On the optimal thresholds in remote state estimation with communication costs

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Overview

Motivation

Applications

- Smart grid
- Environmental monitoring
- Sensor networks

Salient features

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical

The remote-state estimation setup

State process $X_t \in \mathbb{Z}$
Uncontrolled *symmetric* Markov process.

Transmitter $U_t = f_t(X_{1:t}, U_{1:t-1})$ and $Y_t = \begin{cases} X_t, & \text{if } U_t = 1; \\ \mathfrak{E}, & \text{if } U_t = 0, \end{cases}$

Receiver $\hat{X}_t = g_t(Y_{1:t})$
Distortion: $d(X_t - \hat{X}_t)$,
$d(-e) = d(e) \le d(e+1)$, $e \in \mathbb{Z}_{\ge 0}$

Communication strategies
Transmission strategy $f = \{f_t\}_{t=0}^{\infty}$
Estimation strategy $g = \{g_t\}_{t=0}^{\infty}$

The constrained optimization problem

$$D^*_\beta(\alpha) \coloneqq D_\beta(f^*, g^*) \coloneqq \inf_{(f,g):N_\beta(f,g)\le\alpha} D_\beta(f,g), \quad \beta \in (0,1]$$

Minimize expected distortion such that expected number of transmissions is less than α

1. Discounted setup

- $D_\beta(f,g) \coloneqq (1-\beta)\mathbb{E}^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t d(X_t - \hat{X}_t) \,\middle|\, X_0 = 0\right]$
- $N_\beta(f,g) \coloneqq (1-\beta)\mathbb{E}^{(f,g)}\left[\sum_{t=0}^{\infty} \beta^t U_t \,\middle|\, X_0 = 0\right]$

The constrained optimization problem

$$D_\beta^*(\alpha) := D_\beta(f^*, g^*) := \inf_{(f,g):N_\beta(f,g)\le\alpha} D_\beta(f,g), \quad \beta \in (0,1]$$

Minimize expected distortion such that expected number of transmissions is less than α

2. Long-term average setup

- $D_1(f,g) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \,\Big|\, X_0 = 0\Big]$
- $N_1(f,g) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{(f,g)}\Big[\sum_{t=0}^{T-1} U_t \,\Big|\, X_0 = 0\Big]$

Literature overview

- [Imer-Basar 2005 and 2010]
  Remote estimation problem with communication a finite number of times.
- [Lipsa-Martins 2009 and 2011], [Molin-Hirche 2009]
  Remote estimation with communication cost for finite horizon LQG setup.
- [Nayyar-Basar-Teneketzis-Veeravalli 2013]
  Remote estimation with communication cost for finite horizon Markov chain setup. Also considered energy harvesting at the transmitter.
- A long list of literature on event-driven communication

Key differences in our model

- Infinite horizon setup
- Constrained formulation

Salient features

Decentralized control

- Two decision makers – the transmitter and the receiver.
- (One-sided) nested information structure: the transmitter knows all the information available to the receiver.
- Non-classical information structure.

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Our contributions

- Identify qualitative properties of optimal strategies
- Identify a dynamic programming decomposition
- Determine optimal strategies in closed form based on the DP.

Assumptions on the model

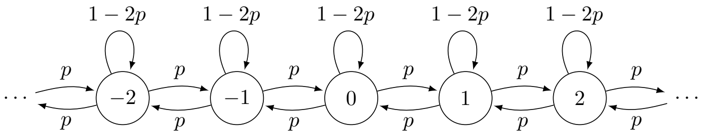
(A_0) $X_t \in \mathbb{Z}$, $X_0 = 0$

(A_1) Toeplitz transition matrix with decaying off-diagonal terms

$$\begin{bmatrix} \ddots & p_0 & \ddots & & & \\ \cdots & p_1 & p_0 & p_1 & \cdots & \\ & \ddots & p_1 & p_0 & p_1 & \cdots \\ & & \ddots & \ddots & p_0 & \ddots \end{bmatrix}, \text{ where where } p_0 \geq p_1 \geq \cdots$$

(A_2) The distortion function is even and increasing on $\mathbb{Z}_{\geq 0}$.
$\forall e \in \mathbb{Z}_{\geq 0}$: $d(e) = d(-e)$ and $d(e) \leq d(e+1)$.

