Remote estimation over control area networks

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The road to self-driving cars . . .

Level 1 — Control either speed or steering
- Cruise control
- Automatic breaking

Level 2 — Control both speed or steering
- Automatic lane control (Tesla's autopilot)

Level 3 — Car can handle "dynamic driving tasks" but still need human intervention
- . . .

Level 4 — Fully autonomous in certain situations
- . . .

Level 5 — Fully autonomous in all situations
- . . .

The road to self-driving cars . . .

Level 1 Control either speed or steering

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Lev … tion

These advances are driven by sophisticated algorithms that rely on measurements from multiple sensors

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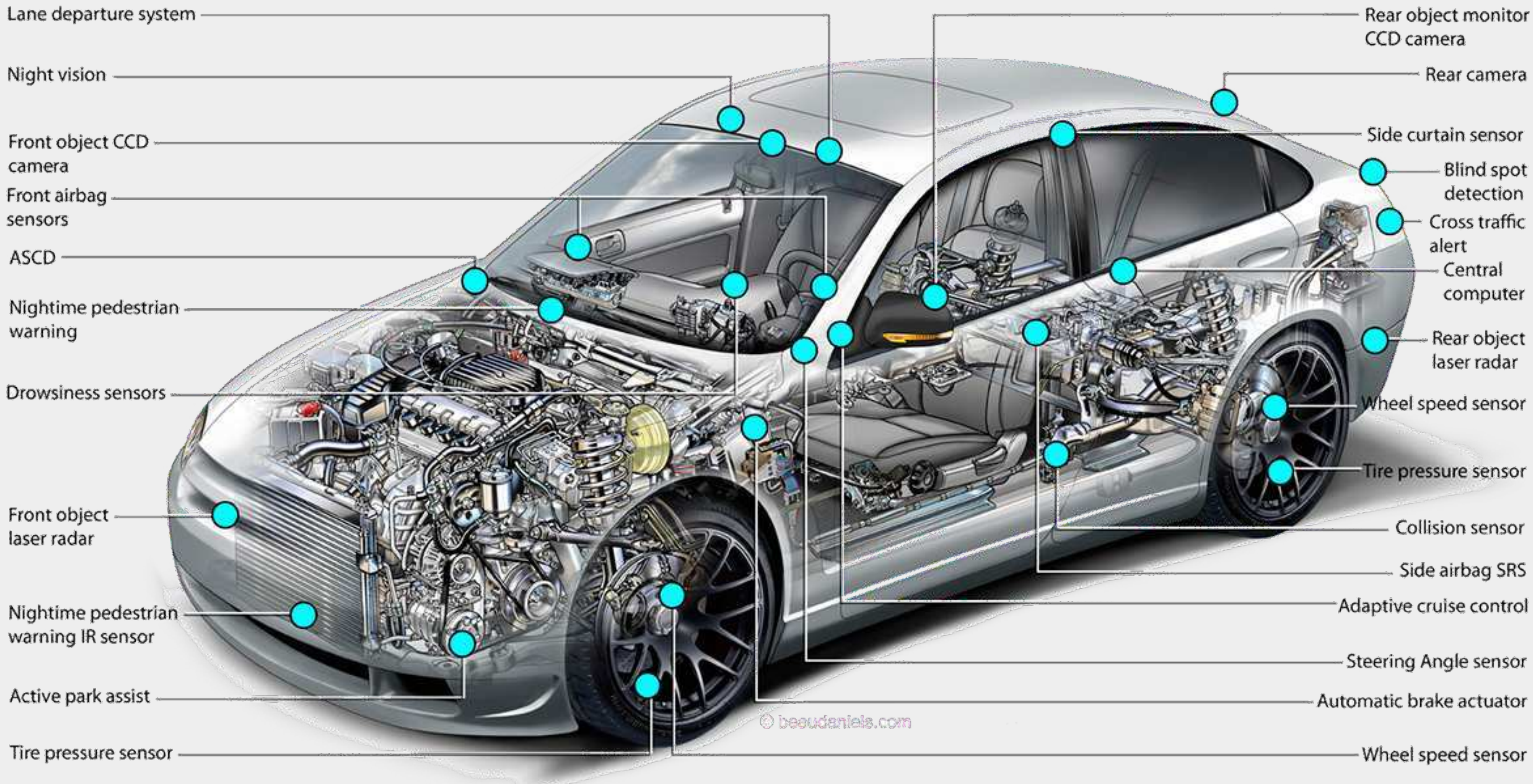
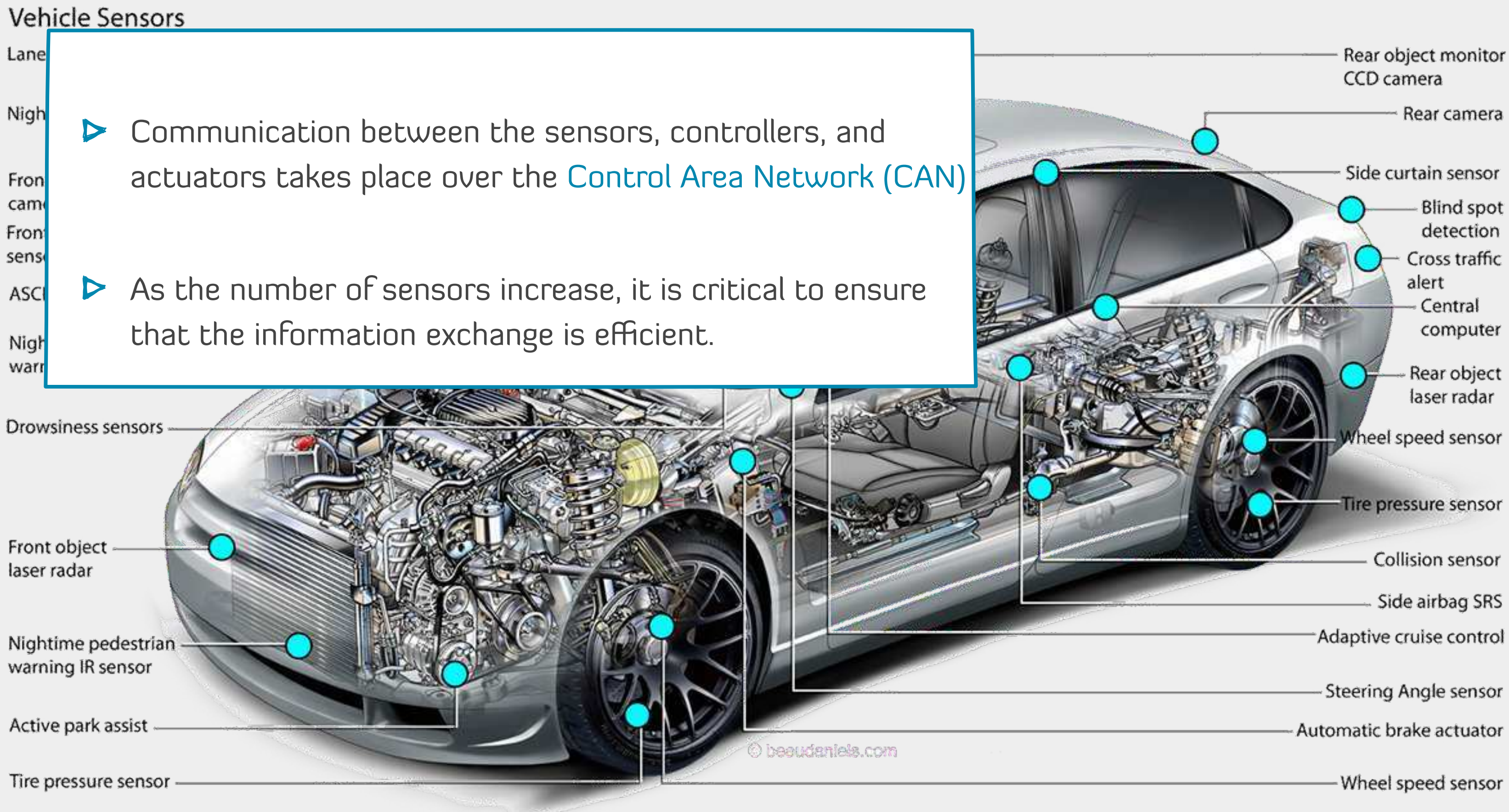
Level 5 Fully autonomous in all situations

- . . .

