Optimal sampling of multiple linear processes over a shared medium

Sebin Mathew[a], Karl H. Johansson[b], Aditya Mahajan[a]

[a] McGill University

[b] KTH Royal Institute of Technology

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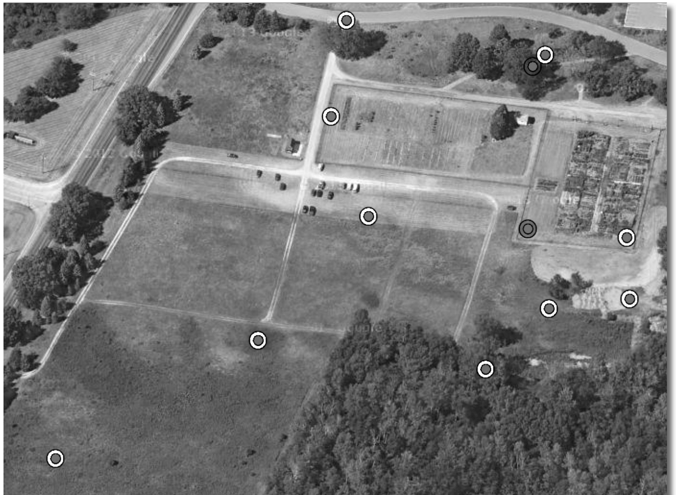
Many remote estimation applications where:

- Multiple sensors transmit over shared links
- Link capacity varies exogenously



Sensor Networks

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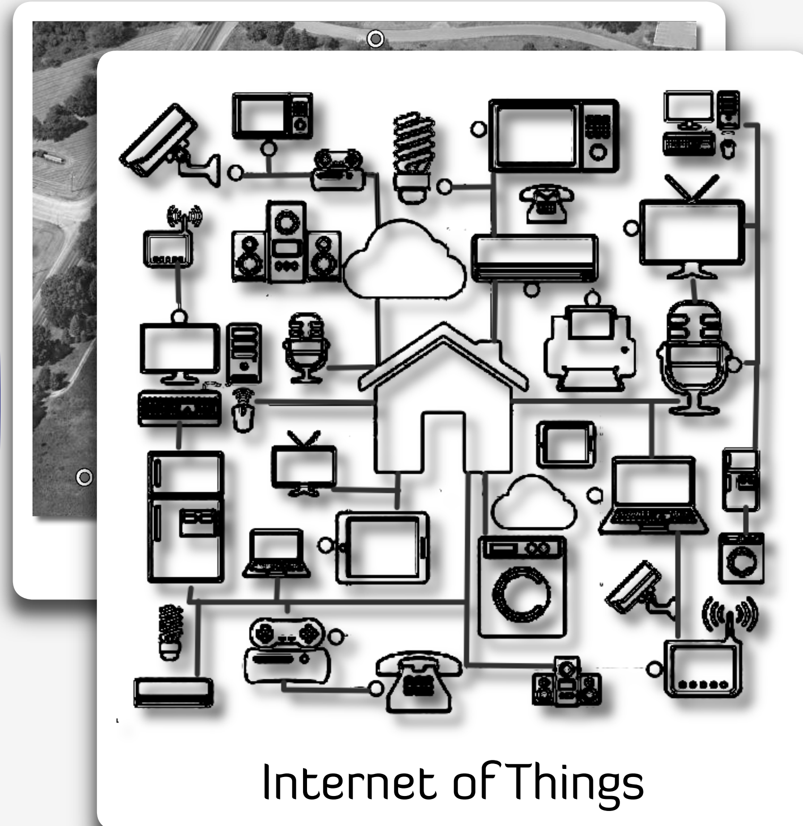

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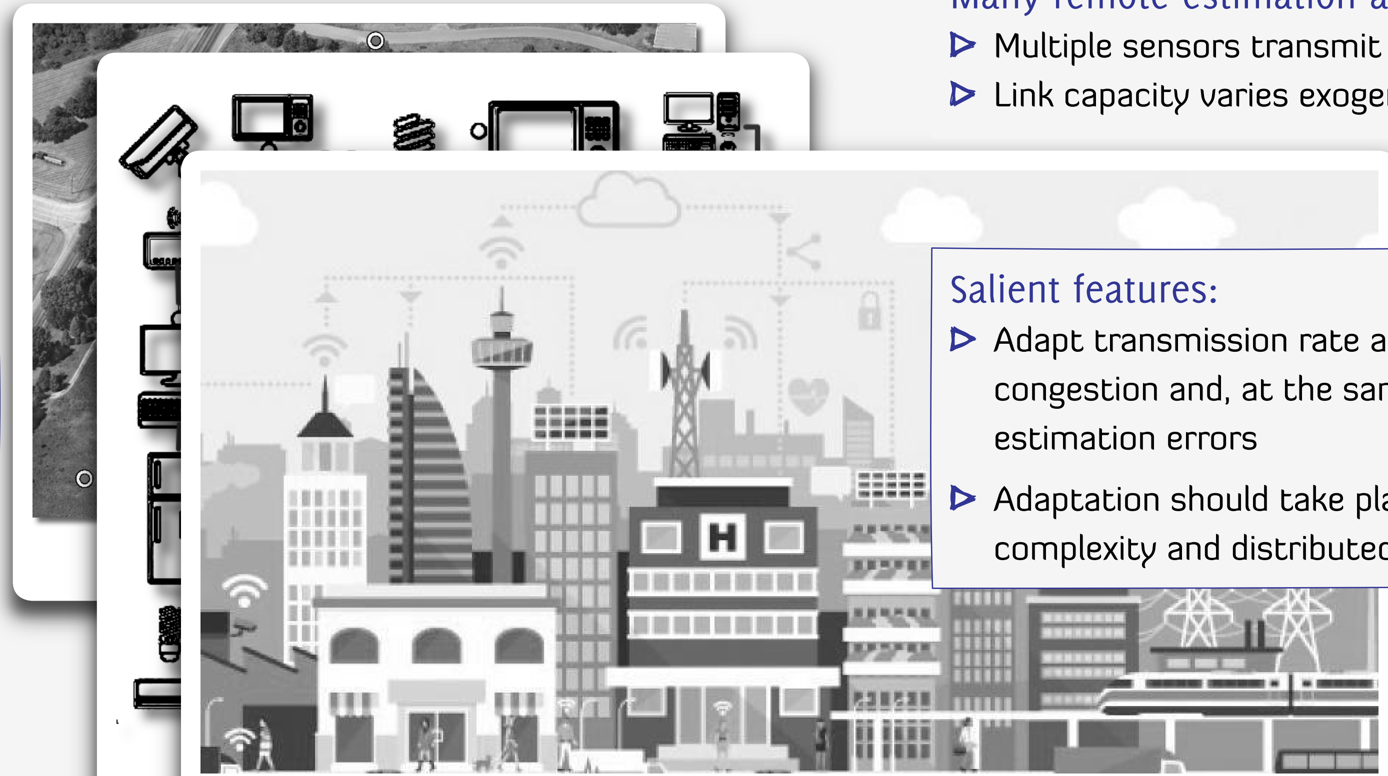

