1)$,

$$D^{(k)}_\beta(0) := D_\beta(f^{(k)}, g^*) = \frac{L^{(k)}_\beta(0)}{M^{(k)}_\beta(0)},$$

$$N^{(k)}_\beta(0) := N_\beta(f^{(k)}, g^*) = \frac{1}{M^{(k)}_\beta(0)} - (1-\beta),$$

$$C^{(k)}_\beta(0;\lambda) := C_\beta(f^{(k)}, g^*;\lambda) = \frac{L^{(k)}_\beta(0) + \lambda}{M^{(k)}_\beta(0)} - \lambda(1-\beta).$$

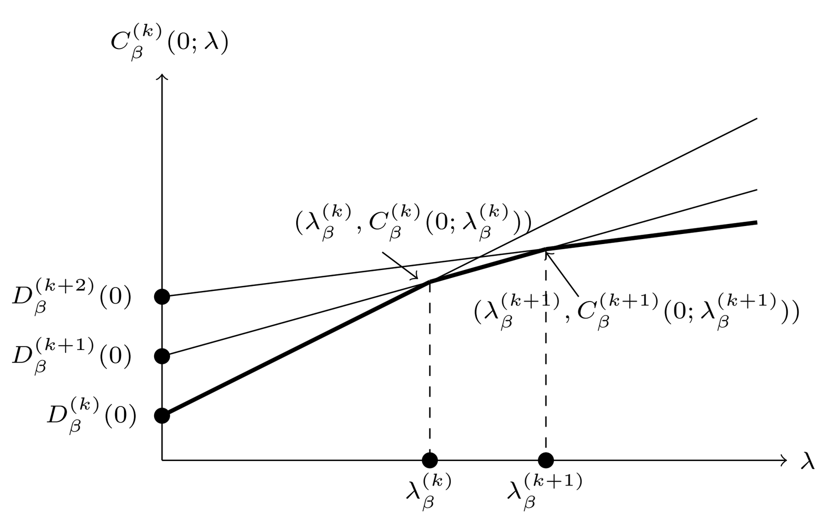
## Optimal threshold

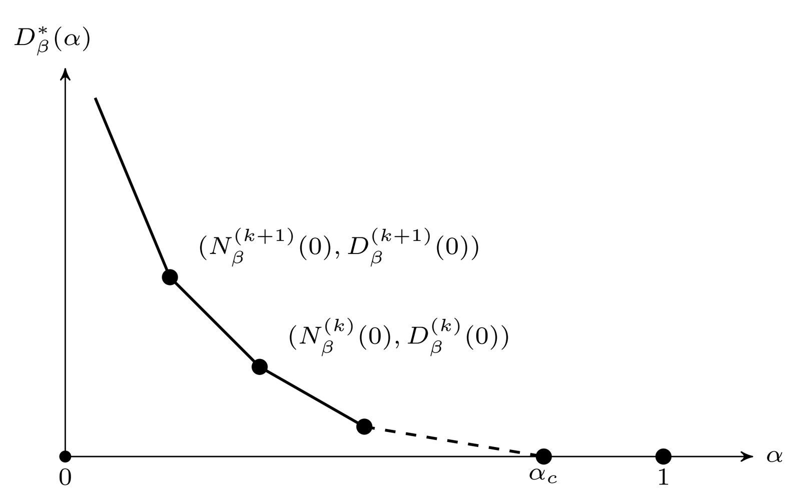
- **Costly**: For $\beta \in (0,1]$, $\mathbb{K} := \{k \in \mathbb{Z}_{\ge0} : D^{(k+1)}_\beta(0) > D^{(k)}_\beta(0)\}$. For $k_n \in \mathbb{K}$,

$$\lambda^{(k_n)}_\beta := \frac{D^{(k_{n+1})}_\beta(0) - D^{(k_n)}_\beta(0)}{N^{(k_n)}_\beta(0) - N^{(k_{n+1})}_\beta(0)}$$

  - For any $k_n \in \mathbb{K}$ and any $\lambda \in (\lambda^{(k_{n-1})}_\beta, \lambda^{(k_n)}_\beta]$, $f^* = f^{(k_n)}$
  - $C^*_\beta(\lambda)$: continuous, concave, increasing and piecewise linear in $\lambda$.
- **Constrained**: $k^* := \sup\Big\{k \in \mathbb{Z}_{\ge0} : M^{(k)}_\beta \le \frac{1}{1+\alpha-\beta}\Big\}$, $\theta^* := \frac{M^{(k^*+1)}_\beta - \frac{1}{1+\alpha-\beta}}{M^{(k^*+1)}_\beta - M^{(k^*)}_\beta}$. Then,
  - $f^*(e) = \begin{cases} 0, & \text{if } |e| < k^*; \\ 0, & \text{w.p. } 1-\theta^*, \text{ if } |e| = k^*; \\ 1, & \text{w.p. } \theta^*, \text{ if } |e| = k^*; \\ 1, & \text{if } |e| > k^*. \end{cases}$
  - $\alpha^{(k)} := N_\beta(f^{(k)}, g^*)$. Then, for $\alpha \in (\alpha^{(k+1)}, \alpha^{(k)})$, $k^* = k$ and $\theta^* = (\alpha - \alpha^{(k+1)})/(\alpha^{(k)} - \alpha^{(k+1)})$.
  - $D^*_\beta(\alpha) = \theta^* D^{(k^*)}_\beta + (1-\theta^*)D^{(k^*+1)}_\beta$: continuous, convex, decreasing and piecewise linear in $\alpha$

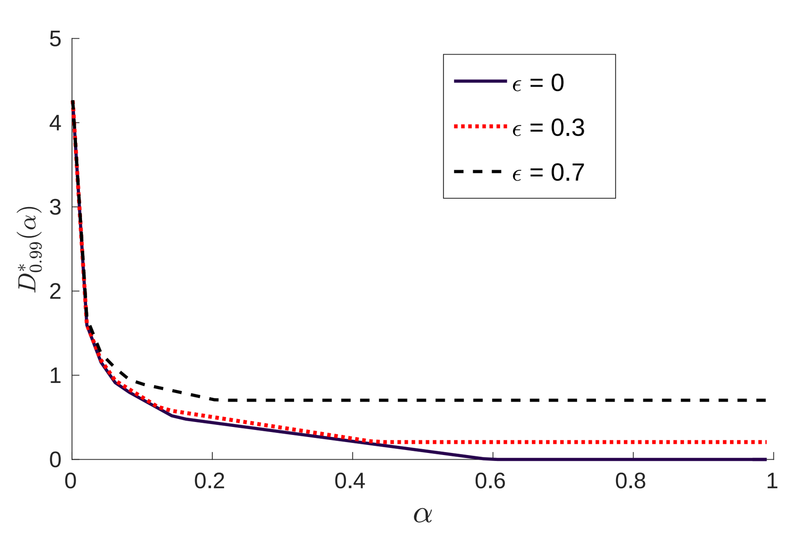
## Figures: Optimal performances

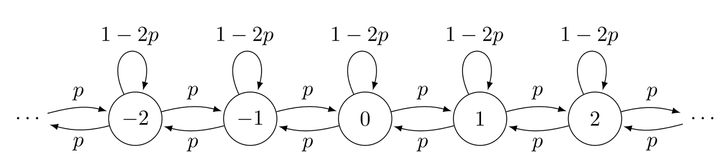



## Example: symmetric birth-death Markov chain

- $p = 0.3$, $\beta = 0.99$, $\varepsilon \in \{0, 0.3, 0.7\}$. **Constrained performance**

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