Filtering Non-Linear Transfer Functions on Surfaces

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Abstract

This document provides additional results in support of the associated paper. We include the full mathematical development of many formulations in the paper, although they are not required to understand the theoretical and practical workings of our method. Pseudocode of the general algorithm is also included, and this supplement will be released with the paper. All ground-truth results were computed by super-sampling pixels with a 32×32 jittered grid.

1 Results for Color Textures

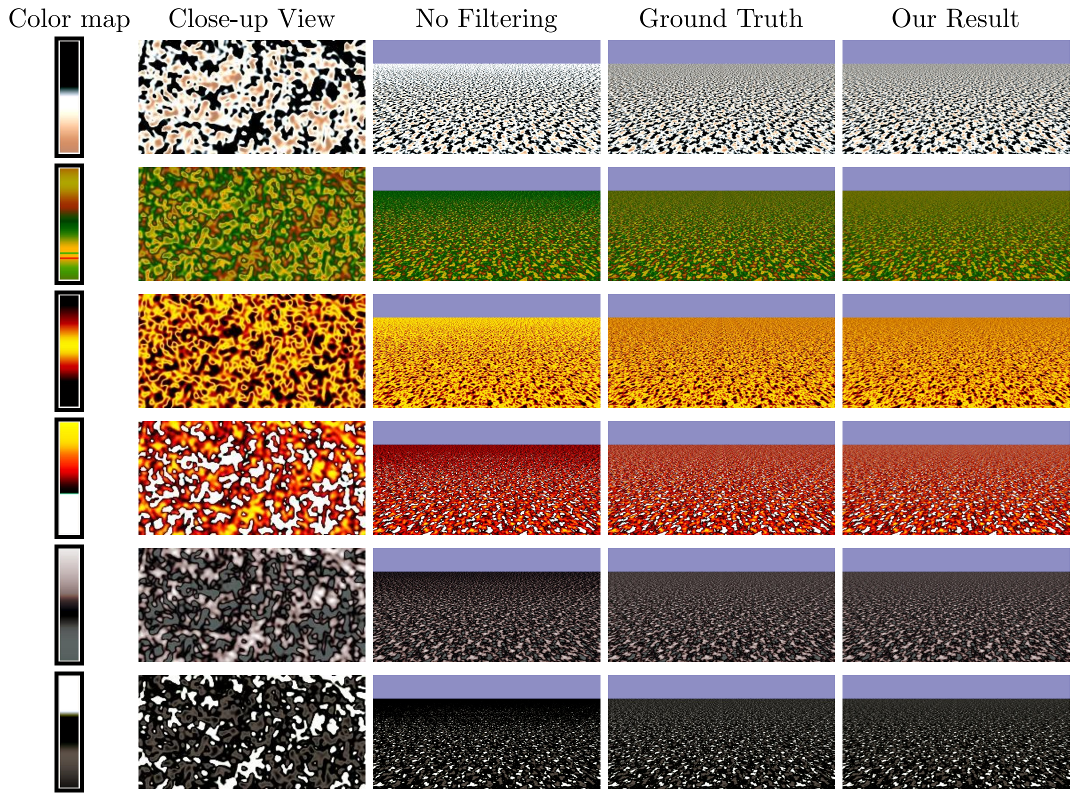


Table 1: Filtering of color mapped textures.