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Salient features:

- Adapt transmission rate at sensors to avoid congestion and, at the same time, minimize estimation errors
- Adaptation should take place in a low complexity and distributed manner

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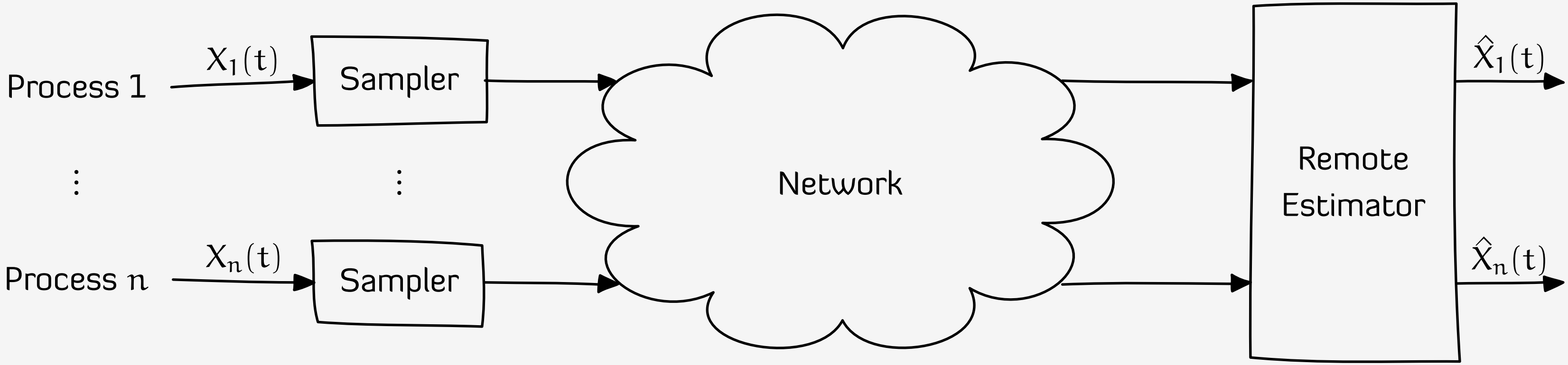
Show that such questions can be answered using dual decomposition theory

System Model



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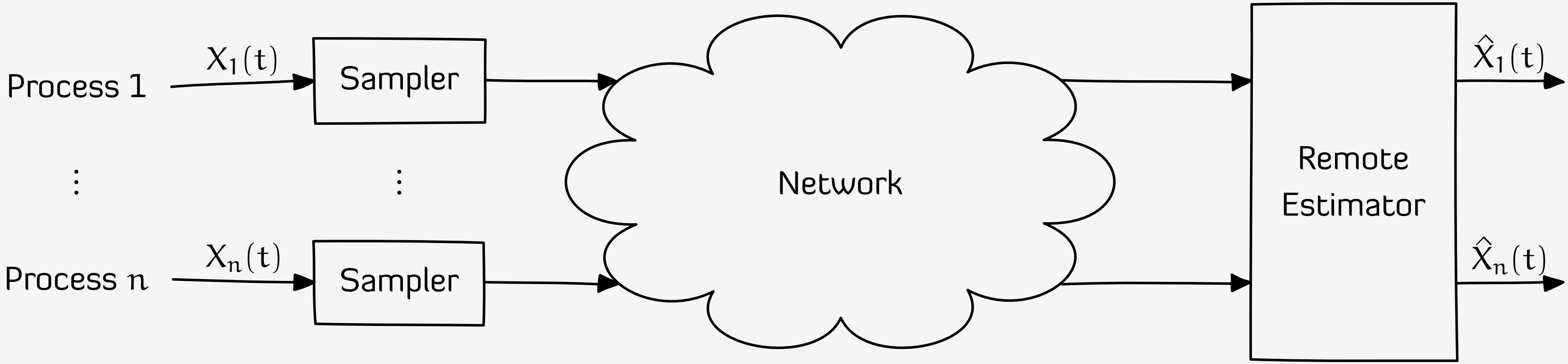

Process dynamics $\quad dX_i(t) = a_i X_i(t)dt + dW_i(t)$. $\quad \{W_i(t)\}_{t \geq 0}$ is stationary and indep across sensors.

