Simultaneous real-time transmission of multiple Markov sources over a shared channel

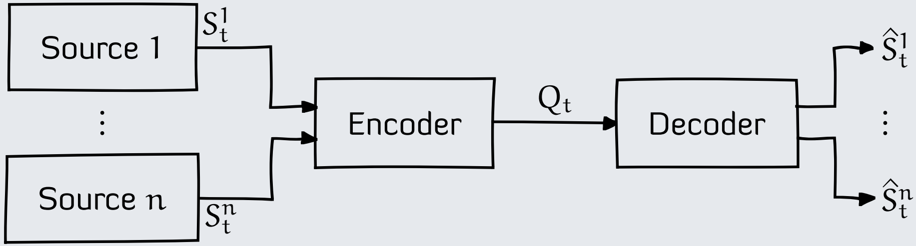
Mehnaz Mannan and Aditya Mahajan

McGill University

ISIT 2014

July 3, 2014

The communication system



Sources n independent Markov sources $\{S_t^i\}_{t=0}^{\infty}$, $i \in \{1, \dots, n\}$

Quantizer $Q_t = f_t(\mathbf{S}_{1:t}, Q_{1:t-1})$, $Q_t \in \mathcal{Q}$, where $\mathbf{S}_t = (S_t^1, \dots, S_t^n)$.

Receiver $\hat{\mathbf{S}}_t = (\hat{S}_t^1, \dots, \hat{S}_t^n) = g_t(Q_{1:t})$.

Separable distortion $d(\mathbf{S}_t, \hat{\mathbf{S}}_t) = \sum_{i=1}^{n} d^i(S_t^i, \hat{S}_t^i)$.

Objective Choose encoding-decoding strategy $(\mathbf{f}, \mathbf{g}) = (\{f_t\}_{t=1}^{\infty}, \{g_t\}_{t=1}^{\infty})$ to minimize

$$J_\beta(\mathbf{f}, \mathbf{g}) = \mathbb{E}^{(\mathbf{f},\mathbf{g})}\Big[\sum_{t=1}^{\infty} \beta^{t-1} d(\mathbf{S}_t, \hat{\mathbf{S}}_t) \,\Big|\, \mathbf{S}_0 = s_0\Big], \quad \text{where } \beta \in (0,1).$$

Literature overview

Witsenhausen 1979

No loss of optimality in using encoding strategies of the form

$$Q_t = f_t(S_t, Q_{1:t-1})$$

- Generalizes to higher order Markov sources and source coding with lookahead (i.e., finite decoding delay)

Walrand-Varaiya 1983

Define $\Pi_{t|t-1}$ and $\Pi_{t|t}$ as follows: for $\mathbf{s} = (s^1, \dots, s^n)$

$$\Pi_{t|t-1}(\mathbf{s}) = \mathbb{P}(\mathbf{S}_t = \mathbf{s} \mid Q_{1:t-1}); \quad \Pi_{t|t}(\mathbf{s}) = \mathbb{P}(\mathbf{S}_t = \mathbf{s} \mid Q_{1:t}).$$

Then, there is no loss of optimality in restricting attention to encoding and decoding strategies of the form

$$Q_t = f_t(\mathbf{S}_t, \Pi_{t|t-1}), \qquad \hat{\mathbf{S}}_t = g_t(\Pi_{t|t}).$$

- Linder-Yüksel 2013 showed that such results hold under quite general assumptions on the Markov source and distortion function.
- Similar result under some restrictive assumptions was also established by Borkar-Mitter-Tatikonda 2001.

- Witsenhausen, "On the structure of real-time source coders," BSTJ 1979.
- Walrand and Varaiya, "Optimal causal coding-decoding problems," IT 1983.

Literature overview (continued)

Generalizations

- Joint source-channel coding (Teneketzis 2006; M-Teneketzis 2009)
- Coding with side-information (Teneketzis 2006)
- Variable rate quantization (Kaspi-Merhav 2012)
- Finite lookahead (Asnana-Weissman 2013)
- Multi-terminal setups (Nayyar-Teneketzis 2011; Yüksel 2013).

Other ways to model real-time communication

- Zero-delay coding of individual sequences
- Causal coding and sequential coding
- Finite block length coding
- Zero-delay streaming
- . . .

The main idea of this paper

Comments on structural results

- Structural results identify time-homogeneous sufficient statistic of the data available at the transmitter and the receiver.
- Simplify implementation complexity.
- Identify dynamic program to search for optimal strategies.

- These results have been on limited use because of the inherent computational complexity in solving the resultant dynamic programs
- In the model presented above, the source is a collection of n sources. Thus solving the corresponding dynamic program will be an order of magnitude more difficult than than of a single Markov source.

