4 Analog modulation

4.1 Modulation formats

The message waveform is represented by a low-pass real signal $m(t)$ such that

$$M(f) = 0 \quad |f| \geq W$$

where W is the message bandwidth. $m(t)$ is called the **modulating signal**.
Carrier modulation: Reversible transformation of $m(t)$ into a bandpass signal $x(t)$ centered around $f_c \gg W$ (f_c is the carrier frequency). Demodulation is the inverse transformation of $x(t)$ into $m(t)$. $x(t)$ is the **modulated signal**.

$$x(t) = x_I(t) \cos 2\pi f_c t - x_Q(t) \sin 2\pi f_c t$$

Two types of modulation schemes:

- Linear modulation: linear relationship between the modulated signal and the message signal (ex: AM, DSB-SC, SSB, VSB).
- Angle modulation: the angle of the carrier wave is varied according to the message signal (ex: FM,PM).

4.2 Linear modulation

a) Amplitude modulation AM

$$\begin{aligned}
x(t) &= A_c\big(1 + k_a m(t)\big) \cos 2\pi f_c t \\
&= A_c\big(1 + \mu m_n(t)\big) \cos 2\pi f_c t \\
x_I(t) &= A_c\big(1 + \mu m_n(t)\big) \qquad x_Q(t) = 0 \\
m_n(t) &= \frac{m(t)}{\max |m(t)|}
\end{aligned}$$

k_a is the *amplitude sensitivity*, $\mu = k_a \max |m(t)|$ is *modulation index (factor)* ($0 \leq \mu \leq 1$).
Exercise: Plot a graph representing $x(t)$ and identify the message signal $m(t)$.

The Fourier transform of $x(t)$ is given by

$$X(f) = \frac{A_c}{2}\big[\delta(f-f_c) + \mu M_n(f-f_c)\big] + \frac{A_c}{2}\big[\delta(f+f_c) + \mu M_n(f+f_c)\big]$$

where $M_n(f) = \mathcal{F}\{m_n(t)\}$.

Exercise: Plot $X(f)$ and find the condition on f_c to avoid distortion.

Let W be the bandwidth of $M(f) = \mathcal{F}\{m(t)\}$ and $M_n(f)$, then the bandwidth of the modulated signal $x(t)$ is $B = 2W$.

Generation of AM:

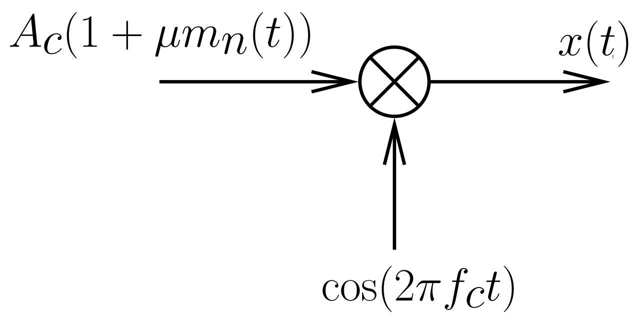


Figure 4: Generation of an AM modulated signal

Demodulation of AM:

- Synchronous (coherent) detection
- Envelope detection

Coherent detection:

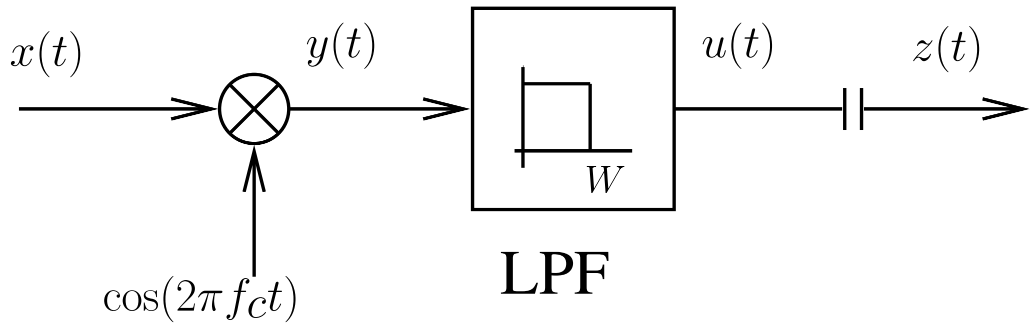


Figure 5: Coherent detection of an AM modulated signal

$$
\begin{aligned}
y(t) &= A_c\big(1+\mu m_n(t)\big)\cos^2(2\pi f_c t) \\
&= \frac{A_c}{2}\big(1+\mu m_n(t)\big) + \frac{A_c}{2}\left(1+\mu m_n(t)\right)\cos(4\pi f_c t) \\
u(t) &= \frac{A_c}{2}\big(1+\mu m_n(t)\big) \quad \text{(Output of low pass)} \\
z(t) &= \frac{A_c \mu}{2} m_n(t) = \frac{A_c}{2} k_a m(t)
\end{aligned}
$$

Exercise: Assume that the local carrier is not synchronized to the signal, show that the output of the coherent demodulator is distorted.

Envelope detection:

The output of an envelope detector is the natural envelope $|\tilde{x}(t)|$. Envelope detection is feasible since $1 + \mu m_n(t) \geq 0$.

Draw an example of a RC circuit implementing envelope detection. What conditions on the resistors and capacitor are necessary to ensure the envelope detector will function properly ?

Envelope detection is feasible due to the inclusion of the carrier but the transmission of the carrier represents a waste of power (contains no information).

b) Double sideband-suppressed carrier: DSB-SC

$$x(t) = A_c m(t) \cos 2\pi f_c t$$
$$x_I(t) = A_c m(t) \qquad x_Q(t) = 0$$

Exercise: Plot $x(t)$ and identify the message signal $m(t)$. What phenomenon characterizes DSB-SC ?

The Fourier transform of $x(t)$ is given by

$$X(f) = \frac{A_c}{2} M(f - f_c) + \frac{A_c}{2} M(f + f_c)$$

where $M(f) = \mathcal{F}\{m(t)\}$.

Exercise: Plot $X(f)$ and find the condition on f_c to avoid distortion.

Let W be the bandwidth of $M(f) = \mathcal{F}\{m(t)\}$, then the bandwidth of the modulated signal $x(t)$ is $B = 2W$.

Generation of DSB-SC:

Draw a block diagram of a DSB-SC modulator.

Demodulation of DSB-SC:

- **Only** Synchronous (coherent) detection

Coherent detection: Using a local carrier synchronized to the received signal carrier, draw a block diagram of a coherent detector for DSB-SC.

Envelope detection:

The output of an envelope detector is the natural envelope $|\tilde{x}(t)| = A_c|m(t)| \neq A_c m(t)$.

AM and DSB-SC are wasting bandwidth, thus filtering of sidebands to reduce bandwidth results in a more bandwidth efficient scheme.

c) Single sideband modulation: SSB

$$x(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t - \frac{A_c}{2} \hat{m}(t) \sin 2\pi f_c t \qquad \text{(Upper sideband SSB)}$$
$$x(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t + \frac{A_c}{2} \hat{m}(t) \sin 2\pi f_c t \qquad \text{(Lower sideband SSB)}$$
$$x_I(t) = \frac{A_c}{2} m(t) \qquad x_Q(t) = \pm \frac{A_c}{2} \hat{m}(t)$$

The Fourier transform of $x(t)$ is given by

$$\begin{aligned} X(f) &= \frac{A_c}{2} \left\{ \frac{1}{2} M(f - f_c) + \frac{1}{2} M(f + f_c) - \frac{1}{2j} \left[\hat{M}(f - f_c) - \hat{M}(f + f_c) \right] \right\} \\ &= \frac{A_c}{4} M(f - f_c) \big(1 + \operatorname{sgn}(f - f_c)\big) + \frac{A_c}{4} M(f + f_c) \big(1 - \operatorname{sgn}(f + f_c)\big) \end{aligned}$$

where $M(f) = \mathcal{F}\{m(t)\}$.

