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**Distortion-transmission trade-off in real-time transmission of Gauss-Markov sources**

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Motivation

- Sequential transmission of data
- Zero delay in reconstruction
Motivation

- Sequential transmission of data
- Zero delay in reconstruction

Applications

- Smart grids
- Environmental monitoring
- Sensor networks

- Sensing is cheap
- Transmission is expensive
- Size of data-packet is not critical
The remote-state estimation setup

Source process \( X_{t+1} = X_t + W_t, \ W_t \sim \mathcal{N}(0, \sigma^2), \ i.i.d. \)

Uncontrolled Gauss-Markov process.

Transmitter \( U_t = f_t(X_1:t, U_1:t-1) \) and \( Y_t = \begin{cases} X_t, & \text{if } U_t = 1; \\ \mathcal{E}, & \text{if } U_t = 0, \end{cases} \)

Receiver \( \hat{X}_t = g_t(Y_1:t) \)
Distortion: \((X_t - \hat{X}_t)^2\)

Communication
Transmission strategy \( f = \{f_t\}_{t=0}^\infty \)
Estimation strategy \( g = \{g_t\}_{t=0}^\infty \)
The optimization problem

- \( D(f, g) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^{(f,g)} \left[ \sum_{t=0}^{T-1} d(X_t - \hat{X}_t) \mid X_0 = 0 \right] \)

- \( N(f, g) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^{(f,g)} \left[ \sum_{t=0}^{T-1} U_t \mid X_0 = 0 \right] \)
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The Distortion-Transmission function

\[ D^*(\alpha) := D(f^*, g^*) := \inf_{(f, g): N(f, g) \leq \alpha} D(f, g) \]

Minimize expected distortion such that expected number of transmissions is less than \( \alpha \)
Costly communication: analysis of optimal performance

- **Estimation with measurement cost**: estimator decides whether the sensor should transmit - Athans, 1972; Geromel, 1989; Wu et al, 2008.

- **Sensor sleep scheduling**: sensor is allowed to sleep for a pre-specified amount of time - Shuman and Liu, 2006; Sarkar and Cruz, 2004, 2005; Federgruen and So, 1991.

- **Censoring sensors**: sequential hypothesis testing setup; sensor decides whether to transmit or not - Rago et al, 1996; Appadwedula et al, 2008.
Literature overview

Remote state estimation: focus on structure of optimal strategies

- **Gauss-Markov source with finite number of transmissions** - Imer and Basar, 2005.


- **Countable Markov source with costly communication (finite horizon)** - Nayyar et al, 2013.
Remote state estimation: focus on structure of optimal strategies

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Gauss-Markov source; infinite horizon setup; constrained optimization.
Main result: the Distortion-Transmission function

Variance: \( \sigma^2 = 1 \)
Main result: the Distortion-Transmission function

How to compute $D^*(\alpha)$ for a given $\alpha \in (0, 1)$?
Main result: the Distortion-Transmission function

How to compute $D^*(\alpha)$ for a given $\alpha \in (0, 1)$?

- Find $k^*(\alpha) \in \mathbb{R}_{\geq 0}$ such that $M^{(k^*(\alpha))}(0) = 1/\alpha$, where
  $$M^{(k)}(e) = 1 + \int_{-k}^{k} \phi(w - e)M^{(k)}(w)dw.$$  

- Compute $L^{(k^*(\alpha))}(0)$ where
  $$L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w - e)L^{(k)}(w)dw.$$  

- $D^*(\alpha) = L^{(k^*(\alpha))}(0) / M^{(k^*(\alpha))}(0).$

- **Scaling** of distortion-transmission function with variance.
  $$D^*_\sigma(\alpha) = \sigma^2 D^*_1(\alpha).$$
## An illustration

**Comparison with periodic strategy**
Source process $X_t$
An illustration

$\alpha = 1/6$, Periodic strategy

Source process $X_t$

Error process $E_t$

Distortion = 2.083
An illustration

\[ \alpha = \frac{1}{6}, \text{ Threshold strategy; Threshold}=2 \]

Source process \( X_t \)

Error process \( E_t \)

Distortion = 1.5
An illustration

- **Threshold strategy**
- **Periodic strategy**

The system

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Graph showing the relationship between distortion and the parameter \( \alpha \).
Proof outline

We don’t proceed in the usual way to find the achievable scheme and a converse! Instead,
We don’t proceed in the usual way to find the achievable scheme and a converse! Instead,

- Identify structure of optimal strategies.
- Find the best strategy with that structure.
Lagrange relaxation

\[ C^*(\lambda) := \inf_{(f,g)} C(f, g; \lambda), \]

where \( C(f, g; \lambda) = D(f, g) + \lambda N(f, g), \lambda \geq 0. \)
Structure of optimal strategies

The structure of optimal transmitter and estimator follows from [Lipsa-Martins 2011] and [Nayyar-Basar-Teneketzis-Veeravalli 2013].