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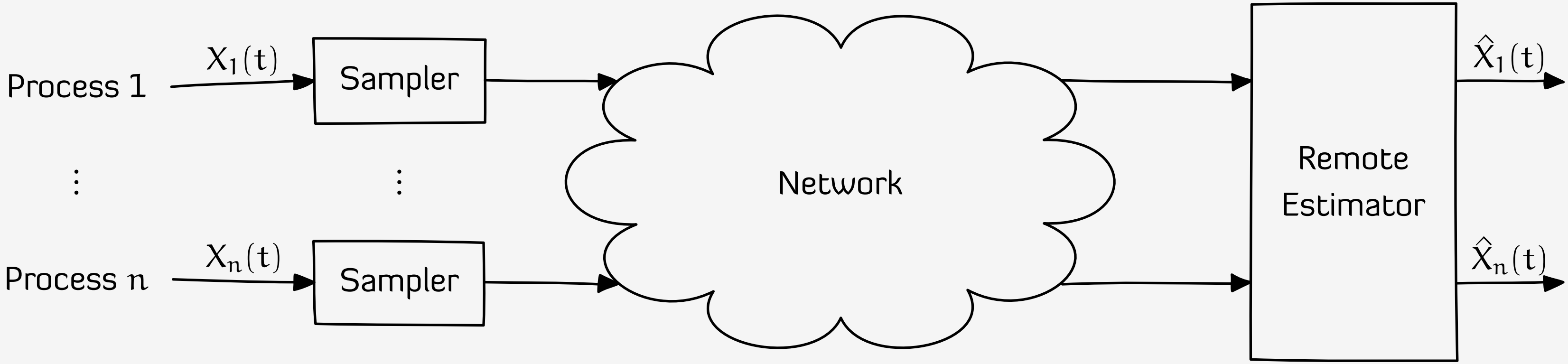

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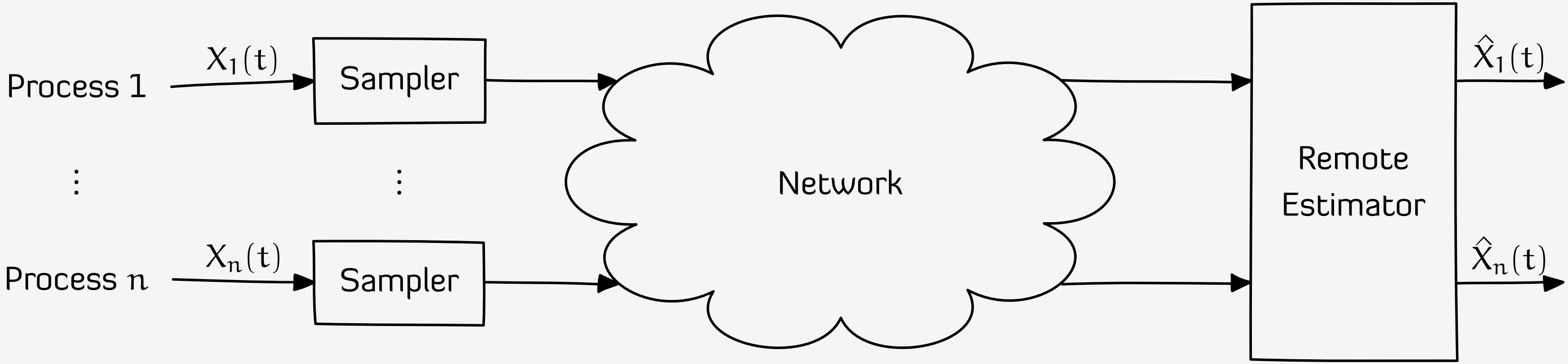

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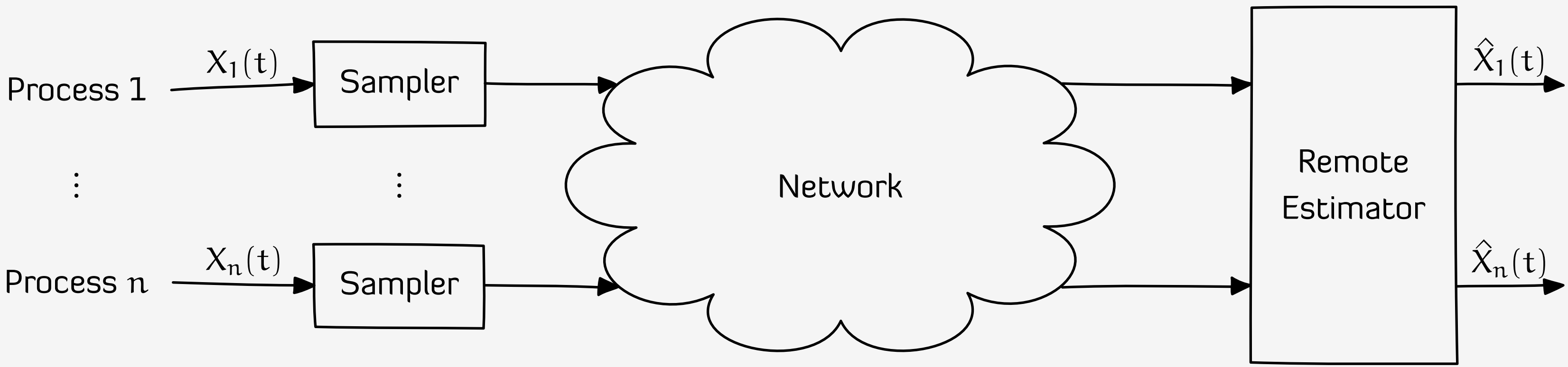

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Estimated process At a sampling time: $\hat{X}_i(t) = X_i(t)$. At other times: $d\hat{X}_i(t) = a_i \hat{X}_i(t)dt$

System Performance and Optimization Problem

Mean-square error $M_i(R_i) = R_i \int_0^{1/R_i} \left(X_i(t) - \hat{X}_I(t)\right)^2 dt$ when sensor i is sampling at rate R_i.

Example If the noise process is a Wiener process with variance σ_i^2, then the state process is a Gauss-Markov (or Ornstein-Uhlenbeck) process, and $M_i(R_i) = \frac{\sigma_i^2}{2a_i}\left[\frac{e^{2a_i/R_i - 1}}{2a_i/R_i} - 1\right]$.

