A Decision Theoretic Framework for Real-Time Communication

Aditya Mahajan    Demosthenis Teneketzis

Department of EECS,
University of Michigan,
Ann Arbor, MI 48105-2122, USA

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What is Real-Time Communication?

- Real-Time (zero or finite delay) encoding,
- Real-Time (zero or finite-delay) decoding.

Why consider Real-Time Communication?

Motivated by informationally decentralized system
- QoS (delay) requirements in communication networks,
- Sensor networks,
- Traffic flow control in transportation networks,
- Decentralized resource allocation (decentralized routing)
Literature Overview

Problem has received considerable attention in past.

- Zero-delay and finite-delay source coding.
- Causal Source coding.
- Performance bounds of systems with a real-time or finite-delay constraint.
- Zero-delay joint source channel coding.
- Real-time quantization of Markov sources (noiseless channel).
- Real-time encoding/decoding for noisy channels with noiseless feedback.
- Real-time encoding/decoding for noisy channels (no feedback)
Problem has received considerable attention in past.

Different approaches can be classified into two categories:
- Information Theoretic approach.
- Decision Theoretic approach

**Limitations of Standard Results of Information Theory**

- Fundamental Results of Information Theory are asymptotic.
- Based on encoding/decoding of typical sequences.
- Small delay schemes work only when the time horizon goes to infinity.
System Model

Source $X_t$ → Encoder $Z_t$ → Channel $Y_t$ → Memory $M_{t-1}$ → Decoder $\hat{X}_t$
• **Source** is first order Markov with known statistics.
- Source is first order Markov with known statistics.
- **Encoder** is real-time

\[ Z_t = c_t(X_1, X_2, \ldots, X_t) \]
Source is first order Markov with known statistics.
Encoder is real-time

\[ Z_t = c_t(X_1, X_2, \ldots, X_t) \]

Discrete Memoryless Channel, known statistics.

\[ \Pr(y_t \mid x^t, z^t) = \Pr(y_t \mid z_t) \]
Source is first order Markov with known statistics.
Encoder is real-time

\[ Z_t = c_t(X_1, X_2, \ldots, X_t) \]

Discrete Memoryless Channel, known statistics.

\[ \Pr(y_t | x^t, z^t) = \Pr(y_t | z_t) \]

Finite memory receiver.

\[ M_t = l_t(Y_t, M_{t-1}) \]
System Model

- Source is first order Markov with known statistics.
- Encoder is real-time

\[ Z_t = c_t(X_1, X_2, \ldots, X_t) \]

- Discrete Memoryless Channel, known statistics.

\[ \Pr(y_t \mid x^t, z^t) = \Pr(y_t \mid z_t) \]

- Finite memory receiver.

\[ M_t = l_t(Y_t, M_{t-1}) \]

- Real-time decoder.

\[ \hat{X}_t = g_t(Y_t, M_{t-1}) \]
System Performance

- Distortion measure

\[ \rho_t : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty) \]

- **Design:** Choice of \( c \triangleq (c_1, c_2, \ldots, c_T), \ g \triangleq (g_1, g_2, \ldots, g_T) \) and \( l \triangleq (l_1, l_2, \ldots, l_T) \).

- Performance measure

\[ \mathcal{J}(f, g, l) = \mathbb{E} \left\{ \sum_{t=1}^{T} \rho_t(X_t, \hat{X}_t) \right\} \]
Problem Formulation

Problem

Assume that both encoder and decoder know

- statistics of the source,
- statistics of the channel,
- and the time horizon $T$

choose an optimal design $(c^*, g^*, l^*)$ such that

$$J^* = J(c^*, g^*, l^*) = \min_{(c, g, l)} J(c, g, l)$$

Salient Features

- dynamic team problem
- non-classical information structure
- non-convex (in policy space) optimization problem
D. Teneketzis.