2 Results for Height Correlated Color Mapped Surfaces

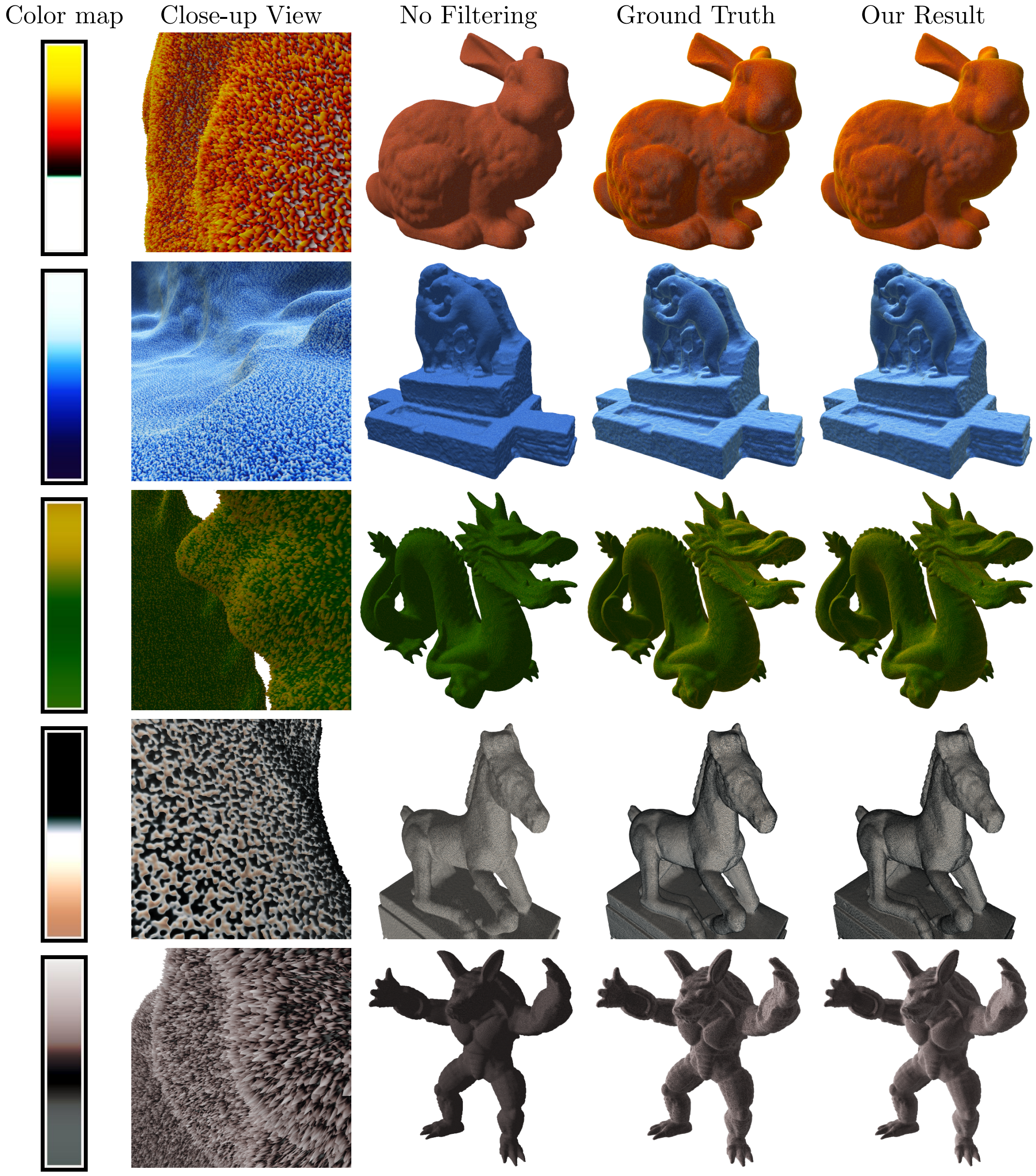


Table 2: Filtering of color mapped surfaces with correlated microsurface heights.

3 Gaussian Surface Slope Distribution

In this section we review the definition of a Gaussian microsurface local-orientation/slope generation process, as well as its transformation after applications of rotation operators.

Definition: A Gaussian microsurface local-orientation/slope distribution $\mathcal{N}(\bar{s}, \Sigma)$ expresses the probability with which a microfacet of slope s is present on the microsurface. It is defined by the following *probability distribution function* (PDF):

$$p_s(s) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(s-\bar{s})^T \Sigma^{-1} (s-\bar{s})\right). \tag{1}$$

Expressed about a local orthonormal basis $(\vec{x}, \vec{y})$ at a microsurface point, the average slope and covariance matrix are:

$$\bar{s} = \begin{pmatrix} \bar{s}_x \\ \bar{s}_y \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_x^2 & c_{xy} \\ c_{xy} & \sigma_y^2 \end{pmatrix}.$$

Rotation: The computation of view-dependent slope distributions can be simplified by expressing the distribution in a view-aligned coordinate frame, rotating it to align with $(\vec{o}, \vec{\perp})$, where $\vec{o}$ is the view direction and $\vec{\perp}$ is a direction orthogonal to the view direction. The slope s_o in the view direction and the orthogonal direction $s_\perp$ are defined as

$$s_o = s_x \cos\phi_o + s_y \sin\phi_o \tag{2}$$
$$s_\perp = -s_x \sin\phi_o + s_y \cos\phi_o \tag{3}$$

and the inverse transformation is obtained as

$$s_x = s_o \cos\phi_o - s_\perp \sin\phi_o \tag{4}$$
$$s_y = s_o \sin\phi_o + s_\perp \cos\phi_o \ . \tag{5}$$