Exercise: Plot $X(f)$.

Let W be the bandwidth of $M(f) = \mathcal{F}\{m(t)\}$, then the bandwidth of the modulated signal $x(t)$ is $B = W$.

Generation of SSB:

- Using a Hilbert transformer (but wideband $\pi/2$ shifter difficult to implement)

- Using sideband filtering (but demands a very sharp filter if $M(f)$ contains very low frequencies components, hence Vestigial sideband modulation (VSB) is also used)

Demodulation of SSB:

- **Only** Synchronous (coherent) detection
- Envelope detection by adding a strong carrier to the SSB signal but not a regular SSB anymore (see VSB subsection)

Coherent detection: Using a local carrier synchronized to the received signal carrier, draw a block diagram of a coherent detector for SSB.

SSB is difficult to implement if the message signal $m(t)$ has a large bandwidth and it is rich in low frequency components. In this case vestigial sideband modulation is used.

d) Vestigial sideband modulation: VSB

$$x(t) = \frac{A_c}{2} K m(t) \cos 2\pi f_c t - \frac{A_c}{2} m_\nu(t) \sin 2\pi f_c t$$
$$x_I(t) = \frac{A_c}{2} K m(t) \qquad x_Q(t) = \frac{A_c}{2} m_\nu(t) = \frac{A_c}{2} m(t) * h_\nu(t)$$

Rationale (generation of VSB): Assume that a DSB-SC signal is passed through a general bandpass filter to alter its sidebands (ex. for SSB half of the sidebands filtered out).

The Fourier transforms of the DSB-SC signal $y(t)$ and $x(t)$ are given by

$$Y(f) = \frac{A_c}{2}\big\{M(f - f_c) + M(f + f_c)\big\} \quad \text{DSB-SC}$$
$$X(f) = \frac{A_c}{2}\big\{M(f - f_c) + M(f + f_c)\big\} H(f)$$

where $M(f) = \mathcal{F}\{m(t)\}$. The filter $H(f)$ must have spectral characteristics such that the original message signal $m(t)$ can be recovered from $x(t)$ by coherent detection.
Demodulation of VSB:

- **Only** Synchronous (coherent) detection
- Envelope detection by adding a strong carrier to the VSB signal but not a regular VSB anymore

Coherent detection:

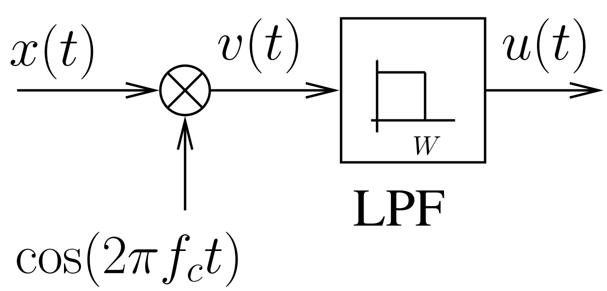


Figure 6: Coherent detection of VSB

$v(t) = x(t) \cos 2\pi f_c t$

$$
\begin{aligned}
V(f) &= \frac{1}{2}\big[X(f-f_c)+X(f+f_c)\big]\\
&= \frac{A_c}{4}\Big\{\big[M(f-2f_c)+M(f)\big]H(f-f_c)+\big[M(f)+M(f+2f_c)\big]H(f+f_c)\Big\}\\
&= \frac{A_c}{4}M(f)\big[H(f-f_c)+H(f+f_c)\big]+\frac{A_c}{4}\big[M(f-2f_c)H(f-f_c)+M(f+2f_c)H(f+f_c)\big]
\end{aligned}
$$