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Assumptions

(A1) For any sensor i and rate $R_i > 0$, $M_i(R_i)$ is strictly decreasing and convex in R_i.

(A2) $M_i(R_i)$ is twice differentiable and there exists a positive constant c_i such that

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Solution approach

Proposition Under assumptions (A1) and (A2), the optimization problem has a unique solution which we denote by $\mathbf{R}^* = (R_1^*, \ldots, R_n^*)$.

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How do we find $\mathbf{R}^*$ in a distributed manner?

Distributed Solution via Dual Decomposition

Primal Problem

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Dual Decomposition

Decomposes into two parts: Network and Sensor i

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Syncronous Algorithm

- Network starts with a guess λ_0.
- At each iteration $k = 0, 1, \ldots$

At each sensor i

Pick $R_{i,k}$ to min $M_i(R_i) + \lambda_k R_{i,k}$

At the network

$$\lambda_{k+1} = \left[\lambda_k - \alpha_k \left(C - \sum_{i=1}^{n} R_{i,k} \right) \right]^+$$

Properties of synchronous algorithm

Theorem 1 Under (A1) and (A2), for any initial guess λ_0 and appropriately chosen step sizes α_k,

$$\lim_{k\to\infty} \mathbf{R}_k := \lim_{k\to\infty} (R_{1,k}, \ldots, R_{n,k}) = \mathbf{R}^*.$$

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Implementation The synchronous algorithm can be implemented as part of the initial handshaking protocol when the sensors come online.

Drawbacks

- Large signaling overhead.
- Algorithm needs to be rerun when:
  - a sensor leaves, or
  - a new sensor comes on board, or
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> Salient feature: The network doesn't need to know $R_{i,k}$. It only needs an estimate of $\sum R_i$, which it can infer from the received packets.

Asynchronous algorithm for choosing sampling rates

At the network

Initialize $\lambda > 0$

Upon event $\langle$new packet received$\rangle$ **do**

- Estimate received sum rate $\hat{C}_k$ based on packets received in a sufficiently large sliding window of time.
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Properties of asynchronous algorithm

Assumption (A3) The time between the consecutive samples is bounded.

Theorem 2 Under (A1)–(A3), for any initial guess λ_0 and appropriately chosen step sizes α_k,

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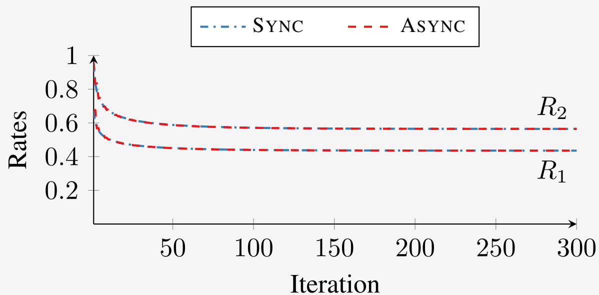
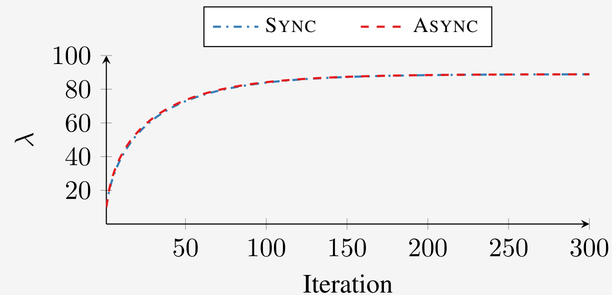
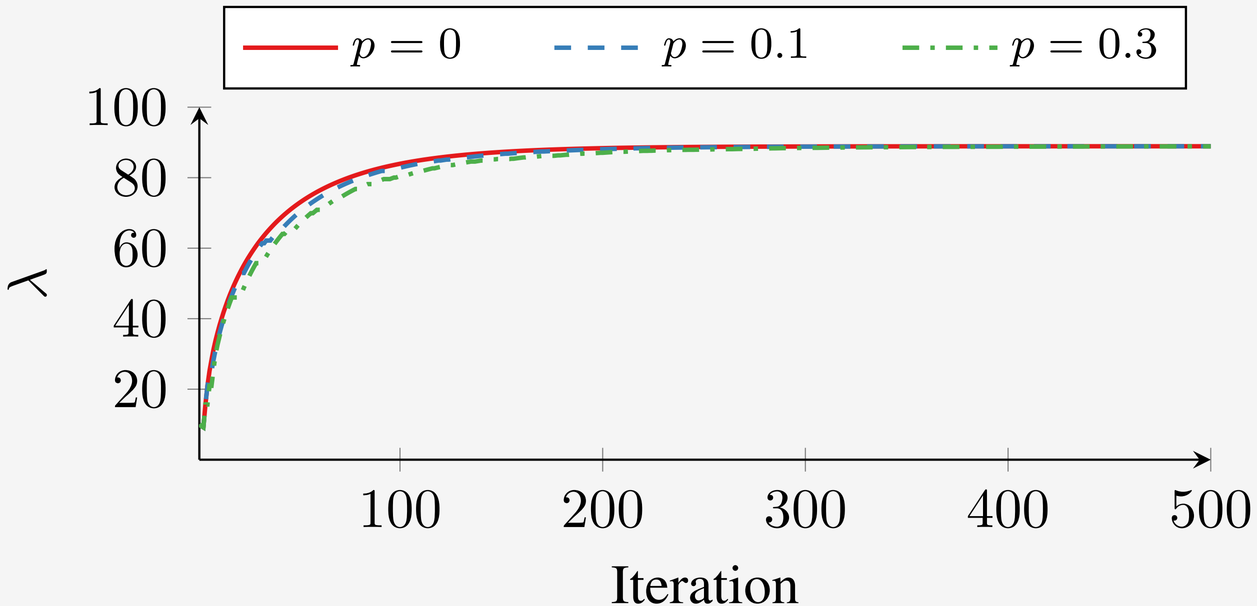
Example 2 sensors: GaussMarkov$(1,1)$ and GaussMarkov$(1,2)$. Network capacity $C = 1$.