In order to compute the PDF p_s of the slope distribution *in the rotated view-aligned frame*, we need only compute the new average slope and covariance matrix, taking advantage of the fact that a rotated Gaussian distribution remains a Gaussian distribution:

$$\begin{pmatrix} \bar{s}_o \\ \bar{s}_\perp \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sigma_o^2 & c_{o\perp} \\ c_{o\perp} & \sigma_\perp^2 \end{pmatrix}$$

The average values are rotated in a similar manner,

$$\bar{s}_o = \bar{s}_x \cos\phi_o + \bar{s}_y \sin\phi_o \tag{6}$$
$$\bar{s}_\perp = -\bar{s}_x \sin\phi_o + \bar{s}_y \cos\phi_o \ , \tag{7}$$

and the rotated variances and covariance are given by

$$\begin{aligned}
\sigma_o^2 &= \mathrm{var}(s_x \cos\phi_o + s_y \sin\phi_o) && (8)\\
&= \sigma_x^2 \cos^2\phi_o + \sigma_y^2 \sin^2\phi_o + 2\, c_{xy} \cos\phi_o \sin\phi_o \\
\sigma_\perp^2 &= \mathrm{var}(-s_x \sin\phi_o + s_y \cos\phi_o) \\
&= \sigma_x^2 \sin^2\phi_o + \sigma_y^2 \cos^2\phi_o - 2\, c_{xy} \cos\phi_o \sin\phi_o && (9)\\
c_{o\perp} &= \mathrm{cov}(s_x \cos\phi_o + s_y \sin\phi_o, -s_x \sin\phi_o + s_y \cos\phi_o) \\
&= (\sigma_y^2 - \sigma_x^2) \cos\phi_o \sin\phi_o + c_{xy}(\cos^2\phi_o - \sin^2\phi_o) \ . && (10)
\end{aligned}$$

4 Visible Slope Distribution

In the previous section we defined the slope PDF of the microsurface. The visible slope PDF describes the probability that a microfacet with slope s is visible from the view direction $\omega_o(\theta_o, \phi_o)$, and we detail its derivation below.

Definition: The PDF of the visible slope distribution D_s depends on the PDF of the microsurface slope PDF p_s and of the visible (foreshortened) area of the microfacets with respect to the viewing direction. These values depend on the normals of the microfacets. A microfacet with slope $s = (s_x, s_y)$ has normal

$$n = [n_x, n_y, n_z] = \frac{[-s_x, -s_y, 1]^T}{\sqrt{s_x^2 + s_y^2 + 1}} \ . \tag{11}$$

Since the Gaussian slope process is defined on the plane (in 2D) of normal $[0, 0, 1]^T$ the presence of the microfacet (in 3D) has to be weighted by the jacobian of the 2D $\rightarrow$ 3D projection which is $\frac{1}{n_z}$. The foreshortening weight induced by projection towards the viewing direction is the clamped cosine $\max(0, n(s) \cdot \omega_o)$, and so the PDF of the visible slope distribution is

$$D_s(s) = \frac{1}{N} \frac{p_s(s) \ \max(0, n(s) \cdot \omega_o)}{n_z} \tag{12}$$

$$= \frac{1}{N} \ p_s(s) \ W(s) \ H(s), \tag{13}$$

where

$$W(s) = -s_x \, \omega_{ox} - s_y \, \omega_{oy} + \omega_{oz} \tag{14}$$

$$H(s) = \text{Heaviside}(-s_x \, \omega_{ox} - s_y \, \omega_{oy} + \omega_{oz}) \tag{15}$$

and N is the normalization factor of the distribution

$$N = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_s(s) \ W(s) \ H(s) \ d^2 s. \tag{16}$$

Unfortunately, D_s no longer remains a Gaussian distribution after this modification. However, D_s can be well approximated with a Gaussian distribution parameterized by an average vector and covariance matrix. We will detail the derivation of such an approximation below. The starting point of our derivation is given by Ross et al. [RDP05] who derive N for a Gaussian Distribution p_s centered at $(0, 0)$. We use a similar derivation to compute the first and second moments of D_s for an arbitrarily centered p_s, generalizing Ross et al.'s derivation.