Finite horizon setup; results for Lagrange relaxation

Optimal estimation  Let $Z_t$ be the most recently transmitted symbol.
strategy  $\hat{X}_t = g_t^*(Z_t) = Z_t$; Time homogeneous!

Optimal transmission  Let $E_t = X_t - Z_{t-1}$ be the error process and
strategy  $f_t$ be the threshold based strategy such that

$$f_t(X_t, Y_{0:t-1}) = \begin{cases} 1, & \text{if } |E_t| \geq k_t \\ 0, & \text{if } |E_t| < k_t. \end{cases}$$
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Finite horizon setup; results for Lagrange relaxation

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\[
  f_t(X_t, Y_{0:t-1}) = \begin{cases} 
  1, & \text{if } |E_t| \geq k_t \\
  0, & \text{if } |E_t| < k_t.
  \end{cases}
\]

We prove that the results generalize to infinite horizon setup; the optimal thresholds are time - homogeneous.
Performance of threshold based strategies

Fix a threshold based strategy $f^{(k)}$. Define

- $D^{(k)}$: the expected distortion.
- $N^{(k)}$: the expected number of transmissions.
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$\{E_t\}_{t=0}^{\infty}$ is regenerative process.
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$\{E_t\}_{t=0}^\infty$ is regenerative process.

$\tau^{(k)}$: stopping time when the Gauss-Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{R} : |e| \geq k\}$.
Performance of threshold based strategies

Fix a threshold based strategy $f^{(k)}$. Define

- $D^{(k)}$: the expected distortion.
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$$\{E_t\}_{t=0}^{\infty}$$ is regenerative process.

- $L^{(k)}(e)$: the expected distortion until the first transmission, starting from state $e$.
- $M^{(k)}(e)$: the expected time until the first transmission, starting from state $e$. 
Performance of threshold based strategies

Fix a threshold based strategy $f^{(k)}$. Define

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- $L^{(k)}(e)$: the expected distortion until the first transmission, starting from state $e$.
- $M^{(k)}(e)$: the expected time until the first transmission, starting from state $e$.

**Renewal relationship**

$$D^{(k)} = \frac{L^{(k)}(0)}{M^{(k)}(0)}, \quad N^{(k)} = \frac{1}{M^{(k)}(0)}$$
Performance of threshold based strategies

\[ L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w - e)L^{(k)}(w)dw; \]
\[ M^{(k)}(e) = 1 + \int_{-k}^{k} \phi(w - e)M^{(k)}(w)dw. \]

- Derived using balance equations.
- Solutions of Fredholm Integral Equations of second kind.
Performance of threshold based strategies

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- Derived using balance equations.
- Solutions of Fredholm Integral Equations of second kind.

**Contraction.** Use Banach fixed point theorem to show that
- Fredholm Integral Equations have a solution.
- the solution is unique.
Performance of threshold based strategies

\[ L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w - e)L^{(k)}(w)dw; \]
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- Derived using balance equations.
- Solutions of Fredholm Integral Equations of second kind.

Computation

- Well-studied numerical methods.
- Examples - use the resolvent kernel of the integral equation - the Liouville-Neumann series; use quadrature method to discretize the integral.
Main theorem

Properties

- $L^{(k)}$, $M^{(k)}$, $D^{(k)}$ and $N^{(k)}$ are continuous, differentiable in $k$.
- $L^{(k)}$, $M^{(k)}$ and $D^{(k)}$ monotonically increasing in $k$.
- $N^{(k)}$ is strictly monotonically decreasing in $k$. 
Main theorem

Properties

- $L^{(k)}, M^{(k)}, D^{(k)}$ and $N^{(k)}$ are continuous, differentiable in $k$.
- $L^{(k)}, M^{(k)}$ and $D^{(k)}$ monotonically increasing in $k$.
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Theorem