*On the structure of optimal real-time encoders and decoders in noisy communication.*


**Structure of Optimal Encoder**

Consider any fixed (but arbitrarily) $g \triangleq (g_1, \ldots, g_T)$ and $l \triangleq (l_1, \ldots, l_T)$. 
Structural Results

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**Structure of Optimal Encoder**

Consider any fixed (but arbitrarily) \( g \triangleq (g_1, \ldots, g_T) \) and \( l \triangleq (l_1, \ldots, l_T) \). then,

There is no loss of optimality in restricting attention to encoding rules of the form

\[
Z_t = c_t(X_t, P_{M_{t-1}}), \quad t = 2, 3, \ldots, T
\]

where,

\[
P_{M_t}(m) = \Pr(M_t = m \mid X^t, Z^t, c^t, l^{t-1})
\]
Structure of Optimal Decoder

Consider any fixed (but arbitrarily) $c \triangleq (c_1, \ldots, c_T)$ and $l \triangleq (l_1, \ldots, l_T)$,
Structural Results

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Structure of Optimal Decoder

Consider any fixed (but arbitrarily) $c \triangleq (c_1, \ldots, c_T)$ and $l \triangleq (l_1, \ldots, l_T)$, then

- Obtaining the optimal decoder is a filtering problem — At each $t$ obtain $g_t$ to minimize

$$J_t = \mathbb{E} \left\{ \rho_t(X_t, \hat{X}_t) \mid Y_t = y, M_{t-1} = m \right\}$$

- An optimal decoding rule $g^* \triangleq (g_1^*, g_2^*, \ldots, g_T^*)$ is given by

$$g_t^*(y_t, m_{t-1}) = \tau_t(\xi_t(y_t, m_{t-1}))$$

where

$$\xi_{t}^{f,l}(y, m)(x) = \Pr (X_t = x \mid Y_t = y, m_{t-1} = m)$$

and

$$\tau_t(\xi_t(y, m)) = \arg\min_a \sum_x \rho_t(x, a) \xi_t(y, m)(x)$$
Simplification of the Problem

Implication of Structural Results

- Without loss of optimality we can restrict attention to encoders of the form $Z_t = c_t(X_t, P_{M_{t-1}})$.
- Structure of optimal decoder depends only on the distortion measure and the conditional PMF $\xi_t$.
- $\xi_t$ depends on choice of $c^t, l^{t-1}$.
- $g^*_t = g^*_t(c^t, l^{t-1})$
- $g^* = g^*(c, l)$

Equivalent Problem

$$\min_{c, g, l} J(c, g, l) = \min_{c, l} J(c, g^*(c, l), l)$$
Information States

Properties of Information States

- Need to obtain information states for both agents sufficient for performance evaluation.
- Let $\pi_t$ and $\varphi_t$ be information states of encoder and memory update respt. They should satisfy
  
  \begin{align*}
  (S1a) & \quad \pi_t \text{ is a function of } x^t, c^{t-1} \text{ and } l^{t-1}. \\
  (S1b) & \quad \varphi_t \text{ is a function of } y_t, m_{t-1}, c^t \text{ and } l^{t-1}. \\
  (S2a) & \quad \varphi_t \text{ can be determined from } \pi_t \text{ and } c_t. \\
  (S2b) & \quad \pi_{t+1} \text{ can be determined from } \varphi_t \text{ and } l_t. \\
  (S3) & \quad \ldots 
  \end{align*}
Properties of Information States

\((S3)\) \(\pi_t\) absorbs the effect of \(c^{t-1}, l^{t-1}\) and \(\varphi_t\) absorbs the effect of \(c^t, l^{t-1}\) on expected future distortion, i.e.
Information States

Properties of Information States

(S3) \( \pi_t \) absorbs the effect of \( c^{t-1}, l^{t-1} \) and \( \varphi_t \) absorbs the effect of \( c^t, l^{t-1} \) on expected future distortion, i.e.

\[
\mathbb{E} \left\{ \sum_{s=t}^{T} \rho_s(X_s, \hat{X}_s) \bigg| c, g, l \right\} = \mathbb{E} \left\{ \sum_{s=t}^{T} \rho_s(X_s, \hat{X}_s) \bigg| \pi_t, c^T, l^T \right\} = \mathbb{E} \left\{ \sum_{s=t}^{T} \rho_s(X_s, \hat{X}_s) \bigg| \varphi_t, c_{t+1}^T, l^T \right\}
\]
Consider the random vectors

\[
P_{M_t}(m) = \Pr(M_t = m \mid X^t, Z^t, c^t, l^{t-1})
\]
\[
P_{Y_t,M_{t-1}}(y, m) = \Pr(Y_t = y, M_{t-1} = m \mid X^t, Z^t, c^t, l^{t-1})
\]