Simplification and Derivation: The equations above simplify when expressed in the view-aligned coordinate frame $(\vec{o}, \vec{\perp})$ as

$$W(s) = -s_o \sin\theta_o + \cos\theta_o \tag{17}$$

$$H(s) = \text{Heaviside}(-s_o \sin\theta_o + \cos\theta_o) \ . \tag{18}$$

The average, variance and covariance moments of D_s are given by

$$\bar{s}_o(D_s) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_o \ p_s(s) \ W(s) \ H(s) \ d^2 s \tag{19}$$

$$\bar{s}_\perp(D_s) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_\perp \ p_s(s) \ W(s) \ H(s) \ d^2 s \tag{20}$$

$$\bar{s^2}_o(D_s) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_o^2 \ p_s(s) \ W(s) \ H(s) \ d^2 s \tag{21}$$

$$\bar{s^2}_\perp(D_s) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_\perp^2 \ p_s(s) \ W(s) \ H(s) \ d^2 s \tag{22}$$

$$\bar{s}_{o\perp}(D_s) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s_o \ s_\perp \ p_s(s) \ W(s) \ H(s) \ d^2 s \ . \tag{23}$$

Since $W(s)$ can be decomposed into an sum of two terms (Equation (17)), all these integrals can be expressed as sums of two terms as

$$N = -\sin\theta_o I_{s_o} + \cos\theta_o I \tag{24}$$

$$\bar{s}_o(D_s) = \frac{1}{N}(-\sin\theta_o I_{s_o^2} + \cos\theta_o I_{s_o}) \tag{25}$$

$$\bar{s}_\perp(D_s) = \frac{1}{N}(-\sin\theta_o I_{s_o s_\perp} + \cos\theta_o I_{s_\perp}) \tag{26}$$

$$\bar{s^2}_o(D_s) = \frac{1}{N}(-\sin\theta_o I_{s_o^3} + \cos\theta_o I_{s_o^2}) \tag{27}$$

$$\bar{s^2}_\perp(D_s) = \frac{1}{N}(-\sin\theta_o I_{s_o s_\perp^2} + \cos\theta_o I_{s_\perp^2}) \tag{28}$$

$$\bar{s}_{o\perp}(D_s) = \frac{1}{N}(-\sin\theta_o I_{s_o^2 s_\perp} + \cos\theta_o I_{s_o s_\perp})\ , \tag{29}$$

where each of the component integrals are defined as

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_s(s)\ H(s)\ d^2s \tag{30}$$

$$I_{s_o} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o\ p_s(s)\ H(s)\ d^2s \tag{31}$$

$$I_{s_\perp} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_\perp\ p_s(s)\ H(s)\ d^2s \tag{32}$$

$$I_{s_o^2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o^2\ p_s(s)\ H(s)\ d^2s \tag{33}$$

$$I_{s_\perp^2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_\perp^2\ p_s(s)\ H(s)\ d^2s \tag{34}$$

$$I_{s_o s_\perp} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o\ s_\perp\ p_s(s)\ H(s)\ d^2s \tag{35}$$

$$I_{s_o^2 s_\perp} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o^2\ s_\perp\ p_s(s)\ H(s)\ d^2s \tag{36}$$

$$I_{s_o s_\perp^2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o\ s_\perp^2\ p_s(s)\ H(s)\ d^2s \tag{37}$$

$$I_{s_o^3} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_o^3\ p_s(s)\ H(s)\ d^2s. \tag{38}$$

We describe the solution to these integrals in the following section. Given the solution for the moments, we obtain the following covariance matrix entries

$$\sigma_o^2(D_s) = \bar{s^2}_o(D_s) - \bar{s}_o(D_s)^2 \tag{39}$$

$$\sigma_\perp^2(D_s) = \bar{s^2}_\perp(D_s) - \bar{s}_\perp(D_s)^2 \tag{40}$$

$$c_{o\perp}(D_s) = \bar{s}_{o\perp}(D_s) - \bar{s}_o(D_s)\bar{s}_\perp(D_s)\ . \tag{41}$$