An example: aperiodic, symmetric birth-death Markov chain



$$P_{ij} = \begin{cases} p, & \text{if } j = i+1, i-1 \\ 1-2p, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}, \quad p \in (0,1) \text{ and } d(e) = |e|.$$

Main results: structural of optimal strategies

Lagrange relaxation $C^*_\beta(\lambda) := \inf_{(f,g)} C_\beta(f,g;\lambda)$,
where $C_\beta(f,g;\lambda) = D_\beta(f,g) + \lambda N_\beta(f,g)$

Optimal estimation strategy Let Z_t be the most recently transmitted symbol.
$\hat{X}_t = g^*_t(Z_t) = Z_t$; Time homogeneous!

Optimal transmission strategy Let $E_t = X_t - Z_{t-1}$ be the error process and $f^{(k)}$ be the threshold based strategy such that

$$f^{(k)}(X_t, Y_{0:t-1}) = \begin{cases} 1, & \text{if } |E_t| \geq k \\ 0, & \text{if } |E_t| < k. \end{cases}$$

Main results: constrained optimization

Optimal strategy

The optimal transmission strategy is a possibly randomized strategy that, at each stage picks

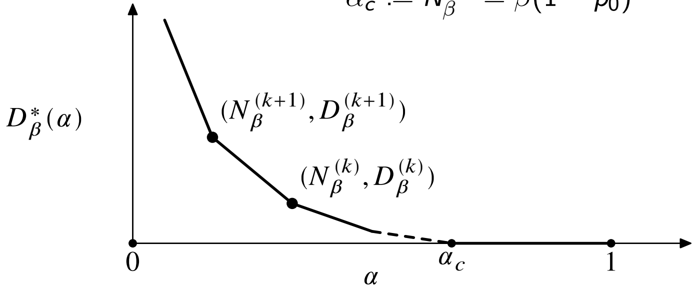
- $f^{(k^*)}$ w.p. θ^*
- $f^{(k^*+1)}$ w.p. $1 - \theta^*$

- Let $D_\beta^{(k)} = D_\beta(f^{(k)}, g^*)$, $N_\beta^{(k)} = N_\beta(f^{(k)}, g^*)$, then
  - k^*: Largest k such that $N_\beta^{(k)} \geq \alpha$
  - θ^*: Solution of $\theta^* N_\beta^{(k^*)} + (1 - \theta^*) N_\beta^{(k^*+1)} = \alpha$

Main results: constrained optimization

- Distortion-transmission trade-off:
  $D^*_\beta(\alpha) = \theta^* D_\beta^{(k^*)} + (1-\theta^*) D_\beta^{(k^*+1)}$

$\alpha_c := N_\beta^{(1)} = \beta(1-p_0)$



Proof outline

The structure of optimal transmitter and estimator follows from [Lipsa-Martins 2011] and [Nayyar-Basar-Teneketzis-Veeravalli 2013].

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Proof outline

Step 1 DP to identify best response transmitter.

- Key concern: the cost may be unbounded.

Step 2 Closed form expressions for $D^{(k)}_\beta = D_\beta(f^{(k)}, g^*)$ and $N^{(k)}_\beta = N_\beta(f^{(k)}, g^*)$.

Step 3 Identify

$$\Lambda(k) = \Big\{\lambda \ge 0 : C^*_\beta(\lambda) = C_\beta(f^{(k)}, g^*; \lambda)\Big\} = (\lambda^{(k)}_\beta, \lambda^{(k+1)}_\beta].$$

Step 4 Identify optimal randomized strategy for constrained setup.

Step 1: Main idea

Main idea

- Restrict attention to time-homogeneous estimation strategy $\hat{X}_t = g^*_t(Z_t) = Z_t$
- Consider the problem of finding the "best response" transmission strategy.
- Centralized stochastic control problem with countable state space and unbounded cost.
- Standard MDP results apply under mild technical assumptions.