Vehicle Sensors



- Communication between the sensors, controllers, and actuators takes place over the Control Area Network (CAN)
- As the number of sensors increase, it is critical to ensure that the information exchange is efficient.



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- Bitwise transmission can be **dominant** (high voltage) or **recessive** (low voltage)
- If any node transmits at a dominant level, the voltage of the bus is high.

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- Nodes monitor the voltage on the bus. If a node transmitting at a recessive level detects a dominant voltage on the bus, it immediately quits transmitting.

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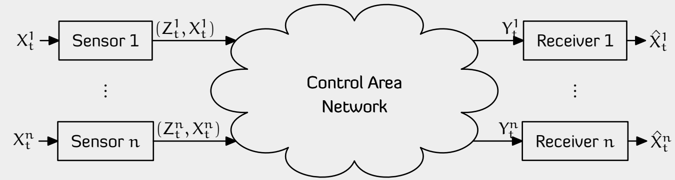
Scheduling sensor measurements is different from scheduling data packets

- Suppose a sensor does not get access to the channel.
- Then, it should simply discard the previous measurement rather than buffering it.
- Transmit fresh measurement at the next transmission instant.

Model and Problem Formulation

Model and Problem Formulation

System Model



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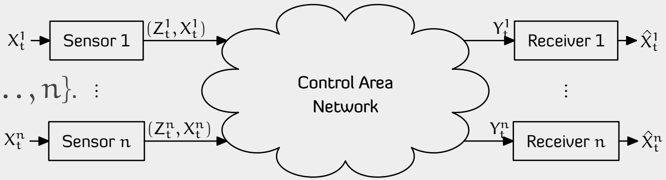

Sensors

- n sensors indexed by $N = \{1, \dots, n\}$.
- $X_{t+1}^i = a^i X_t^i + W_t^i$,
  $a^i, X_t^i, W_t^i \in \mathbb{R}, \quad W_t^i \sim \varphi^i(\cdot)$.

Assumptions

- The observation processes across sensors are independent.
- The noise process is independent across time (and independent of initial state)
- The density $\varphi^i(\cdot)$ is even and unimodal.

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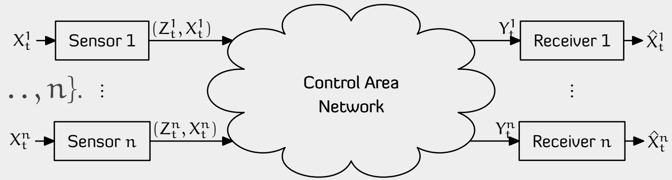

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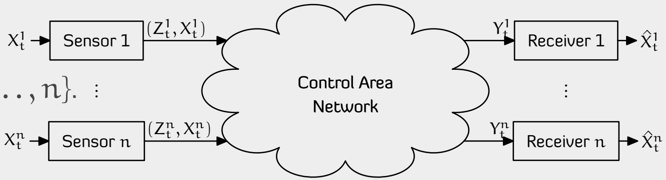

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Receivers

Estimate $\hat{X}_t^i = \begin{cases} Y_t^i, & \text{if } Y_t^i \neq \mathfrak{E} \\ a^i \hat{X}_{t-1}^i, & \text{if } Y_t^i = \mathfrak{E} \end{cases}$

Distortion $d^i(X_t^i - \hat{X}_t^i)$, where $d^i(\cdot)$ is an even and increasing function.

Model and Problem Formulation

System Model

Problem formulation

Information structure

- S_t: Sensor with highest priority. All sensors observe S_t.
- Priority Assignment rule $g^i_t \colon (X^i_{1:t}, S_{1:t-1}) \mapsto Z^i_t$.
- $S_t = \arg\max_{i \in N} Z^i_t$

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Objective

$$\min \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\sum_{t=0}^{T-1} \sum_{\substack{i \in N \\ i \neq S_t}} d^i(X_t^i - \hat{X}_t^i) \right]$$

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Salient Features

- Decentralized stochastic control problem.
- Finding optimal solution is notoriously difficult. Use a heuristic policy instead.
- Motivated by **value of information** in economics.