Finally, the average vector and covariance matrix in the canonical surface-aligned coordinate frame is obtained by applying the inverse rotation operator:

$$\bar{s}_x(D_s) = \bar{s}_o(D_s)\cos\phi_o - \bar{s}_\perp(D_s)\sin\phi_o \tag{42}$$

$$\bar{s}_y(D_s) = \bar{s}_o(D_s)\sin\phi_o + \bar{s}_\perp(D_s)\cos\phi_o \tag{43}$$

$$\sigma_x^2(D_s) = \sigma_o^2(D_s)\cos^2\phi_o + \sigma_\perp^2(D_s)\sin^2\phi_o - 2\ c_{o\perp}(D_s)\cos\phi_o\sin\phi_o \tag{44}$$

$$\sigma_y^2(D_s) = \sigma_o^2(D_s)\sin^2\phi_o + \sigma_\perp^2(D_s)\cos^2\phi_o + 2\ c_{o\perp}(D_s)\cos\phi_o\sin\phi_o \tag{45}$$

$$c_{xy}(D_s) = (\sigma_o^2(D_s) - \sigma_\perp^2(D_s))\cos\phi_o\sin\phi_o + c_{o\perp}(D_s)(\cos^2\phi_o - \sin^2\phi_o). \tag{46}$$

5 Solving for Integrals in Equations (30) to (38)

Given the PDF

$$p_s(s) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}[s_o - \bar{s}_0, s_\perp - \bar{s}_\perp]\Sigma^{-1}[s_o - \bar{s}_0, s_\perp - \bar{s}_\perp]^T\right) \ , \tag{47}$$

the inverse of the covariance matrix expressed in the coordinate frame of $(\vec{o}, \vec{\perp})$ is

$$\Sigma^{-1} = \frac{1}{\sigma_o^2\sigma_\perp^2 - c_{o\perp}^2} \begin{pmatrix} \sigma_\perp^2 & -c_{o\perp} \\ -c_{o\perp} & \sigma_o^2 \end{pmatrix}$$

and by expanding the argument in the exponential in Equation (47) we obtain

$$p_s(s) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{a}{2}(s_o - \bar{s}_o)^2 - b(s_o - \bar{s}_o)(s_\perp - \bar{s}_\perp) - \frac{c}{2}(s_\perp - \bar{s}_\perp)^2\right), \tag{48}$$

where

$$a = \frac{\sigma_\perp^2}{\sigma_o^2\sigma_\perp^2 - c_{o\perp}^2}, \quad b = -\frac{c_{o\perp}}{\sigma_o^2\sigma_\perp^2 - c_{o\perp}^2}, \quad c = \frac{\sigma_o^2}{\sigma_o^2\sigma_\perp^2 - c_{o\perp}^2}$$

and we have

$$\sigma_o^2 = \frac{c}{ac - b^2}, \quad c_{o\perp} = -\frac{b}{ac - b^2}, \quad \sigma_\perp^2 = \frac{a}{ac - b^2}.$$

From the partial derivative of p_s

$$\frac{\partial p_s(s)}{\partial s_o} = (-a(s_o - \bar{s}_o) - b(s_\perp - \bar{s}_\perp))\ p_s(s) \tag{49}$$

$$\frac{\partial p_s(s)}{\partial s_\perp} = (-b(s_o - \bar{s}_o) - c(s_\perp - \bar{s}_\perp))\ p_s(s) \tag{50}$$

we obtain

$$s_o\ p_s(s) = -c_{o\perp}\frac{\partial p_s(s)}{\partial s_\perp} - \sigma_o^2\frac{\partial p_s(s)}{\partial s_o} + \bar{s}_o\ p_s(s) \tag{51}$$

$$s_\perp\ p_s(s) = -\sigma_o^2\frac{\partial p_s(s)}{\partial s_\perp} + c_{o\perp}\frac{\partial p_s(s)}{\partial s_o} + \bar{s}_\perp\ p_s(s)\ , \tag{52}$$

and after multiplication by $H(s)$ and integration by parts, we obtain

$$s_o\ p_s(s)\ H(s) = -c_{o\perp}\frac{\partial p_s(s)H(s)}{\partial s_\perp} + c_{o\perp}p_s(s)\frac{\partial H(s)}{\partial s_\perp} - \sigma_o^2\frac{\partial p_s(s)H(s)}{\partial s_o} + \sigma_o^2 p_s(s)\frac{\partial H(s)}{\partial s_o} + \bar{s}_o\ p_s(s) \tag{53}$$