Step 1: DP for discounted setup

Structure Under a standard technical assumption (A_3), the optimal transmission strategy is characterized by time-homogeneous threshold k, i.e.,

$$U_t = f(E_t) = \begin{cases} 1, & \text{if } |E_t| \geq k \\ 0, & \text{if } |E_t| < k. \end{cases}$$

Dynamic program For $\beta \in (0,1)$, the optimal strategy is determined by the unique fixed point of the following DP:

$$V_\beta(e;\lambda) = \min \Big\{ \lambda + \beta \textstyle\sum_{n=-\infty}^{\infty} P_{0n} V_\beta(n;\lambda), \text{ Transmit} \\ d(e) + \beta \textstyle\sum_{n=-\infty}^{\infty} P_{en} V_\beta(n;\lambda) \Big\} \text{ Don't transmit.}$$

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- Note that $C^*_\beta(\lambda) = V_\beta(0;\lambda)$.

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$V_\beta(\cdot;\lambda)$ satisfies SEN conditions. Therefore, the vanishing discount approach is applicable.

- Let $f_1^*(\cdot;\lambda)$ be any limit point of $f_\beta^*(\cdot;\lambda)$ as $\beta \uparrow 1$.
  Then the time-homogeneous transmission strategy $f_1^*(\cdot;\lambda)$ is optimal for $\beta = 1$ (the long-term average setup).
- Performance of optimal strategy:
  $$C_1^*(\lambda) \coloneqq C_1(f^*, g^*; \lambda) \coloneqq \inf_{(f,g)} C_1(f, g; \lambda)$$
  $$= \lim_{\beta \uparrow 1} V_\beta(0;\lambda) = \lim_{\beta \uparrow 1} C_\beta^*(\lambda)$$

Step 1: The SEN conditions

For any $\lambda \geq 0$, the value function $V_\beta(\cdot;\lambda)$ satisfies the SEN conditions:

SEN conditions

(S1) There exists a reference statee $e_0 \in \mathbb{Z}$ such that $V_\beta(e_0;\lambda) < \infty$ for all $\beta \in (0,1)$.

(S2) Define $h_\beta(e;\lambda) = (1-\beta)^{-1}[V_\beta(e;\lambda) - V_\beta(e_0;\lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e;\lambda) \leq K_\lambda(e)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

(S3) There exists a non-negative (finite) constant L_λ such that $-L_\lambda \leq h_\beta(e;\lambda)$ for all $e \in \mathbb{Z}$ and $\beta \in (0,1)$.

Step 2: Performance of threshold based strategies

Cost until first transmission

Let $S^{(k)} := \{e \in \mathbb{Z} : |e| \leq k-1\}$ and let $\tau^{(k)}$ be the stopping time when the Markov process starting at state 0 at time $t = 0$ escapes the set $S^{(k)}$. Then, for $\beta \in (0, 1)$,

$$L_\beta^{(k)} := \mathbb{E}\Big[\textstyle\sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \,\Big|\, E_0 = 0\Big]$$

$$M_\beta^{(k)} := \frac{1-\mathbb{E}[\beta^{\tau^{(k)}} \mid E_0=0]}{1-\beta}.$$

Step 2: Performance of threshold based strategies

Renewal relationship

$$D^{(k)}_\beta = \frac{L^{(k)}_\beta}{M^{(k)}_\beta}, \quad N^{(k)}_\beta = \frac{1}{M^{(k)}_\beta} - (1-\beta)$$

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Renewal relationship

$$D_\beta^{(k)} = \frac{L_\beta^{(k)}}{M_\beta^{(k)}}, \quad N_\beta^{(k)} = \frac{1}{M_\beta^{(k)}} - (1-\beta)$$