### Information States

\[
\pi_t = \Pr(X_t, P_{M_{t-1}}), \quad \text{(Info. state for Encoder)}
\]
\[
\varphi_t = \Pr(X_t, P_{Y_t,M_{t-1}}), \quad \text{(Info. state for Memory Update)}
\]
\( \pi_t \) and \( \varphi_t \) satisfy (S1)–(S3), i.e.

1. there is a linear transformation \( Q_t(c_t) \) such that

   \[ \varphi_t = Q_t(c_t)\pi_t \]

2. there is a linear transformation \( \hat{Q}_t(l_t) \) such that

   \[ \pi_{t+1} = \hat{Q}_t(l_t)\varphi_t \]

3. for any choice of \( c \) and \( l \), the expected conditional instantaneous cost can be expressed as

   \[ \mathbb{E}\left\{ \rho_t(X_t, \hat{X}_t) \, \bigg| \, c^t, g_t^*(c^t, l^{t-1}), l^{t-1} \right\} = \tilde{\rho}_t(\varphi_t) \]

where \( g_t^*(c^t, l^{t-1}) \) is an optimal decoding rule corresponding to \( c^t, l^{t-1} \) and \( \tilde{\rho}_t(\cdot) \) is a deterministic function.
Equivalent Deterministic Problem

- System Equations

\[ \varphi_t = Q_t(c_t)\pi_t, \quad t = 1, \ldots, T \]
\[ \pi_{t+1} = \hat{Q}_t(l_t)\varphi_t, \quad t = 1, \ldots, T - 1 \]

\(Q_t(\cdot)\) and \(\hat{Q}_t(\cdot)\) are deterministic transformations depending on \(c_t\) and \(l_t\) respt.

- Initial state \(\pi_1\) is known.

- Instantaneous cost \(\tilde{\rho}_t(\varphi_t)\).

- Optimization criterion

\[ \inf_{c,l} \sum_{t=1}^{T} \tilde{\rho}_t(\varphi_t) \]
\[ \hat{V}_T(\varphi) \equiv 0 \]

\[ V_t(\pi) = \inf_{c_t} \left[ \tilde{\rho}_t(Q_t(c_t)\pi) + \hat{V}_t(Q_t(c_t)\pi) \right], \quad t = 1, \ldots, T \]

\[ \hat{V}_t(\varphi) = \min_{l_t} \left[ V_{t+1}(\hat{Q}_t(l_t)\varphi) \right], \quad t = 1, \ldots, T - 1 \]
Nested Optimality Equations

\( \hat{V}_T(\varphi) \equiv 0 \)

\[ V_t(\pi) = \inf_{c_t} \left[ \tilde{\rho}_t(Q_t(c_t)\pi) + \hat{V}_t(Q_t(c_t)\pi) \right], \quad t = 1, \ldots, T \]

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\[ \hat{V}_t(\varphi) = \min_{l_t} \left[ V_{t+1}(\hat{Q}_t(l_t)\varphi) \right], \quad t = 1, \ldots, T - 1 \]
Time Homogenous Case

- source transition matrix
- channel
- noise statistics
- distortion measure

are time invariant. Then, the same methodology works for
- finite time horizon,
- infinite time horizon with an expected discounted distortion criterion.
Extensions

A. Mahajan and D. Teneketzis

*On jointly optimal encoding, decoding and memory update for noisy real-time communication*

Control Group Report CGR-05-07, Department of EECS, University of Michigan, Ann Arbor, MI.

- $k$-th order Markov source.
- Finite delay $\rho_t(X_{t-\delta}, \hat{X}_t)$.
- Channels with memory.
Summary

- Provide a decision theoretic framework to study real-time communication.
- Use the structural results of Teneketzis 2004, to obtain jointly optimal real-time encoding, decoding and memory update rules.
- Extend the methodology to infinite horizon problems.

Future Work

- Extend the methodology to multi-terminal systems.
- Performance bounds.
- Computational issues.