and

$$s_\perp\ p_s(s)\ H(s) = -\sigma_o^2\frac{\partial p_s(s)H(s)}{\partial s_\perp} + \sigma_o^2 p_s(s)\frac{\partial H(s)}{\partial s_\perp} + c_{o\perp}\frac{\partial p_s(s)H(s)}{\partial s_o} + -c_{o\perp}p_s(s)\frac{\partial H(s)}{\partial s_o} + \bar{s}_\perp\ p_s(s)\,. \tag{54}$$

We substitute the integrands in Equations (30)-(38) with these results and leverage the following properties of the terms in Equations (53) and (54) to aid in our analytic integration:

- Since H depends only on s_o,

$$\frac{\partial H(s)}{\partial s_\perp} = 0\ ; \tag{55}$$

- Since p_s tends toward 0 at infinity and H is a binary function,

$$\int_{-\infty}^{+\infty} \frac{\partial p_s(s)H(s)}{\partial s_o}\, ds_o = 0 \text{ and} \tag{56}$$

$$\int_{-\infty}^{+\infty} \frac{\partial p_s(s)H(s)}{\partial s_\perp}\, ds_\perp = 0\ ; \tag{57}$$

- The derivative of a Heaviside function is a Dirac, and so

$$\int_{-\infty}^{+\infty} p_s(s)\ \frac{\partial H(s)}{\partial s_o}\ ds_o = -p_s(s_o = \cot\theta_o, s_\perp)\ ; \tag{58}$$

and by integrating over $s_\perp$ we obtain

$$E_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_s(s)\ \frac{\partial H(s)}{\partial s_o}\ d^2s = -\frac{\sigma_o}{\sqrt{2\pi}}\exp\left(-\frac{(\cot\theta_o - \bar{s}_o)^2}{2\sigma_o^2}\right)\ . \tag{59}$$

- The integral below can be simplified by noting, as mentioned above, that H depends only on s_o:

$$E_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_s(s)\ H(s)\ d^2s = 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\cot\theta_o - \bar{s}_o}{\sqrt{2}\sigma_0}\right)\ . \tag{60}$$

By applying Equations (55) to (60), we can integrate Equations (30)-(38) and obtain

$$I = E_2 \tag{61}$$

$$I_{s_o} = -E_1 + \bar{s}_o\ E_2 \tag{62}$$

$$I_{s_\perp} = -\frac{c_{o\perp}}{\sigma_o^2}E_1 + \bar{s}_\perp E_2 \tag{63}$$

$$I_{s_o^2} = -\cot\theta_o\ E_1 + \sigma_o^2\ E_2 + \bar{s}_o\ I_{s_o} \tag{64}$$

$$I_{s_\perp^2} = -\frac{c_{o\perp}}{\sigma_o^2}(\bar{s}_\perp + \frac{c_{o\perp}}{\sigma_o^2}(\cot\theta_o - \bar{s}_0))E_1 + \sigma_\perp^2\ E2 + \bar{s}_\perp\ I_{s_\perp} \tag{65}$$

$$I_{s_o s_\perp} = -\frac{c_{o\perp}}{\sigma_o^2}\ \cot\theta_o\ E_1 + c_{o\perp}\ E_2 + \bar{s}_\perp I_{s_o} \tag{66}$$

$$I_{s_o^2 s_\perp} = -\frac{c_{o\perp}}{\sigma_o^2}\ \cot^2\theta_o\ E_1 + 2\ c_{o\perp}\ I_{s_o} + \bar{s}_\perp\ I_{s_o^2} \tag{67}$$

$$I_{s_o s_\perp^2} = -(\frac{\sigma_o^2\sigma_\perp^2 - c_{o\perp}^2}{\sigma_o^2} + (\bar{s}_\perp + \frac{c_{o\perp}}{\sigma_o^2}(\cot\theta_o - \bar{s}_0))^2)\ E_1 + \bar{s}_o\ I_{s_\perp^2} + 2\ c_{o\perp}\ I_{s_\perp} \tag{68}$$

$$I_{s_o^3} = 2\ \sigma_o^2\ I_{s_o} - \cot^2\theta_o\ E_1 + \bar{s}_o\ I_{s_o^2}\ . \tag{69}$$

6 Implementation and Pseudocode

We propose an example pseudocode implementation (see following page) to compute the parameters of D_s given a p_s distribution and (distant) point of view (θ, ϕ) as input. Table 3 illustrates the results of our approximation.