Vanishing discount relationships

$$L_1^{(k)} := \lim_{\beta \uparrow 1} L_\beta^{(k)}, \quad M_1^{(k)} := \lim_{\beta \uparrow 1} M_\beta^{(k)},$$

$$D_1^{(k)} := \lim_{\beta \uparrow 1} D_\beta^{(k)} = \frac{L_1^{(k)}}{M_1^{(k)}}$$

$$N_1^{(k)} := \lim_{\beta \uparrow 1} N_\beta^{(k)} = \frac{1}{M_1^{(k)}}$$

Step 2: Closed form expressions

Analytic expressions for performance Define

$$P_{ij}^{(k)} := P_{ij}, \quad i, j \in S^{(k)};$$
$$Q_{\beta}^{(k)} := [I_{2k-1} - \beta P^{(k)}]^{-1};$$
$$d^{(k)} := [d(-k+1), \cdots, d(k-1)]^{\intercal}.$$

Then

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Then

- $L^{(k)}_\beta = \left\langle [Q^{(k)}_\beta]_0, d^{(k)} \right\rangle$; $M^{(k)}_\beta = \left\langle [Q^{(k)}_\beta]_0, 1_{2k-1} \right\rangle$.
- $D^{(k)}_\beta$ and $N^{(k)}_\beta$ can be computed using these expressions.

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- $D^{(k)}_{\beta}$ and $N^{(k)}_{\beta}$ can be computed using these expressions.

Some inequalities $L^{(k)}_{\beta} < L^{(k+1)}_{\beta}$, $M^{(k)}_{\beta} < M^{(k+1)}_{\beta}$, $D^{(k)}_{\beta} < D^{(k+1)}_{\beta}$.

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Critical Lagrange multipliers

Let $\lambda_\beta^{(k)}$ be the value of the Lagrange multiplier for which, starting from state 0, one is indifferent between transmission strategies $f^{(k)}$ and $f^{(k+1)}$

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$$\lambda_\beta^{(k)} = \frac{D_\beta^{(k+1)} - D_\beta^{(k)}}{N_\beta^{(k)} - N_\beta^{(k+1)}}.$$

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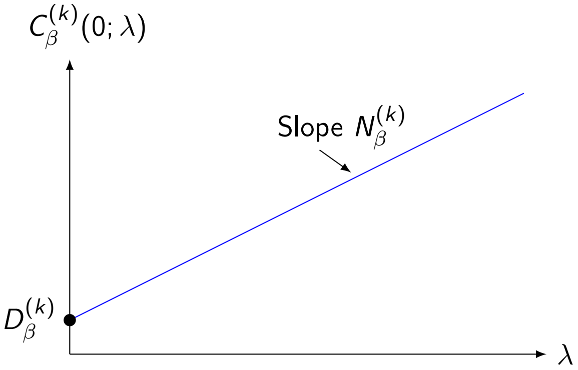

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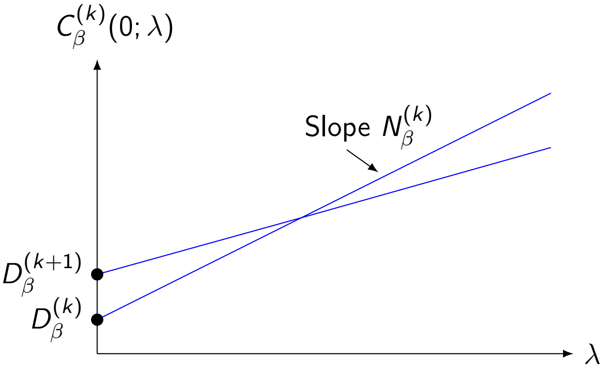

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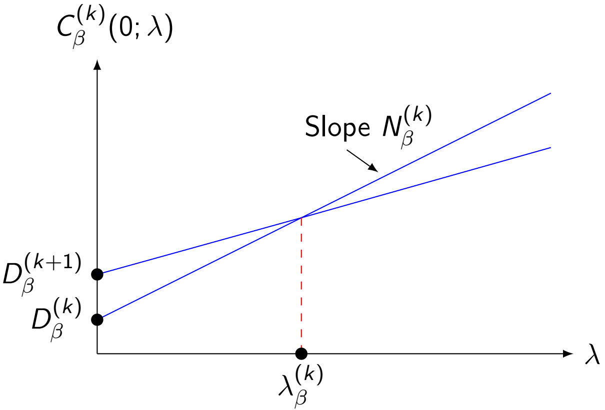