The output of the lowpass filter is given by

$$U(f)=\frac{A_c}{4}M(f)\big[H(f-f_c)+H(f+f_c)\big]$$

For distortionless transmission

$$U(f)=\frac{A_c}{4}KM(f) \qquad \text{i.e. } u(t)=\frac{A_c}{4}Km(t)$$

where K is a constant. Thus the filter $H(f)$ must satisfy the so-called vestigial symmetry condition:

$$H(f-f_c)+H(f+f_c)=K=\text{const.} \qquad |f|\le W$$

Time domain representation of VSB signals:

$$x(t)=h(t)*y(t) \quad \Longrightarrow \tilde{x}(t)=\frac{1}{2}\tilde{h}(t)*\tilde{y}(t)$$

Since $y(t)=A_c\cos(2\pi f_c t)m(t)$, its complex envelope is given by

$$\tilde{y}(t)=A_c m(t) \quad \Longrightarrow \tilde{Y}(f)=A_c M(f)$$

The Fourier transform of $\tilde{h}(t)$ is

$$\tilde{H}(f)=2H(f+f_c) \quad f>-f_c$$

hence

$$\tilde{X}(f)=\frac{1}{2}\tilde{H}(f)\tilde{Y}(f)=A_cM(f)H(f+f_c) \quad f>-f_c \tag{9}$$

The inphase and quadrature components of $x(t)$ are

$$x_I(t)=\Re\{\tilde{x}(t)\}=\frac{1}{2}\left[\tilde{x}(t)+\tilde{x}^*(t)\right]$$

$$x_Q(t) = \Im\{\tilde{x}(t)\} = \frac{1}{2j}\left[\tilde{x}(t) - \tilde{x}^*(t)\right]$$

their Fourier transform are then given by

$$\begin{aligned}
X_I(f) &= \frac{1}{2}\left[\tilde{X}(f) + \tilde{X}^*(-f)\right] \\
&= \frac{A_c}{2}\left[M(f)H(f+f_c) + M^*(-f)H^*(-f+f_c)\right] \quad \text{from (9)} \\
&= \frac{A_c}{2}M(f)\left[H(f+f_c) + H(f-f_c)\right] \quad \text{(since } m(t) \text{ and } h(t) \text{ are real.)} \\
&= \frac{A_c}{2}M(f)K \quad \text{from the vestigial symmetry} \qquad (10) \\
X_Q(f) &= \frac{1}{2j}\left[\tilde{X}(f) - \tilde{X}^*(-f)\right] \\
&= \frac{A_c}{2j}M(f)\left[H(f+f_c) - H(f-f_c)\right] \\
&= \frac{A_c}{2}M(f)H_\nu(f) \qquad (11)
\end{aligned}$$

where the filter $H_\nu(f)$ is defined as

$$H_\nu(f) = \frac{1}{j}\left[H(f+f_c) - H(f-f_c)\right]$$

From (10) and (11), the inphase and quadrature components of a VSB signal are given by

$$\begin{aligned}
x_I(t) &= \frac{A_c}{2}Km(t) \\
x_Q(t) &= \frac{A_c}{2}m(t) * h_\nu(t) = \frac{A_c}{2}m_\nu(t)
\end{aligned}$$

Bandwidth of $m(t)$: W
Bandwidth of $x(t)$: $W < B < 2W$ (typically $B_{\text{VSB}} = 1.25B_{\text{SSB}}$)

If $H_\nu(f) = -j\,\text{sgn}(f)$, then we obtain the upper sideband SSB.