Robustness to packet drops and delays



Illustrative example

Changing network conditions

- Sensors arrive according to a Poisson process and stay in the system for an exponentially distributed amount of time.
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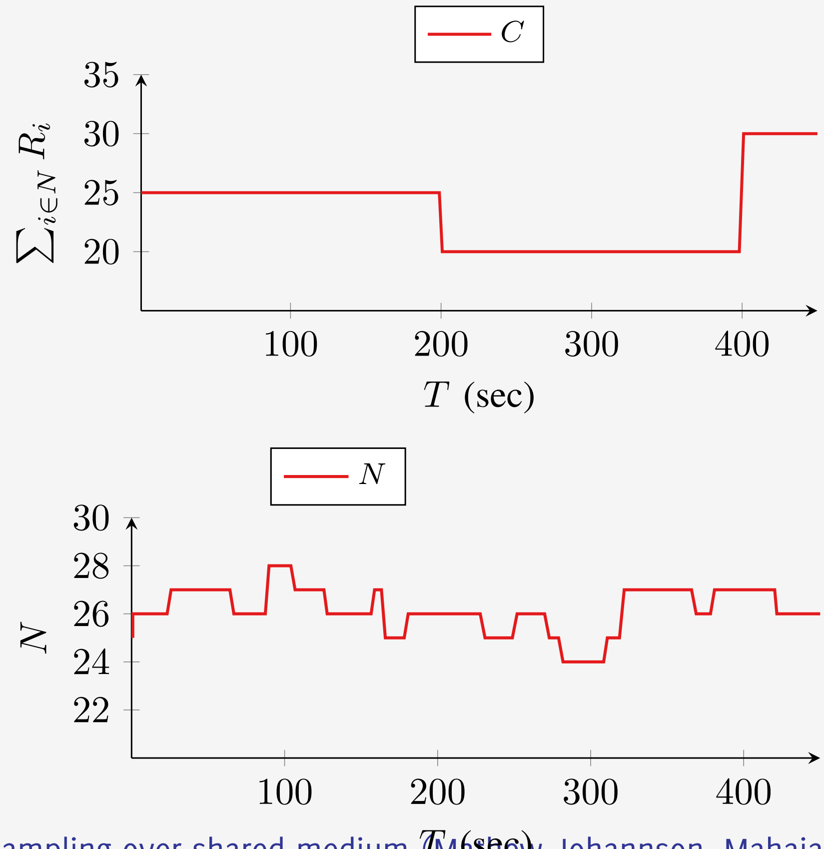
Network

- Network is not aware of the number of sensors.
- Adapts λ according to the asynchronous algorithm
- Broadcasts the value of λ.

Sensors

- Sensors don't know the network capacity.
- Run the asynchronous algorithm to adapt rate R_i.

Illustrative example: System parameters vs time



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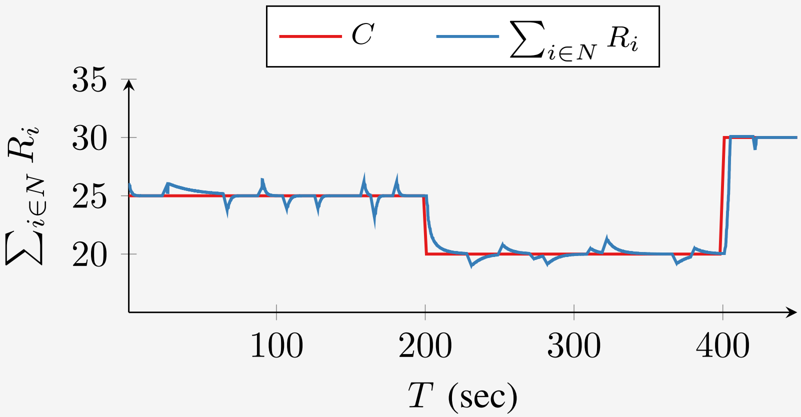