The main idea of this paper

Comments on structural results

- Structural results identify time-homogeneous sufficient statistic of the data available at the transmitter and the receiver.
- Simplify implementation complexity.
- Identify dynamic program to search for optimal strategies.

- These results have been on limited use because of the inherent computational complexity in solving the resultant dynamic programs
- In the model presented above, the source is a collection of n sources. Thus solving the corresponding dynamic program will be an order of magnitude more difficult than than of a single Markov source.

Outline of the approach

- Simplify the problem by imposing assumptions on the structure of the encoding-decoding strategies.
- Under these assumptions, the problem reduces to a partially observable scheduling problem.

The main idea of this paper

Comments on structural results

- Structural results identify time-homogeneous sufficient statistic of the data available at the transmitter and the receiver.
- Simplify implementation complexity.
- Identify dynamic program to search for optimal strategies.

- These results have been on limited use because of the inherent computational complexity in solving the resultant dynamic programs
- In the model presented above, the source is a collection of n sources. Thus solving the corresponding dynamic program will be an order of magnitude more difficult than than of a single Markov source.

Outline of the approach

- Simplify the problem by imposing assumptions on the structure of the encoding-decoding strategies.
- Under these assumptions, the problem reduces to a partially observable scheduling problem.
- Convert the resultant POMDP to a countable state MDP.
- Find a sequence of approximating finite state dynamic programs that converge to the solution of countable state MDP.

Assumption A1: Separation of quantization and scheduling

Individual Walrand-Varaiya type strategies

For each source, a Walrand-Varaiya type strategy (for transmitting over alphabet $\mathcal{Q}$) has been specified.

- For every $\pi^i_{t|t-1} \in \Delta(\mathcal{S}^i)$, the encoding strategy prescribes the quantization symbol
$$q^i_t = f^i_t(s^i_t, \pi^i_{t|t-1})$$
- For every $\pi^i_{t|t} \in \Delta(\mathcal{S}^i)$, the decoding strategy prescribes the source reconstruction
$$\hat{s}^i_t = g^i_t(\pi^i_{t|t})$$

Scheduling strategies

- At each time, the encoder chooses an index $U_t \in \{1, \dots, n\}$ according to a scheduling strategy $\{h_t\}_{t=1}^{\infty}$
$$U_t = h_t(\mathbf{S}_t, \Pi_{t|t-1})$$
and transmits
$$Q_t = (U_t, f^{U_t}_t(S^{U_t}, , \Pi^{U_t}_{t|t-1}))$$
- The decoder updates $\Pi_{t|t-1}$ to $\Pi_{t|t}$ and generates
$$\hat{S}^i_t = g^i_t(\Pi^i_{t|t}), \quad \forall i.$$

Assumption A2: Oblivious posterior update

Update of posterior distribution

- In general, the evolution of the posterior distribution $\Pi_{t|t-1}$ to $\Pi_{t|t}$ is coupled with the scheduling strategy $\{h_t\}_{t=1}^{\infty}$.
- Hence, the dynamic program will be similar to that of decentralized stochastic control problems (each step will be a functional optimization problem of choosing h_t).

Oblivious posterior update

The transmitter and receiver keep track of marginal distributions

$$\boldsymbol{\Pi}_{t|t-1} = (\Pi^1_{t|t-1}, \dots, \Pi^n_{t|t-1}), \quad \boldsymbol{\Pi}_{t|t} = (\Pi^1_{t|t}, \dots, \Pi^n_{t|t}).$$

These are updated as follows:

$$\Pi^i_{t|t} = \begin{cases} \ell^i_t(\Pi^i_{t|t-1}, q^i_t), & \text{if } Q_t = (i, q^i_t) \\ \Pi^i_{t|t-1}, & \text{otherwise} \end{cases}$$

and

$$\Pi^i_{t+1|t} = \Pi^i_{t|t} P^i$$

Simplified problem and its solution

Problem formulation Given individual time-homogeneous Walrand-Varaiya-type strategies for all sources and assuming oblivious posterior update, find a scheduling strategy $\mathbf{h}$ to minimize

$$J_\beta(\mathbf{h}) = \mathbb{E}^{\mathbf{h}}\Big[\sum_{t=1}^{\infty} \beta^{t-1} d(\mathbf{S}_t, \hat{\mathbf{S}}_t) \,\Big|\, \mathbf{S}_0 = s_0\Big], \quad \text{where } \beta \in (0,1).$$

Simplified problem and its solution

Problem formulation Given individual time-homogeneous Walrand-Varaiya-type strategies for all sources and assuming oblivious posterior update, find a scheduling strategy $\mathbf{h}$ to minimize

$$J_\beta(\mathbf{h}) = \mathbb{E}^{\mathbf{h}}\Big[\sum_{t=1}^{\infty} \beta^{t-1} d(\mathbf{S}_t, \hat{\mathbf{S}}_t) \,\Big|\, \mathbf{S}_0 = s_0\Big], \quad \text{where } \beta \in (0,1).$$

Definition Let $D^i(\pi^i) = \sum_{s^i} d^i(s^i, g^i(\pi^i))\pi^i(s^i)$ denote the expected distortion for source i when posterior $\Pi^i_{t|t}$ is π^i.