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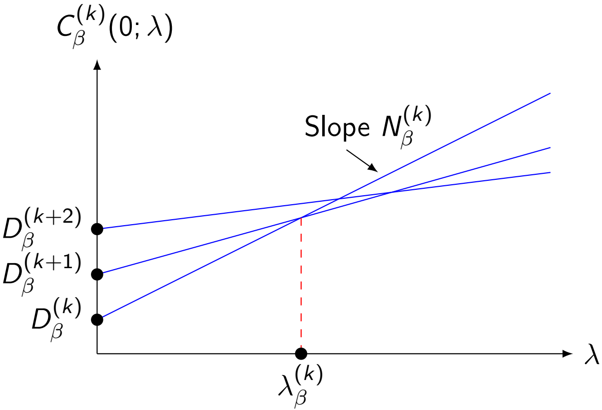

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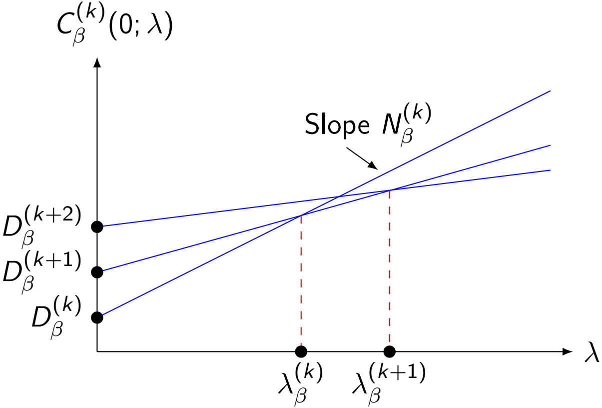

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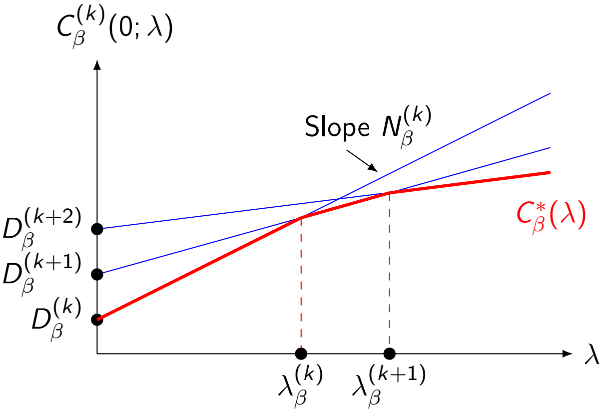

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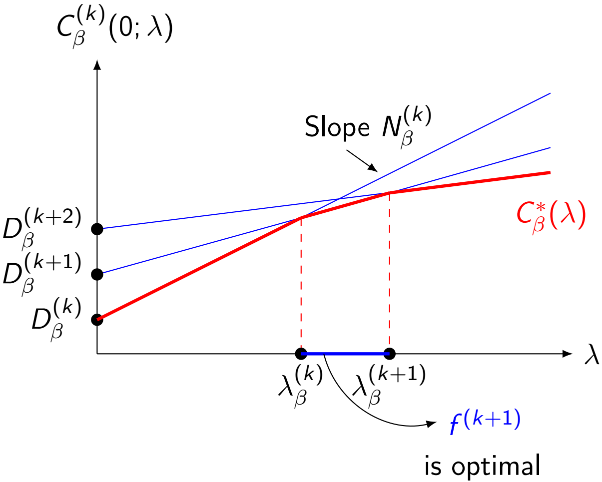

Step 4: The constrained setup

A (possibly randomized) strategy (f°, g°) is optimal for a constrained optimization problem with $\beta \in (0, 1]$, if

Sufficient conditions for optimality [Sennott, 1999]

(C1) $N_\beta(f^\circ, g^\circ) = \alpha$,

(C2) There exists a Lagrange multiplier $\lambda^\circ \geq 0$ such that (f°, g°) is optimal for $C_\beta(f, g; \lambda^\circ)$.