What is Value of Information?

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A change of variables

Error process

Define $E^i_0 = X^i_0$ and for $t \geqslant 0$,

$$E^i_{t+1} = \begin{cases} W^i_t, & \text{if } S_t = i \\ a^i E^i_t + W^i_t & \text{if } S_t \neq i \end{cases}$$

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Called the error process because when $S_t \neq i$, $X_t - \hat{X}^i_t = E^i_t$. Thus, total estimation error can be written as

$$\min \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=0}^{T-1} \sum_{\substack{i\in N \\ i\neq S_t}} d^i(E^i_t) \right]$$

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Defining the value of information

Value of Information (VOI)	The amount of money someone is willing to pay to access that information.

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VOI for remote estimation:

- Suppose that there is a single sensor, say i, and a dedicated communication channel is available.
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- Then, the error process is

$$E^i_{t+1} = \begin{cases} W^i_t, & \text{if } U^i_t = 1 \\ a^i E^i_t + W^i_t & \text{if } U^i_t = 0 \end{cases}$$

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Defining the value of information (cont.)

Optimal policy The objective is a single agent multi-stage optimization problem. Optimal solution is given by dynamic programming.

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$$h^i + v^i(e) = \min \left\{ \lambda^i + \int_{\mathbb{R}} \varphi^i(w) v^i(w) dw, \quad d^i(e) + \int_{\mathbb{R}} \varphi^i(w) v^i(ae + w) dw \right\}$$

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Theorem (Structure of optimal policy)

There exists a threshold $k^i(\lambda^i)$ such that the optimal policy is of the form

$$f^i_*(e) = \begin{cases} 1, & \text{if } |e| < k^i(\lambda^i) \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, at $k^*(\lambda^*)$,

$$\lambda^i + \int_{\mathbb{R}} \varphi^i(w) v^i(w) dw = d^i(e) + \int_{\mathbb{R}} \varphi^i(w) v^i(ae + w) dw$$

Proof relies on stochastic monotonicity, stochastic dominance, and submodularity.

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The objective is a single agent multi-stage optimization problem. Optimal solution

> VOI at e is the smallest value of access fee for which the sensor is indifferent between transmitting and not transmitting when the state is $|e|$, i.e.,
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Naive method

- For a given λ^i, find $k^i(\lambda^i)$ by numerically solving the dynamic program.
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First simplification

Let f^i_k denote the threshold policy with threshold k.

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Then, $J^i(f^i_k) = D^i_k + \lambda^i N^i_k$. The policy is optimal if $\partial_k D^i_k + \lambda^i \partial_k N^i_k = 0$.

Therefore, $$\mathrm{VOI}^i(k) = -\frac{\partial_k D^i_k}{\partial_k N^i_k}$$

How to compute value of information?

Computing $\partial_k D_k^i$ and $\partial_k N_k^i$

Renewal relationships

Let τ denote the stopping time of the first transmission.

Define $L_k^i(x) = \mathbb{E}\left[\sum_{t=0}^{\tau-1} d(E_t^i) \middle| E_0^i = x\right]$ and $M_k^i(x) = \mathbb{E}\left[\tau \middle| E_0^i = x\right]$.

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Then, from renewal theory: $D_k^i = \dfrac{L_k^i(0)}{M_k^i(0)}$ and $N_k^i = \dfrac{1}{M_k^i(0)}$.

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Need to compute $M^i_k(0)$, $L^i_k(0)$, $\partial_k M^i_k(0)$, $\partial_k L^i_k(0)$ to compute VOI.

How to compute value of information?