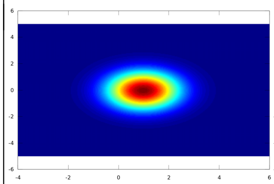
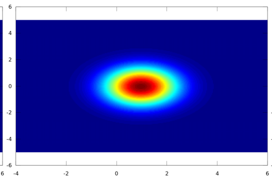
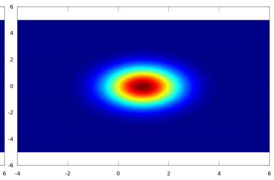
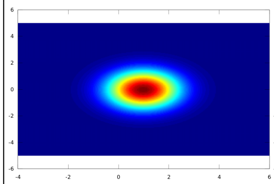
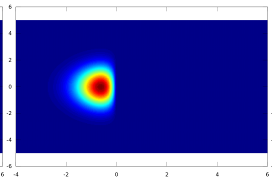
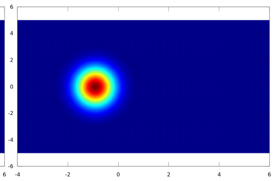
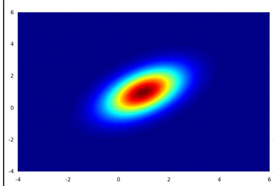
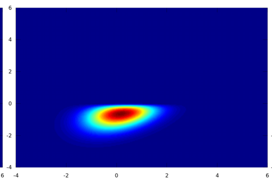
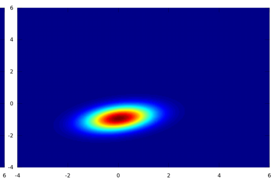
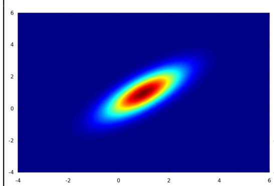
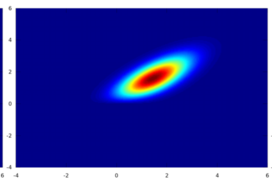
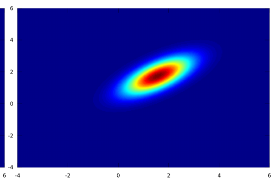
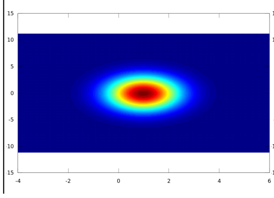
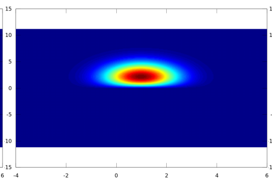
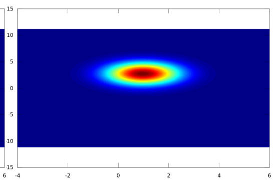
Parameters of p_s and point of view	Surface distribution p_s	View-dependent distribution D_s	Gaussian approximation of D_s
$\bar{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\theta = 0, \phi = 0$			
$\bar{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\theta = 1.56, \phi = 0$			
$\bar{s} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ $\theta = 1.56, \phi = \frac{\pi}{2}$			
$\bar{s} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix}$ $\theta = 1.56, \phi = -\frac{\pi}{2}$			
$\bar{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ $\theta = 1.56, \phi = -\frac{\pi}{2}$			