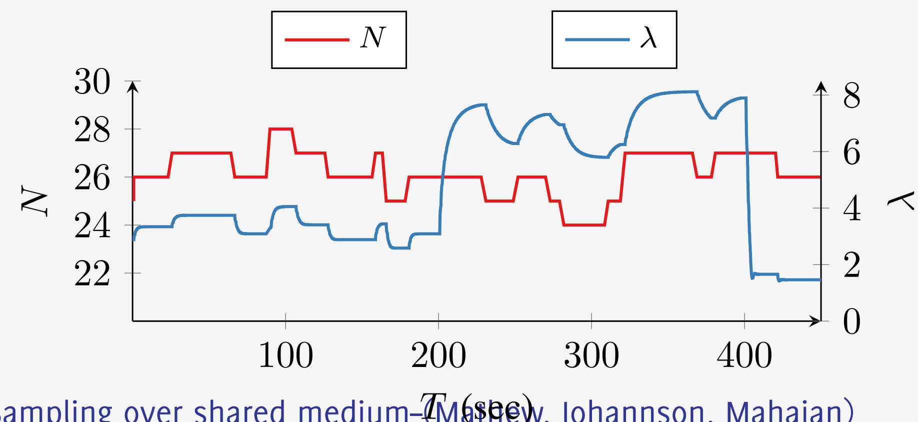

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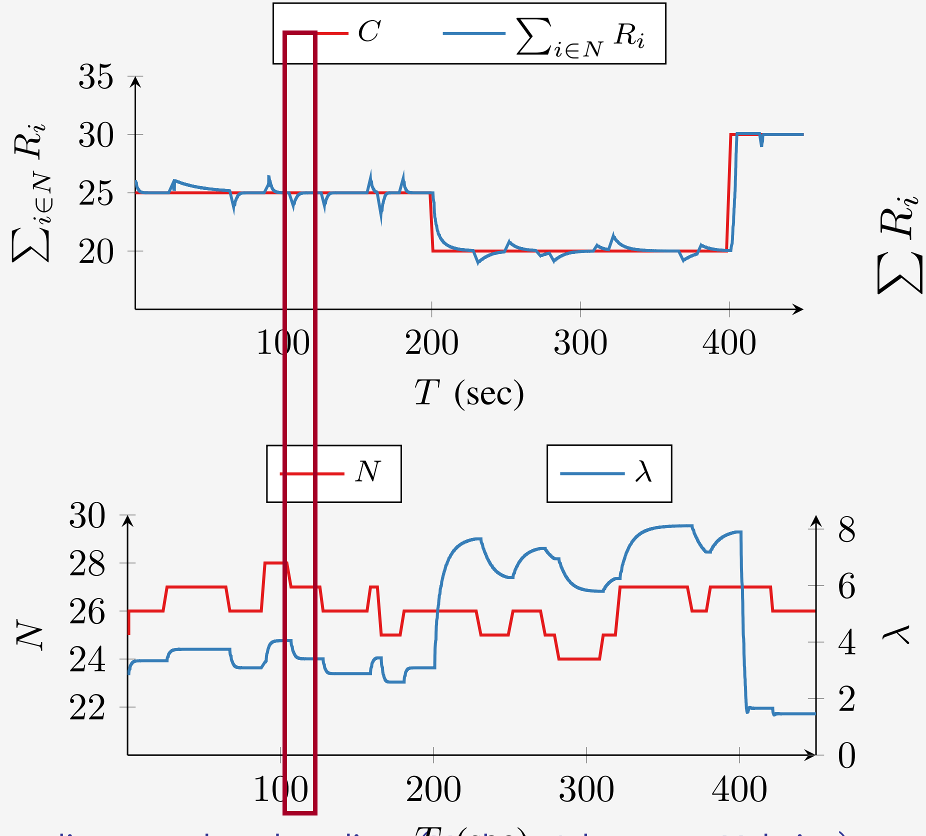

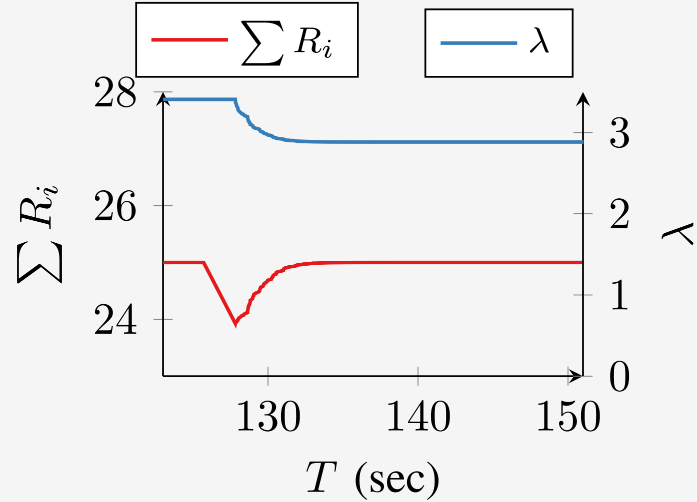

Sensor leaving

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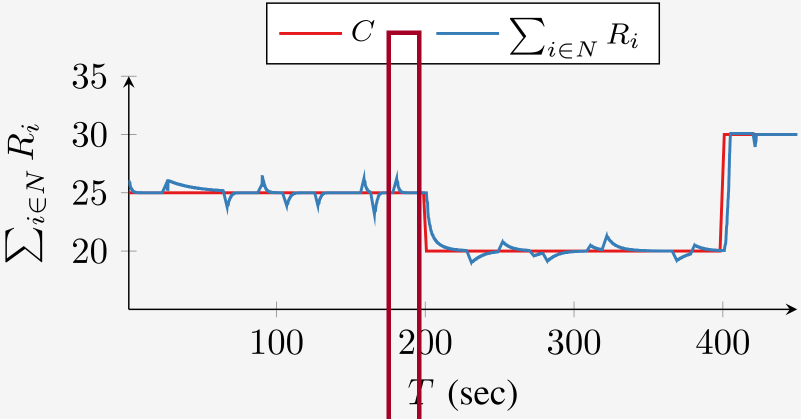