Step 4: The constrained setup

- Let k^* be largest k such that $N_\beta^{(k)} \geq \alpha$. Find k^* for a given α;
- Find θ^* such that $\theta^* N^{(k^*)} + (1-\theta^*)N^{(k^*+1)} = \alpha$;

Optimal randomized strategy $f^* = \theta^* f^{(k^*)} + (1-\theta^*) f^{(k^*+1)}$.

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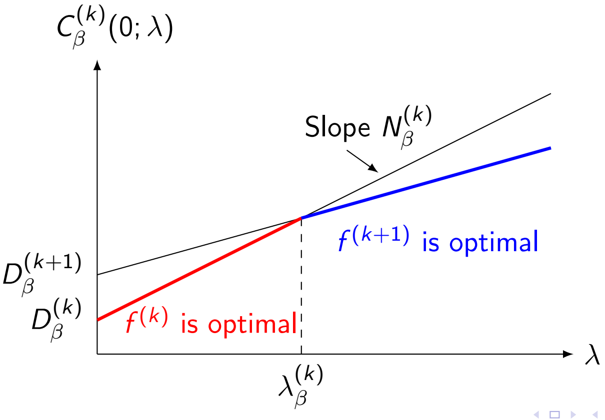

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- Randomization between two strategies that differ only at one state;
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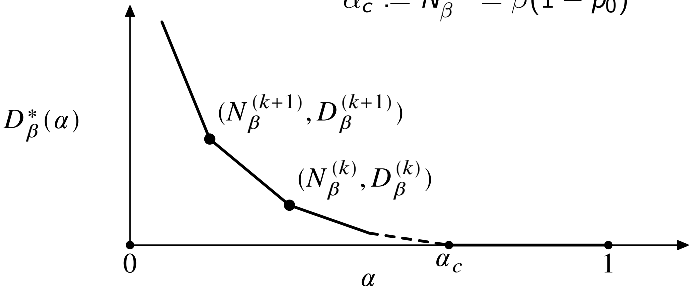
Randomized strategy

$$f^*(e) = \begin{cases} 0, & \text{if } |e| < k^*; \\ 0, & \text{w.p. } 1-\theta^*, \text{ if } |e| = k^*; \\ 1, & \text{w.p. } \theta^*, \text{ if } |e| = k^*; \\ 1, & \text{if } |e| > k^*. \end{cases}$$

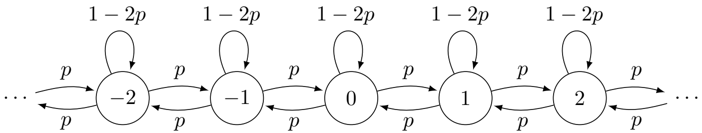
Distortion-transmission trade-off

$$\alpha_c := N_\beta^{(1)} = \beta(1 - p_0)$$



$D^*_\beta(\alpha)$ is piecewise linear, continuous, convex and decreasing in α.

Birth Death Markov Chain



$$P_{ij} = \begin{cases} p, & \text{if } j = i+1, i-1 \\ 1-2p, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}, \; p \in (0,1) \text{ and } d(e) = |e|.$$

Example: discounted setup

Define
$K_\beta = -2 - (1-\beta)/(\beta p)$ and $m_\beta = \cosh^{-1}(-K_\beta/2)$. Then

- $D_\beta^{(k)} = \frac{\sinh(km_\beta) - k\sinh(m_\beta)}{2\sinh^2(km_\beta/2)\sinh(m_\beta)}$;
- $N_\beta^{(k)} = \frac{2\beta p \sinh^2(m_\beta/2)\cosh(km_\beta)}{\sinh^2(km_\beta/2)} - (1-\beta)$.