Table 3: Comparisons of our analytic Gaussian approximations to the exact D_s distributions for several microsurface and viewing configurations.

```
void D_s(
    // input: surface slope distribution p_s
    double ps_sx0, double ps_sy0, double ps_varx, double ps_vary, double ps_cxy,
    // input: view direction
    double theta, double phi,
    // output: visible slope distribution D_s
    double& ds_sx0, double& ds_sy0, double& ds_varx, double& ds_vary, double& ds_cxy) {
      // pre-computations
      double cos_theta = cos(theta);
      double sin_theta = sin(theta);
      double cot_theta = cos_theta / sin_theta;
      double cos_phi = cos(phi);
      double sin_phi = sin(phi);

      // rotate p_s statistics to view-oriented basis
      double ps_para0 = ps_sx0 * cos_phi + ps_sy0 * sin_phi;
      double ps_ortho0= -ps_sx0 * sin_phi + ps_sy0 * cos_phi;
      double ps_varpara = ps_varx*cos_phi*cos_phi + ps_vary*sin_phi*sin_phi + 2*ps_cxy*cos_phi*sin_phi;
      double ps_varortho= ps_varx*sin_phi*sin_phi + ps_vary*cos_phi*cos_phi - 2*ps_cxy*cos_phi*sin_phi;
      double ps_cparaortho= (ps_vary-ps_varx)*cos_phi*sin_phi + ps_cxy * (cos_phi*cos_phi-sin_phi*sin_phi);

      // compute E1 and E2
      double E1 = sqrt(0.5*ps_varpara/M_PI) * exp(-0.5 * (cot_theta-ps_para0)*(cot_theta-ps_para0)/ps_varpara);
      double E2 = 1.0 - 0.5 * erfc( (cot_theta-ps_para0) / sqrt(2.0*ps_varpara) );

      // compute Integrals I
      double I = E2;
      double I_para = -E1 + ps_para0 * E2;
      double I_ortho = -ps_cparaortho/ps_varpara*E1 + ps_ortho0 * E2;
      double I_para2 = -cot_theta * E1 + ps_varpara*E2 + ps_para0*I_para;
      double I_ortho2 = -ps_cparaortho/ps_varpara*(ps_ortho0+ps_cparaortho/ps_varpara*(cot_theta-ps_para0))*E1
                        + ps_varortho*E2 + ps_ortho0*I_ortho;
      double I_para_ortho = -ps_cparaortho/ps_varpara*cot_theta*E1 + ps_cparaortho*E2 + ps_ortho0*I_para;
      double I_para2_ortho = -ps_cparaortho/ps_varpara*cot_theta*cot_theta*E1 + 2.0*ps_cparaortho*I_para
                             + ps_ortho0*I_para2;
      double I_para_ortho2 = -((ps_varpara*ps_varortho-ps_cparaortho*ps_cparaortho)/ps_varpara
                             + (ps_ortho0+ps_cparaortho/ps_varpara*(cot_theta-ps_para0))*(ps_ortho0+
                             ps_cparaortho/ps_varpara*(cot_theta-ps_para0)))*E1 + ps_para0 * I_ortho2 + 2.0 *
                             ps_cparaortho * I_ortho;
      double I_para3 = 2.0 * ps_varpara*I_para - cot_theta*cot_theta * E1 + ps_para0*I_para2;

      // compute D_s in view-oriented basis
      double N = -sin_theta*I_para + cos_theta*I;
      double ds_para0 = (-sin_theta*I_para2 + cos_theta*I_para) / N;
      double ds_ortho0 = (-sin_theta*I_para_ortho + cos_theta*I_ortho) / N;
      double ds_varpara = (-sin_theta*I_para3 + cos_theta*I_para2) / N - ds_para0*ds_para0;
      double ds_varortho = (-sin_theta*I_para_ortho2 + cos_theta*I_ortho2) / N - ds_ortho0*ds_ortho0;
      double ds_cparaortho = (-sin_theta*I_para2_ortho + cos_theta*I_para_ortho) / N - ds_para0*ds_ortho0;

      // rotate D_s statistics to canonic basis
      ds_sx0 = ds_para0 * cos_phi - ds_ortho0 * sin_phi;
      ds_sy0 = ds_para0 * sin_phi + ds_ortho0 * cos_phi;
      ds_varx = ds_varpara * cos_phi*cos_phi + ds_varortho * sin_phi*sin_phi - 2 * ds_cparaortho*cos_phi*sin_phi;
      ds_vary = ds_varpara * sin_phi*sin_phi + ds_varortho * cos_phi*cos_phi + 2 * ds_cparaortho*cos_phi*sin_phi;
      ds_cxy = (ds_varpara-ds_varortho)*cos_phi*sin_phi + ds_cparaortho * (cos_phi*cos_phi - sin_phi*sin_phi);
}
```

References

[RDP05] Vincent Ross, Denis Dion, and Guy Potvin. Detailed analytical approach to the gaussian surface bidirectional reflectance distribution function specular component applied to the sea surface. *J. Opt. Soc. Am. A*, 22(11):2442–2453, Nov 2005.