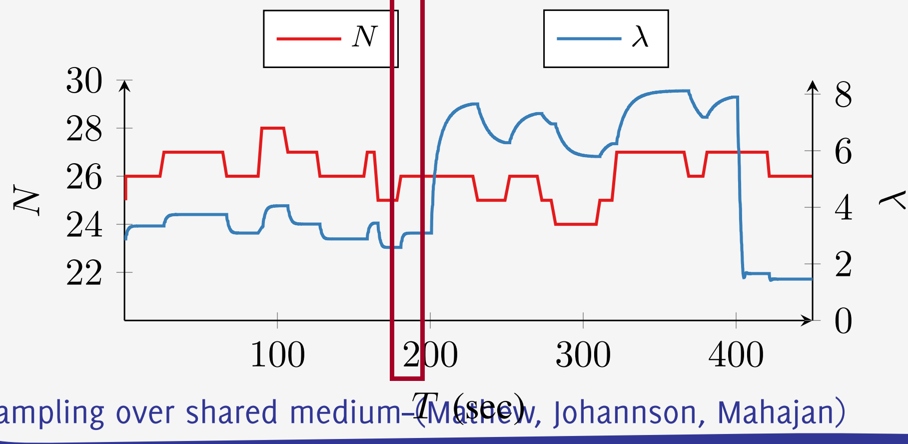

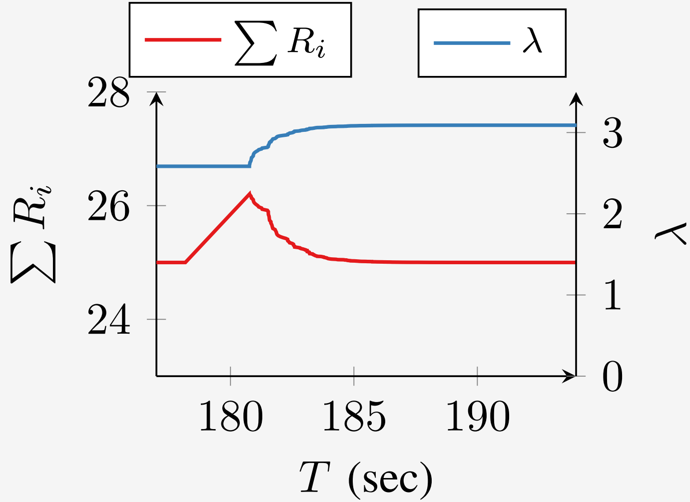

Sensor coming on board

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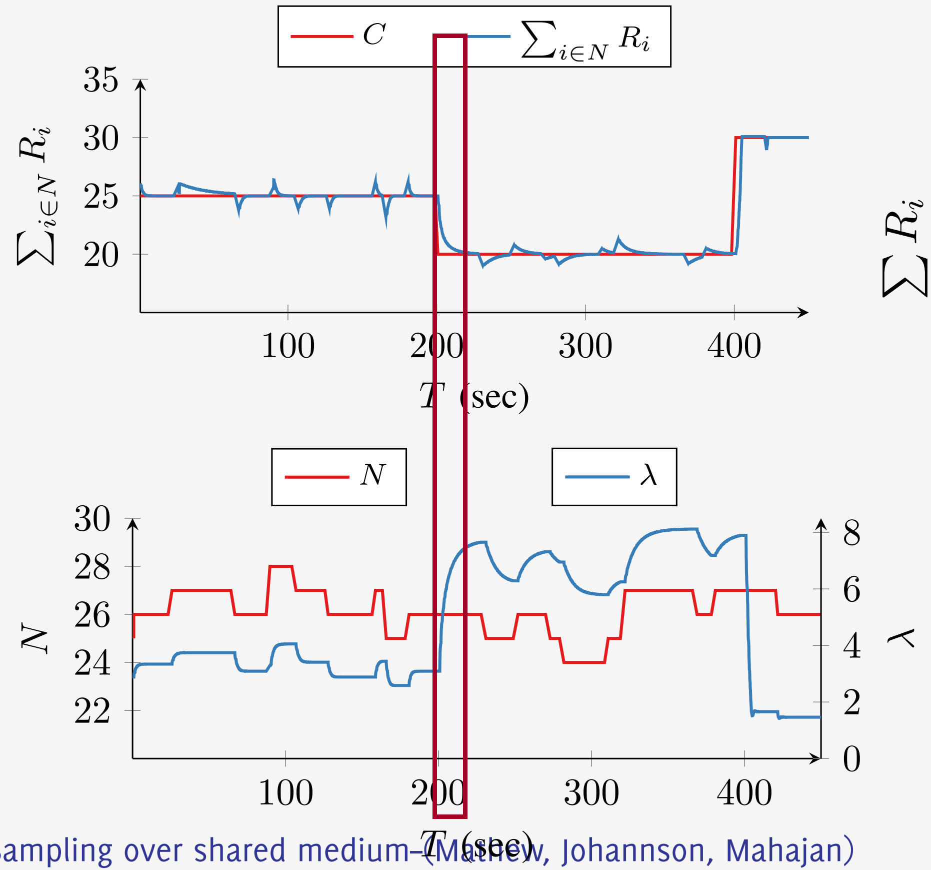