- For any $\alpha \in (0, 1)$, $\exists k^*(\alpha) : N^{(k^*(\alpha))} = \alpha$.
- If the pair $(\lambda, k), \lambda, k \in \mathbb{R}_{\geq 0}$, satisfies $\lambda = -\frac{\partial_k D^{(k)}}{\partial_k N^{(k)}}$, then $C^*(\lambda) = C(f^{(k)}, g^*; \lambda)$.
- $D^*(\alpha) = D^{(k^*(\alpha))}$. 
Scaling with variance

\[ L^{(k)}_\sigma(e) = \sigma^2 L^{(k/\sigma)}_1 \left( \frac{e}{\sigma} \right), \quad M^{(k)}_\sigma(e) = M^{(k/\sigma)}_1 \left( \frac{e}{\sigma} \right), \]
Scaling with variance

\[ L_\sigma^{(k)}(e) = \sigma^2 L_1^{(k/\sigma)} \left( \frac{e}{\sigma} \right), \quad M_\sigma^{(k)}(e) = M_1^{(k/\sigma)} \left( \frac{e}{\sigma} \right), \]

Scaling: distortion-transmission function

\[ D_\sigma^*(\alpha) = \sigma^2 D_1^*(\alpha). \]
## Summary

- Remote state estimation of a Gauss-Markov source under constraints on the number of transmissions.
- Computable expression for distortion-transmission function.
- Simple threshold based strategies are optimal!
Summary

Countable-state Markov chain setup

- Similar results hold - Kalman-like estimator is optimal.
- Randomized threshold based transmission strategy is optimal.
- Distortion-transmission function is piecewise linear, decreasing, convex.

\[ D^*(\alpha) \]

\[ (N^{(k+1)}(0), D^{(k+1)}(0)) \]

\[ (N^{(k)}(0), D^{(k)}(0)) \]
Summary

Countable-state Markov chain setup

- Similar results hold - Kalman-like estimator is optimal.
- Randomized threshold based transmission strategy is optimal.
- Distortion-transmission function is piecewise linear, decreasing, convex.

Future directions

- The results are derived under an idealized system model.
- When the transmitter does transmit, it sends the complete state of the source.
- The channel is noiseless and does not introduce any delay.
Future directions

- The results are derived under an *idealized* system model.
- When the transmitter *does transmit*, it sends the *complete state* of the source.
- The channel is *noiseless* and does *not* introduce any *delay*.

**Future directions**

- Effects of quantization, channel noise and delay.
Future directions

- The results are derived under an idealized system model.
- When the transmitter does transmit, it sends the complete state of the source.
- The channel is noiseless and does not introduce any delay.

Future directions

- Effects of quantization, channel noise and delay.

Some parameters

Let \( \tau^{(k)} \) be the stopping time of first transmission (starting from \( E_0 = 0 \)), under \( f^{(k)} \). Then

\[
L^{(k)}_\beta(e) = (1 - \beta) \mathbb{E} \left[ \sum_{t=0}^{\tau^{(k)}-1} \beta^t d(E_t) \mid E_0 = 0 \right].
\]

\[
M^{(k)}_\beta(e) = (1 - \beta) \mathbb{E} \left[ \sum_{t=0}^{\tau^{(k)}-1} \beta^t \mid E_0 = 0 \right].
\]

Regenerative process: The process \( \{X_t\}_{t=0}^\infty \), if there exist \( 0 \leq T_0 < T_1 < T_2 < \cdots \) such that \( \{X_t\}_{t=T_k+s}^\infty, s \geq 0 \),

- has the same distribution as \( \{X_t\}_{t=T_0+s}^T \),
- is independent of \( \{X_t\}_{t=0}^{T_k} \).
Step 1: Main idea

Proof technique followed after Lerma, Lasserre - Discrete-time Markov control processes: basic optimality criteria, Springer

- The model satisfies certain assumptions (4.2.1, 4.2.2)
- Hence, the structural results extend to the infinite horizon discounted cost setup (Theorem 4.2.3)
- The discounted model satisfies some more assumptions (4.2.1, 5.4.1)
- Hence, structural results extend to long-term average setup (Theorem 5.4.3)
Assumption 4.2.1 - The one-stage cost is l.s.c, non-negative and inf-compact on the set of feasible state-action pairs. The stochastic kernel \( \phi \) is strongly continuous.

Assumption 4.2.2 - There exists a strategy \( \pi \) such that the value function \( V(\pi, x) < \infty \) for each state \( x \in X \).

Theorem 4.2.3 - Suppose Assumptions 4.2.1 and 4.2.2 hold. Then, in the discounted setup, there exists a selector which attains the minimum \( V_\beta^* \) and the optimal strategy, if it exists, is deterministic stationary.