Computing $L^i_k(0)$ and $M^i_k(0)$

Balance equation $$L^i_k(x) = d^i_k(x) + \int_{-k}^{k} \varphi^i(y - ax) L^i_k(y)\,dy$$ (Fredholm integral eqn of the 2nd kind)

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Solution using quadrature method

Let $\{w_{-m}, \dots, w_m\}$ and $\{x_{-m}, \dots, x_m\}$ be the weights and abscissas for any quadrature rule of $2m + 1$ points over $[-k, k]$. Then the above integral equation can be approximated as

$$L_k^i(x_p) \approx d^i(x_p) + \sum_{q=-m}^{m} w_q \varphi^i(x_p - a x_q) L_k^i(x_q)$$

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Take derivative Using Leibniz rule

$$\partial_k L^i_k(x) = \varphi^i(x - ak) L^i_k(x) + \varphi^i(x + ak) L^i_k(-k) + \int_{-k}^{k} \varphi^i(x - ay) L^i_k(y) dy$$

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$$\partial_k L_k^i(x) = \varphi^i(x - ak) L_k^i(x) + \varphi^i(x + ak) L_k^i(-k) + \int_{-k}^{k} \varphi^i(x - ay) L_k^i(y) dy$$

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Rem

How to compute value of information?

Computing $\partial_k L^i_k(0)$ and $\partial_k M^i_k(0)$

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Final expression

$\mathrm{VOI}^i(k) = \mathbf{M}^i_0 \dfrac{\mathbf{L}^i_m}{\mathbf{M}^i_m} - \mathbf{L}^i_0$, where $\mathbf{L}^i = (\mathbf{I} - \mathbf{\Phi}^i)^{-1}\mathbf{d}^i$ and $\mathbf{M}^i = (\mathbf{I} - \mathbf{\Phi}^i)^{-1}\mathbf{1}$.

Numerical example

Scenarios n sensors, each observing a Gauss-Markov process.

- **Scenario A** 50 homogeneous sensors with $(a^i, \sigma^i) = (1, 1)$.
- **Scenario B** 25 sensors with $(a^i, \sigma^i) = (1, 1)$ and 25 sensors with $(a^i, \sigma^i) = (1, 5)$.
- **Scenario C** 20 sensors $(a^i, \sigma^i) = (1, 1)$; 15 with $(1, 5)$; 15 with $(1, 10)$.

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- TDMA: Sensors transmit periodically
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- **TDMA** Sensors transmit periodically
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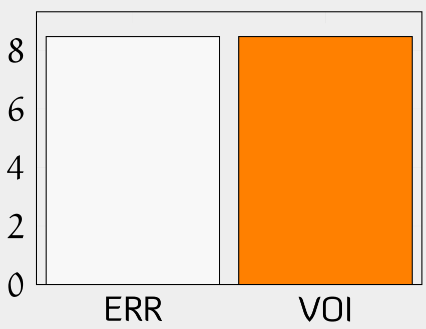
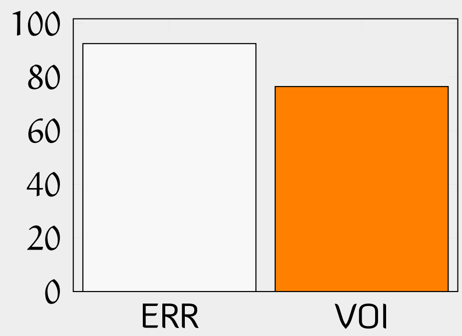


Scenario A

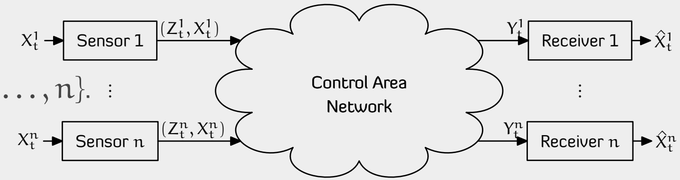


Scenario B

Scenario C

Summary

System Model



Sensors

- n sensors indexed by $N = \{1, \dots, n\}$.
- $X_{t+1}^i = a^i X_t^i + W_t^i$,
  $a^i, X_t^i, W_t^i \in \mathbb{R}, \quad W_t^i \sim \varphi^i(\cdot)$.

Assumptions

- The observation processes across sensors are independent.
- The noise process is independent across time (and independent of initial state)
- The density $\varphi^i(\cdot)$ is even and unimodal.