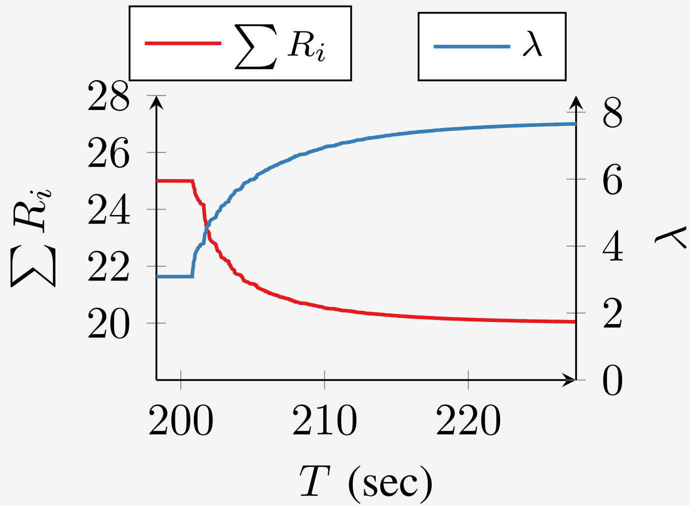

Network capacity changing

Comparison with baseline schemes

- Scheme 1: $R_i = C/30$.
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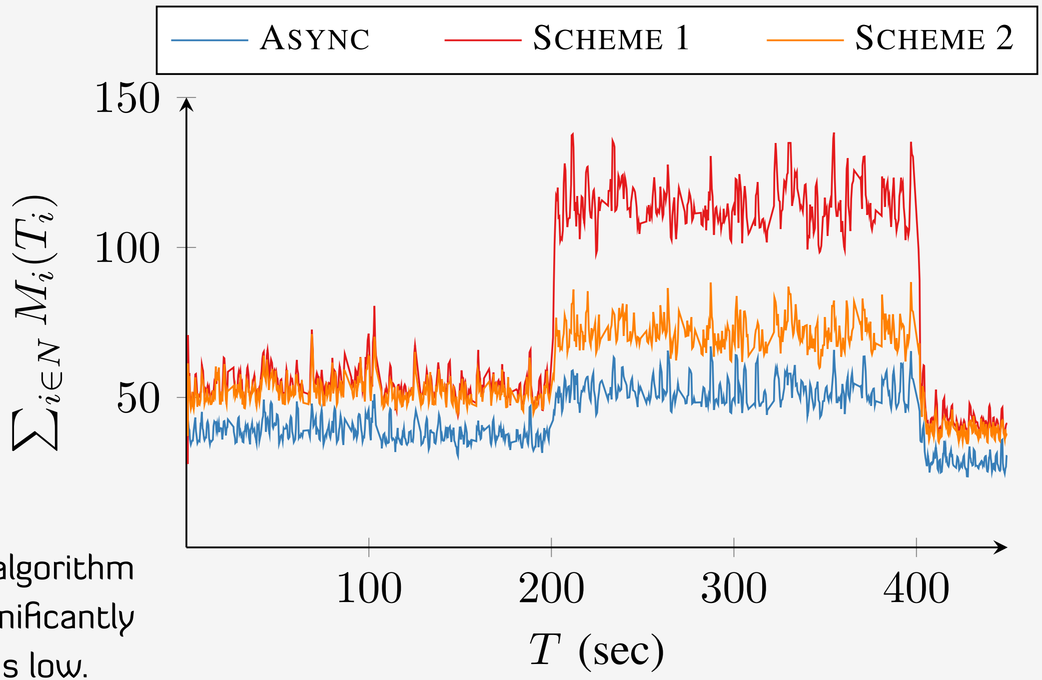

- Performance of Asynchronous algorithm is better than baseline, and significantly so when the network capacity is low.

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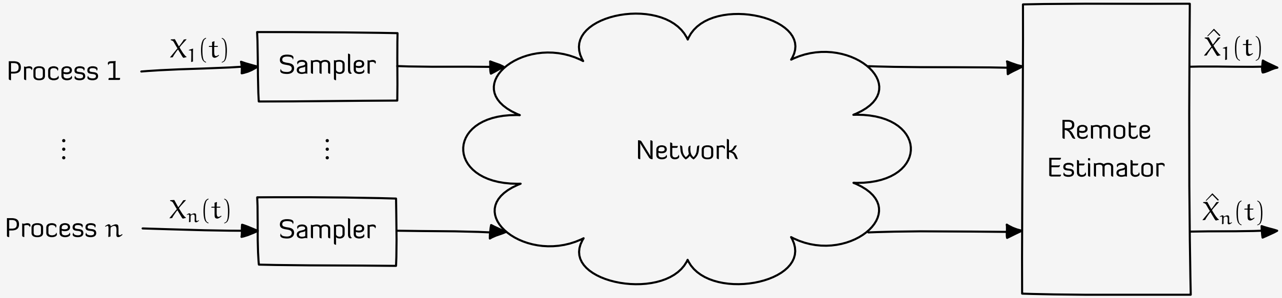

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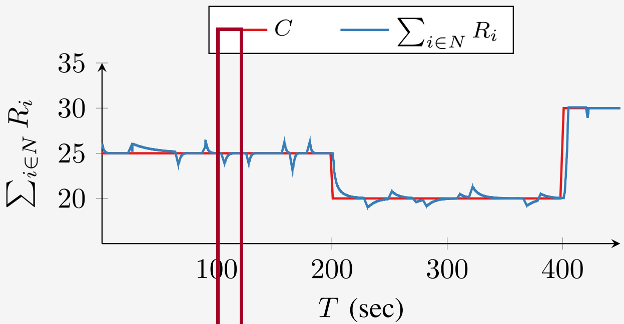

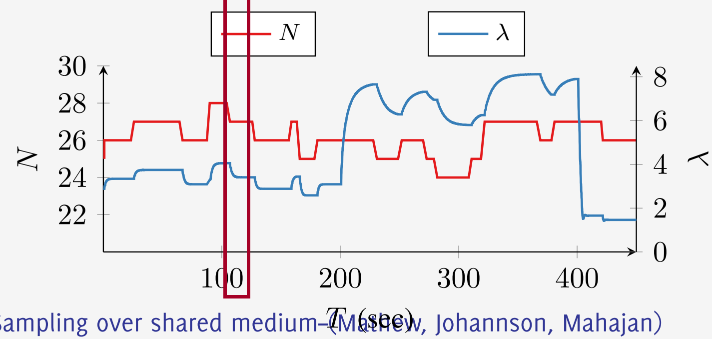

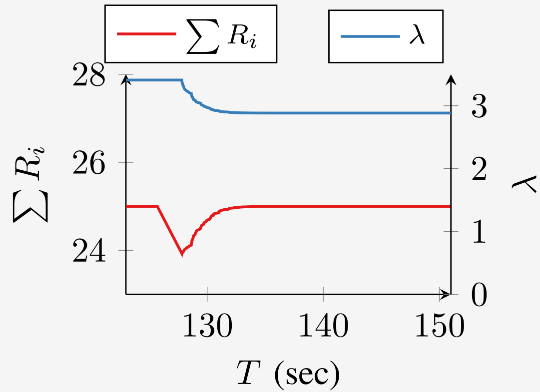

Sensor leaving

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Conclustion

The asynchronous event-driven algorithm can adapt to slowly varying network conditions in a distributed manner. Asymptotically, the algorithm converges to the optimal rates.

The sensors and the estimators don't need synchronous clocks!

Robust to packet drops, delays, and slow variation in system parameters.

Dual decomposition does not ensure that the constraint $\sum R_i \leqslant C$ is satisfied at all iterations. In practice, violation of this constraint will lead to congestion. To understand its impact, we need to consider a more elaborate network model where congestion leads to delay.