Dynamic program Let $V\colon \prod_{i=1}^{n}(\mathcal{S}^i \times \Delta S^i) \to \mathbb{R}$ be the unique bounded fixed point of the following equation: for all $s^i \in \mathcal{S}^i$, $\pi^i \in \Delta(\mathcal{S}^i)$, $i \in \{1, \dots, n\}$

$$V(\mathbf{s}, \boldsymbol{\pi}) = \min_{u \in \{1,\dots,n\}} \Big\{ \sum_{i=1}^{n} D^i(\pi^i_-) + \beta \sum_{\mathbf{s}_+} \boldsymbol{\pi}_+(\mathbf{s}_+) V(\mathbf{s}_+, \boldsymbol{\pi}_+) \Big\}$$

where $\boldsymbol{\pi}_- = (\pi^1_-, \dots, \pi^n_-)$ and $\boldsymbol{\pi}_+ = (\pi^1_+, \dots, \pi^n_+)$.

Let $h^*(\mathbf{s}, \boldsymbol{\pi})$ denote (any of the) arg min of the above equation. Then, the time-homogeneous scheduling strategy $\mathbf{h}^* = (h^*, h^*, \dots)$ is optimal.

Comparison with DP for Walrand-Varaiya setup

- Unlike real-time quantization which is a decentralized stochastic control problem, the above optimal scheduling problem is a centralized stochastic control problem.
- The dynamic program is a standard infinite horizon POMDP and can be solved using standard computational algorithms for POMDPs (piecewise linear and concave approximations, point-based methods, etc.).
- In contrast, the dynamic program for real-time quantization is more complicated (each step is a functional optimization problem) and no efficient computational algorithms exist.

A special case

Assumption A3 The alphabet size of all sources are equal to the quantization alphabet, i.e., $|\mathcal{S}^i| = |\mathcal{Q}|$ for all i.

Optimal quantization

- Uncoded quantization is optimal, i.e.,

$$f^i(S^i_t, \Pi^i_{t|t-1}) = S^i_t$$

- Optimal decoding is the solution to a filtering problem

$$g^i(\Pi^i_{t|t}) = \arg\min_{\hat{s}} \sum_{s} d^i(s, \hat{s}) \Pi^i_{t|t}(s).$$

Dynamic program

$$V(\mathbf{s}, \boldsymbol{\pi}) = \min_{u \in \{1,\dots,n\}} \left\{ \sum_{i \neq u} D^i(\pi^i) + \beta \sum_{\mathbf{s}_+} \boldsymbol{\pi}_+(\mathbf{s}_+) V(\mathbf{s}_+, \boldsymbol{\pi}_+) \right\}$$

where

$$\pi^i_+ = \begin{cases} \delta_{s^i} P^i, & \text{if } u = i \\ \pi^i P^i, & \text{otherwise.} \end{cases}$$

Simplification of the special case

Reachability analysis

Under any scheduling strategy the reachable set of $\boldsymbol{\Pi}_{t|t-1}$ is $\prod_{i=1}^{n} \mathcal{R}^i$ where

$$\mathcal{R}^i = \{\delta_z (P^i)^k \in \Delta(\mathcal{S}^i) : z \in \mathcal{S}^i \text{ and } k \in \mathbb{Z}_{>0}\}$$

Simplification of the special case

Reachability analysis

Under any scheduling strategy the reachable set of $\mathbf{\Pi}_{t|t-1}$ is $\prod_{i=1}^{n} \mathcal{R}^i$ where

$$\mathcal{R}^i = \{\delta_z (P^i)^k \in \Delta(\mathcal{S}^i) : z \in \mathcal{S}^i \text{ and } k \in \mathbb{Z}_{>0}\}$$

Countable state DP

Let $\hat{V}$ be the unique bounded fixed point of the following:
For any $s^i, z^i \in \mathcal{S}^i$ and $k^i \in \mathbb{Z}_{>0}$

$$\hat{V}(s^1, s^2, z^1, k^1, z^2, k^2) = \min \left\{ \hat{W}^1(s^1, z^2, k^2),\ \hat{W}^2(s^2, z^1, k^1) \right\}$$

where $W^i(\cdot, \cdot, \cdot)$ are defined appropriately.