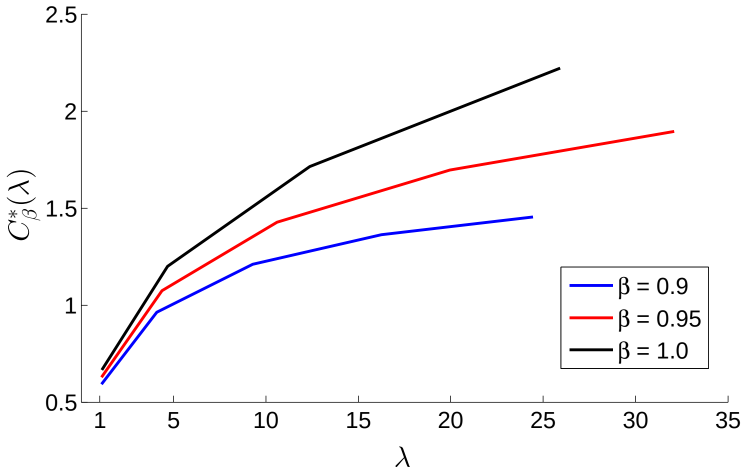
$$k_\beta^* = \sup\left\{k \in \mathbb{Z}_{\geq 0} : \frac{2\cosh(km_\beta)}{\cosh(km_\beta)-1} \geq \frac{1+\alpha-\beta}{\beta p(\cosh(m_\beta)-1)}\right\};$$

Example: long-term average setup

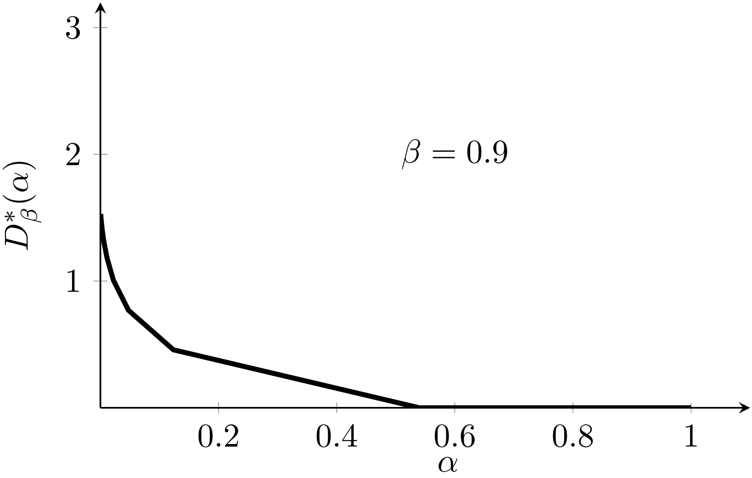
- $D_1^{(k)} = \frac{k^2-1}{3k}$, $N_1^{(k)} = \frac{2p}{k^2}$;
- $\lambda_1^{(k)} = \frac{k(k+1)(k^2+k+1)}{6p(2k+1)}$

$$k_1^* = \left\lfloor \sqrt{\frac{2p}{\alpha}} \right\rfloor;$$

Example: Lagrange cost for different β



Example: distortion-transmission trade-off



Summary and conclusion

Solution approach

- Remote state estimation of a Markov source under constraints on the number of transmissions.
- Investigated both discounted cost and long-term average cost infinite horizon setups.
- Modeled as a decentralized stochastic control problem with two decision maker.
- As long as the transmitter uses a symmetric threshold based strategy, the estimation strategy does not depend on the transmission strategy.
- The problem of finding the "best response" transmitter is a centralized stochastic control problem.

Summary and conclusion

Main results

- Simple Bernoulli randomized strategies $(f^{(k^*)}, f^{(k^*+1)}, \theta^*)$ are optimal
- k^* and θ^* can be computed easily.
- Characterized the distortion-transmission function.
- Closed form expressions of parameters for infinite horizon discounted cost setup.
- Used vanishing discount approach to compute the results for long-term average setup.
- Evaluated the performance for the constrained optimization for both infinite horizon discounted and long-term average setups.

Thank you !

http://arxiv.org/abs/1412.3199