Network

Received packet $Y_t^i = \begin{cases} X_t^I, & \text{if sensor } i \text{ has highest priority} \\ \mathfrak{E}, & \text{otherwise} \end{cases}$

Receivers

Estimate $\hat{X}_t^i = \begin{cases} Y_t^i, & \text{if } Y_t^i \neq \mathfrak{E} \\ a^i \hat{X}_{t-1}^i, & \text{if } Y_t^i = \mathfrak{E} \end{cases}$

Distortion $d^i(X_t^i - \hat{X}_t^i)$, where $d^i(\cdot)$ is an even and increasing function.

Summary

System Model

Defining the value of information

Value of Information (VOI)

The amount of money someone is willing to pay to access that information.

VOI for remote estimation

- Suppose that there is a single sensor, say i, and a dedicated communication channel is available.
- The sensor has to pay an access fee λ^i each time it uses the channel.
- Let $U^i_t \in \{0, 1\}$ denote the sensor's decision.
- Then, the error process is

$$E^i_{t+1} = \begin{cases} W^i_t, & \text{if } U^i_t = 1 \\ a^i E^i_t + W^i_t & \text{if } U^i_t = 0 \end{cases}$$

Objective

$$\min \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=0}^{T-1} \left[\lambda^i U^i + (1 - U^i) d^i(E^i_t) \right] \right]$$

Summary

Naive method

- For a given λ^i, find $k^i(\lambda^i)$ by numerically solving the dynamic program.
- $\text{VOI}^i(e)$ can be computed by doing a binary search of λ^i until $k^i(\lambda^i) = |e|$.
- This method is extremely inefficient because solving DP is hard.

First simplification

Let f^i_k denote the threshold policy with threshold k.

Define $D^i_k = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T-1}(1 - U^i_t)d^i(E^i_t)\right]$ and $N^i_k = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=0}^{T-1} U^i_t\right]$

Then, $J^i(f^i_k) = D^i_k + \lambda^i N^i_k$. The policy is optimal if $\partial_k D^i_k + \lambda^i \partial_k N^i_k = 0$.

Therefore,

$$\boxed{\text{VOI}^i(k) = -\frac{\partial_k D^i_k}{\partial_k N^i_k}}$$

Summary

Computing $\partial_k D_k^i$ and $\partial_k N_k^i$

Renewal relationships

Let τ denote the stopping time of the first transmission.

Define $L_k^i(x) = \mathbb{E}\left[\sum_{t=0}^{\tau-1} d(E_t^i) \,\middle|\, E_0^i = x\right]$ and $M_k^i(x) = \mathbb{E}\left[\tau \,\middle|\, E_0^i = x\right]$.

Then, from renewal theory: $D_k^i = \dfrac{L_k^i(0)}{M_k^i(0)}$ and $N_k^i = \dfrac{1}{M_k^i(0)}$.

Therefore, $$\mathrm{VOI}^i(k) = M_k^i(0)\frac{\partial_k L_k^i(0)}{\partial_k M_k^i(0)} - L_k^i(0)$$

Need to compute $M_k^i(0)$, $L_k^i(0)$, $\partial_k M_k^i(0)$, $\partial_k L_k^i(0)$ to compute VOI.

Summary

Computing $L_k^i(0)$ and $M_k^i(0)$

Balance equation

$$L_k^i(x) = d_k^i(x) + \int_{-k}^{k} \varphi^i(y - ax) L_k^i(y) dy$$ (Fredholm integral eqn of the 2nd kind)

Solution using quadrature method

Let $\{w_{-m}, \dots, w_m\}$ and $\{x_{-m}, \dots, x_m\}$ be the weights and abscissas for any quadrature rule of $2m + 1$ points over $[-k, k]$. Then the above integral equation can be approximated as

$$L_k^i(x_p) \approx d^i(x_p) + \sum_{q=-m}^{m} w_q \varphi^i(x_p - a x_q) L_k^i(x_q)$$

Or, in matrix form, $\mathbf{L}^i = (\mathbf{I} - \mathbf{\Phi}^i)^{-1} \mathbf{d}^i$ and $\mathbf{M}^i = (\mathbf{I} - \mathbf{\Phi}^i)^{-1} \mathbf{1}$

Summary

Computing $\partial_k L_k^i(0)$ and $\partial_k M_k^i(0)$

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Rem