Assumption 5.4.1 - There exists a state \( z \in X \) and scalars \( \alpha \in (0, 1) \) and \( M \geq 0 \) such that

1. \( (1 - \beta) V_\beta^*(z) \leq M, \forall \beta \in [\alpha, 1) \).
2. Let \( h_\beta(x) := V_\beta(x) - V_\beta(z) \). There exists \( N \geq 0 \) and a non-negative (not necessarily measurable) function \( b(\cdot) \) on \( X \) such that \(-N \leq h_\beta(x) \leq b(x), \forall x \in X \) and \( \beta \in [\alpha, 1) \).
Theorem 5.4.3 - Suppose that Assumption 4.2.1 holds. Then the optimal strategy for average cost setup is deterministic stationary and is obtained by taking limit $\beta \uparrow 1$. The vanishing discount method is applicable and is employed to compute the optimal performance.
Step 1: Optimal threshold-type transmitter strategy for long-term average setup

The DP satisfies some suitable conditions so that, the vanishing discount approach is applicable.
Step 1: Optimal threshold-type transmitter strategy for long-term average setup

The DP satisfies some suitable conditions so that, the vanishing discount approach is applicable.

- For discounted setup, $\beta \in (0, 1]$, optimal transmitting strategy $f^*_\beta(\cdot; \lambda)$ is deterministic, threshold-type.

- Let $f^*(\cdot; \lambda)$ be any limit point of $f^*_\beta(\cdot; \lambda)$ as $\beta \uparrow 1$. Then the time-homogeneous transmission strategy $f^*(\cdot; \lambda)$ is optimal for $\beta = 1$ (the long-term average setup).

- Performance of optimal strategy:
  \[ C^*(\lambda) := C(f^*, g^*; \lambda) := \inf_{(f,g)} C(f, g; \lambda) = \lim_{\beta \uparrow 1} C^*_\beta(\lambda) \]
Step 1: The SEN conditions

For any $\lambda \geq 0$, the value function $V_\beta(\cdot; \lambda)$, as given by a suitable DP, satisfies the following SEN conditions of [Lerma, Lasserre]:

**SEN conditions**

(S1) There exists a reference state $e_0 \in \mathbb{R}$ and a non-negative scalar $M_\lambda$ such that $V_\beta(e_0, \lambda) < M_\lambda$ for all $\beta \in (0, 1)$.

(S2) Define $h_\beta(e; \lambda) = (1 - \beta)^{-1}[V_\beta(e; \lambda) - V_\beta(e_0; \lambda)]$. There exists a function $K_\lambda : \mathbb{Z} \to \mathbb{R}$ such that $h_\beta(e; \lambda) \leq K_\lambda(e)$ for all $e \in \mathbb{R}$ and $\beta \in (0, 1)$.

(S3) There exists a non-negative (finite) constant $L_\lambda$ such that $-L_\lambda \leq h_\beta(e; \lambda)$ for all $e \in \mathbb{R}$ and $\beta \in (0, 1)$.
Step 2: Performance of threshold based strategies

Cost until first transmission: solution of FIE

Let $\tau^{(k)}$ be the stopping time when the Gauss-Markov process starting at state 0 at time $t = 0$ enters the set $\{e \in \mathbb{R} : |e| \geq k\}$. Expected distortion incurred until stopping and expected stopping time under $f^{(k)}$ are solutions of Fredholm integral equations of second kind.

$$L^{(k)}(e) = e^2 + \int_{-k}^{k} \phi(w - e)L^{(k)}(w)dw;$$
$$M^{(k)}(e) = 1 + \int_{-k}^{k} \phi(w - e)M^{(k)}(w)dw.$$ 

Note that we have dropped the subscript 1 for ease of notation.
Step 2: Performance of threshold based strategies

Solutions to FIE

Let $\mathcal{C}(k)$ denote the space of bounded functions from $[-k, k]$ to $\mathbb{R}$. Define the operator $\mathcal{B}^{(k)} : \mathcal{C}(k) \rightarrow \mathcal{C}(k)$ as follows. For any $v \in \mathcal{C}(k)$,

$$[\mathcal{B}^{(k)}v](e) = \int_{-k}^{k} \phi(w - e) v(w) dw.$$  

- The operator $\mathcal{B}^{(k)}$ is a contraction.
- Hence, FIE has a unique bounded solution $L^{(k)}$ and $M^{(k)}$. 