- Let $\hat{h}^*(s^1, s^2, z^1, k^1, z^2, k^2)$ denote the arg min of the right hand side.
- For any $s^1 \in \mathcal{S}^i$ and $\pi^i = \delta_{z^i}(P^i)^{k^i} \in \mathcal{R}^i$, define

$$h^*(s^1, s^2, \pi^1, \pi^2) = \hat{h}^*(s^1, s^2, z^1, k^1, z^2, k^2).$$

Then, the stationary strategy $\mathbf{h} = (h^*, h^*, \dots)$ is optimal.

Finite state approximation

Finite state DP

Let $\mathbb{Z}_m$ denote the set $\{1, \dots, m\}$. Let $\hat{V}_m$ be the unique bounded fixed point of the following: For any $s^i, z^i \in \mathcal{S}^i$ and $k^i \in \mathbb{Z}_m$

$$\hat{V}_m(s^1, s^2, z^1, k^1, z^2, k^2) = \min\left\{\hat{W}^1_m(s^1, z^2, k^2),\ \hat{W}^2_m(s^2, z^1, k^1)\right\}$$

where $W^i_m(\cdot, \cdot, \cdot)$ are defined appropriately (see paper for details).

Let $\hat{h}^*_m$ be the corresponding optimal strategy.

Finite state approximation

Finite state DP Let $\mathbb{Z}_m$ denote the set $\{1, \dots, m\}$. Let $\hat{V}_m$ be the unique bounded fixed point of the following: For any $s^i, z^i \in \mathcal{S}^i$ and $k^i \in \mathbb{Z}_m$

$$\hat{V}_m(s^1, s^2, z^1, k^1, z^2, k^2) = \min\left\{\hat{W}^1_m(s^1, z^2, k^2),\ \hat{W}^2_m(s^2, z^1, k^1)\right\}$$

where $W^i_m(\cdot, \cdot, \cdot)$ are defined appropriately (see paper for details).

Let $\hat{h}^*_m$ be the corresponding optimal strategy.

Theorem

- $\lim_{m\to\infty} \hat{V}_m = \hat{V}$.
- Any limit point of sequence $\{\hat{h}^*_m\}_{m=1}^{\infty}$ is an optimal scheduling strategy.

Proof idea Show that the above finite state model forms an augmentation type approximation sequence (Sennott 1999)

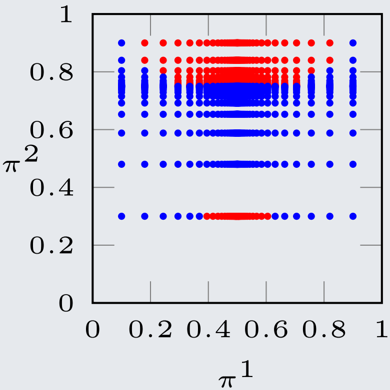
Numerical example

Setup

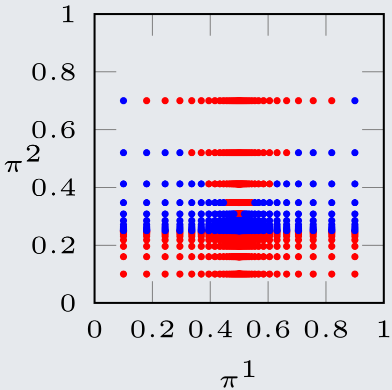
- 2 binary sources
- Hamming distortion
- Discount $\beta = 0.9$
- $P^1 = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$ and $P^2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.7 & 0.3 \end{bmatrix}$.

Optimal strategy

- Value functions converge at $m = 30$. Plot of $h^*_{30}(s^1, s^2, \pi^1, \pi^2)$:



for $s^1 \in \{0, 1\}$ and $s^2 = 0$



for $s^1 \in \{0, 1\}$ and $s^2 = 1$

Conclusion

Summary

- Investigate simultaneous real-time transmimssion of multiple sources over a shared channel.
- Derive a dynamic program under two simplifying assumptions:
  (A1) Separation of quantization and scheduling.
  (A2) Oblivious update of posterior distributions.

Thoughts for future work

- Characterize the degree of sub-optimality due to (A1) and (A2).
- Identify other models where the DP can be solved efficiently.

- Identify the structure of optimal scheduling strategies.
- Relation with bandit problems and Gittins index.

- Generalization to multi-terminal setup.