Step 2: Performance of threshold based strategies

Renewal relationship

\[ D^{(k)}(0) = \frac{L^{(k)}(0)}{M^{(k)}(0)}, \quad N^{(k)}(0) = \frac{1}{M^{(k)}(0)} \]
Step 2: Performance of threshold based strategies

Renewal relationship

\[ D^{(k)}(0) = \frac{L^{(k)}(0)}{M^{(k)}(0)}, \quad N^{(k)}(0) = \frac{1}{M^{(k)}(0)} \]

Properties

- \( L^{(k)} \) and \( M^{(k)} \) are continuous, differentiable and monotonically increasing in \( k \).
- \( D^{(k)}(0) \) and \( N^{(k)}(0) \) are continuous and differentiable in \( k \). Furthermore, \( N^{(k)}(0) \) is strictly decreasing in \( k \).
- \( D^{(k)}(0) \) is increasing in \( k \).
The system

Main result

Optimal strategies

Performance

Step 3: Identify critical Lagrange multipliers

Critical Lagrange multipliers

\[ \lambda = - \frac{D_k^{(k)}(0)}{N_k^{(k)}(0)}, \tag{1} \]
Step 3: Identify critical Lagrange multipliers

Critical Lagrange multipliers

\[ \lambda = - \frac{D_k^{(k)}(0)}{N_k^{(k)}(0)}, \quad (1) \]

Optimal transmission strategy

\((f^{(k)}, g^*)\) is \(\lambda^{(k)}\)-optimal for Lagrange relaxation. Furthermore, for any \(k > 0\), there exists a \(\lambda = \lambda^{(k)} \geq 0\) that satisfies (1).
Step 3: Identify critical Lagrange multipliers

Critical Lagrange multipliers

\[ \lambda = -\frac{D_k^{(k)}(0)}{N_k^{(k)}(0)}, \]  
(1)

Optimal transmission strategy

\((f^{(k)}, g^*)\) is \(\lambda^{(k)}\)-optimal for Lagrange relaxation. Furthermore, for any \(k > 0\), there exists a \(\lambda = \lambda^{(k)} \geq 0\) that satisfies (1).

Proof

- The choice of \(\lambda\) implies that \(C_k^{(k)}(0; \lambda) = 0\). Hence strategy \((f^{(k)}, g^*)\) is \(\lambda\)-optimal.
- \(\lambda^{(k)} \geq 0\), by the properties of \(D^{(k)}(0)\) and \(N^{(k)}(0)\).
Step 4: The constrained setup

A strategy \((f^\circ, g^\circ)\) is optimal for a constrained optimization problem, if

**Sufficient conditions for optimality [Sennott, 1999]**

(C1) \( N(f^\circ, g^\circ) = \alpha \),

(C2) There exists a Lagrange multiplier \( \lambda^\circ \geq 0 \) such that \((f^\circ, g^\circ)\) is optimal for \( C(f, g; \lambda^\circ) \).
Step 4: The constrained setup

For $\alpha \in (0, 1)$, let $k^*(\alpha)$ be such that $N(k^*(\alpha)) = \alpha$. Find $k^*(\alpha)$ for a given $\alpha$;

Optimal deterministic strategy $f^* = f(k^*(\alpha))$. 
Step 4: The constrained setup

- For $\alpha \in (0, 1)$, let $k^*(\alpha)$ be such that $N(k^*(\alpha)) = \alpha$. Find $k^*(\alpha)$ for a given $\alpha$;

Optimal deterministic strategy $f^* = f(k^*(\alpha))$.

Proof

- $(C1)$ is satisfied by $f^o = f(k^*(\alpha))$ and $g^o = g^*$.  
- For $k^*(\alpha)$, we can find a $\lambda$ satisfying (1). Hence we have that $(f(k^*(\alpha)), g^*)$ is optimal for $C(f, g; \lambda)$.
- Thus, $(f(k^*(\alpha)), g^*)$ satisfies $(C2)$.
- $D^*(\alpha) := D(f(k^*(\alpha)), g^*) = D(k^*(\alpha))(0)$
Algorithm 1: Computation of $D^*_\beta(\alpha)$

**input**: $\alpha \in (0, 1), \ \beta \in (0, 1], \ \varepsilon \in \mathbb{R}_{>0}$

**output**: $D^{(k^0)}_\beta(\alpha)$, where $|N^{(k^0)}_\beta(0) - \alpha| < \varepsilon$

Pick $k$ and $\bar{k}$ such that $N^{(k)}_\beta(0) < \alpha < N^{(\bar{k})}_\beta(0)$

$k^0 = (k + \bar{k})/2$

while $|N^{(k^0)}_\beta(0) - \alpha| > \varepsilon$ do

if $N^{(k^0)}_\beta(0) < \alpha$ then

$k = k^0$

else

$\bar{k} = k^0$

$k^0 = (k + \bar{k})/2$

return $D^{(k^0)}_\beta(\alpha)$