Envelope detection of SSB and VSB:

We add a strong carrier $A_c'\cos(2\pi f_c t)$ (with $A_c' \gg \frac{A_c}{2}K|m(t)|, \frac{A_c}{2}|m_\nu(t)|$) such that the signal to be demodulated is given by

$$x(t) = \left(A_c' + \frac{A_c}{2}Km(t)\right)\cos(2\pi f_c t) - \frac{A_c}{2}m_\nu(t)\sin(2\pi f_c t)$$

$$= A_c^{'} \left(1 + \frac{A_c K}{2A_c^{'}} m(t)\right) \cos(2\pi f_c t) - \frac{A_c^{'}}{2} \frac{A_c}{A_c^{'}} m_\nu(t) \sin(2\pi f_c t)$$

For simplicity assume $K = 1$, and let $\frac{A_c}{A_c^{'}} = k_a$

$$x(t) = A_c^{'} \left(1 + \frac{k_a}{2} m(t)\right) \cos(2\pi f_c t) - \frac{A_c^{'}}{2} k_a m_\nu(t) \sin(2\pi f_c t) \tag{12}$$

$$\tilde{x}(t) = A_c^{'} \left(1 + \frac{k_a}{2} m(t)\right) + jA_c^{'} \frac{k_a}{2} m_\nu(t) \quad \text{(complex envelope of } x(t))$$

The output of an envelope detector is the natural envelope of the input, hence the output of the envelope detector is given by

$$\begin{aligned} |\tilde{x}(t)| &= \left[A_c^{'2} \left(1 + \frac{k_a}{2} m(t)\right)^2 + A_c^{'2} \frac{k_a^2}{4} m_\nu^2(t)\right]^{1/2} \\ &= A_c^{'} \left(1 + \frac{k_a}{2} m(t)\right) \left[1 + \frac{\frac{k_a^2}{4} m_\nu^2(t)}{\left(1 + \frac{k_a}{2} m(t)\right)^2}\right]^{1/2} \\ &\approx A_c^{'} \left(1 + \frac{k_a}{2} m(t)\right) \end{aligned}$$

since $\left|\frac{k_a}{2} m(t)\right| \ll 1$ and $\left|\frac{k_a}{2} m_\nu(t)\right| \ll 1$. This method is used in TV systems. Distortion due to the envelope detection of VSB is reduced by reducing k_a ensuring the conditions $\left|\frac{k_a}{2} m(t)\right| \ll 1$ and $\left|\frac{k_a}{2} m_\nu(t)\right| \ll 1$.

4.3 Multiplexing

The purpose of multiplexing is to transmit several signals $\{m_1(t), \ldots, m_N(t)\}$ at the same time by the use of a single communication system. This can be achieved by combining the signals into one signal $s(t)$ such that each of the signals $m_k(t)$ can be extracted from $s(t)$. In this section, we present two types of multiplexing; quadrature carrier multiplexing and frequency division multiplexing (FDM). A third multiplexing technique called Time Division Multiplexing (TDM) will be considered in the context of signal sampling.

a) Quadrature carrier multiplexing

Since $\cos(2\pi f_c t)$ and $\sin(2\pi f_c(t)$ are orthogonal functions, the principle of quadrature multiplexing of two signals is to transmit one signal using a carrier of the form $\cos(2\pi f_c t)$ and to transmit the other signal using a carrier of the form $\sin(2\pi f_c t)$. Let $m_1(t)$ and $m_2(t)$ be two lowpass signals with bandwidth W. The schemes of multiplexing and demultiplexing follows:

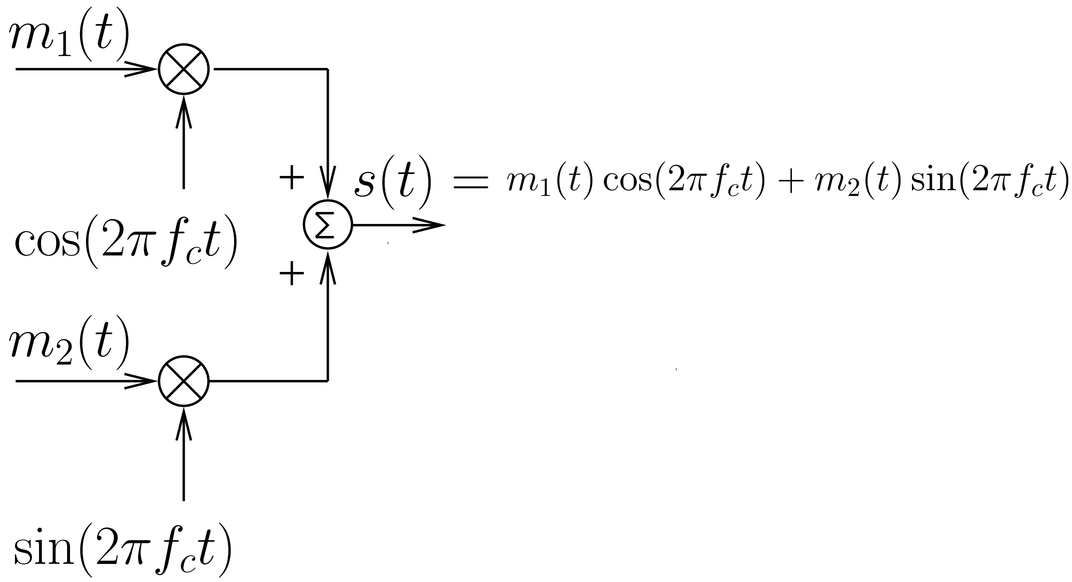
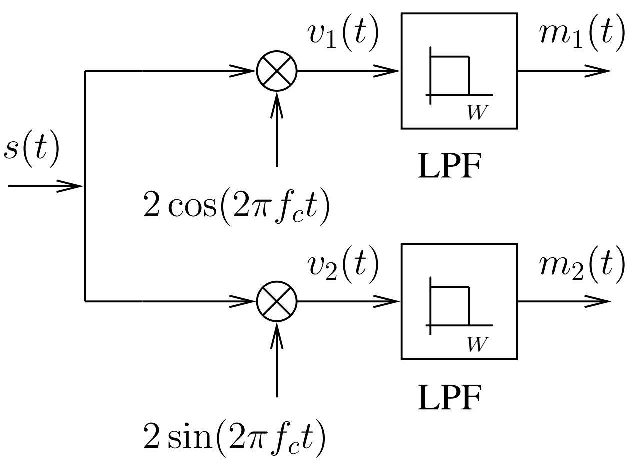


Figure 7: Quadrature carrier multiplexing



Complete the proof that the scheme of Fig. 8 is a quadrature carrier demultiplexer:

$$\begin{aligned} v_1(t) &= 2s(t)\cos(2\pi f_c t) \\ &= \underbrace{\qquad}_{\text{lowpass term}} + \underbrace{\qquad\qquad}_{\text{bandpass term}} \end{aligned}$$

$$\begin{aligned} v_2(t) &= 2s(t)\sin(2\pi f_c t) \\ &= \underbrace{\qquad}_{\text{lowpass term}} + \underbrace{\qquad\qquad}_{\text{bandpass term}} \end{aligned}$$

Since the bandpass terms are removed by the lowpass filter, the result follows.

Figure 8: Quadrature carrier demultiplexing

b) Frequency division multiplexing (FDM)

The principle of frequency division multiplexing is to modulate each signal $m_i^0(t)$ using a different carrier frequency f_{c_i} such that the spectrum of the modulated signals $x_i(t)$ do not overlap. Then a FDM signal is obtained by adding the modulated signal yielding a signal with a higher bandwidth. The multiplexed signal can be further modulated before transmission. Hence the modulation of the signals $m_i(t)$ (bandlimited version of $m_i^0(t)$) to be multiplexed is called **sub-modulation** and the carrier $\{f_{c_i}\}_{i=1,\dots,N}$ are sub-carriers. In a FDM system, the sub-carriers are selected such that the spectrum of the sub-modulated signals do not overlap. Therefore the original message signals $m_i^0(t)$ have to be passed first through a lowpass filter that limit them to a predetermined bandwidth W. If the original message signals are already bandlimited to W, no lowpass filtering is required. Let B be the bandwidth of each of the sub-modulated signals $x_i(t)$. To avoid overlapping of the spectrum of the sub-modulated signals (and hence to ensure distortionless demultiplexing), we must have

$$|f_{c_i} - f_{c_k}| \geq B$$

FDM multipler and demultiplexer follows:

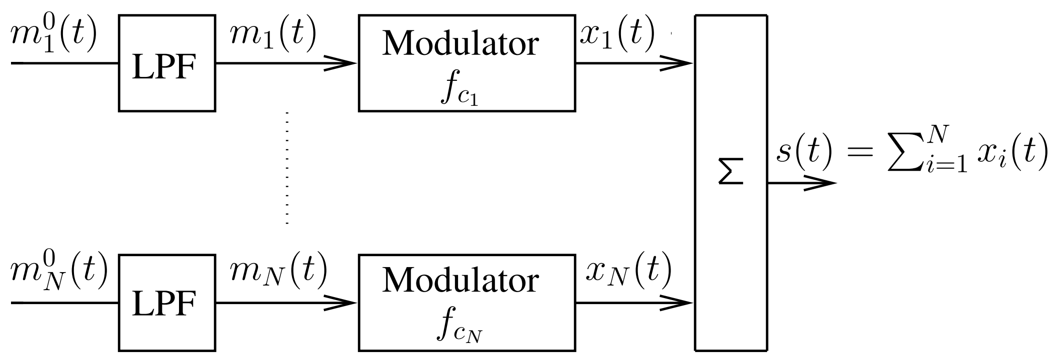


Figure 9: Frequency Division multiplexing

example of FDM:

$$s(t) = \sum_{i=1}^{N} \big(1 + k_{a_i} m_i(t)\big) \cos(2\pi f_{c_i} t) \quad \text{AM sub-modulation (BW of } s(t)\text{:} \geq 2NW)$$

$$s(t) = \sum_{i=1}^{N} \frac{A_c}{2} \big[m_i(t) \cos(2\pi f_{c_i} t) - \hat{m}_i(t) \sin(2\pi f_{c_i} t)\big] \quad \text{SSB sub-modulation (BW of } s(t)\text{: } \geq NW)$$

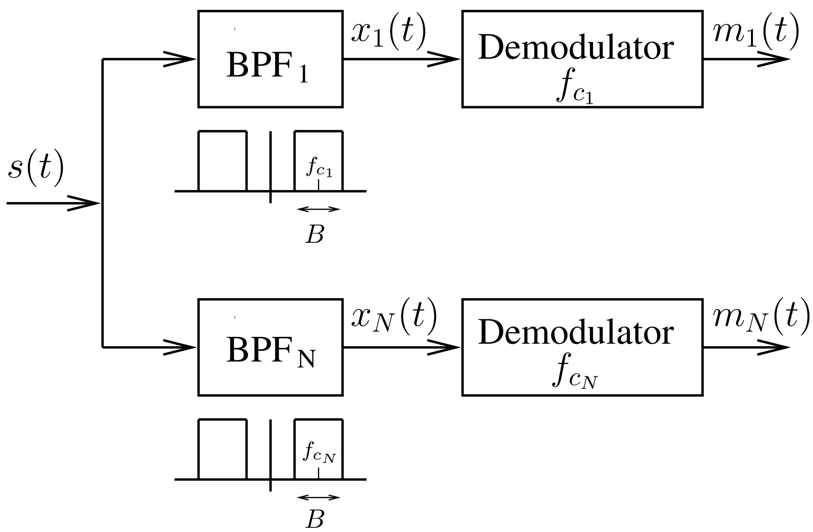


Figure 10: Frequency Division demultiplexing

Complete the proof that the scheme of Fig. 10 is a FDM demultiplexer:

$$s(t) = \underbrace{\qquad\qquad\qquad}_{\text{bandpass term around } f_{c_i}} + \underbrace{\qquad\qquad\qquad}_{\text{sum of bandpass terms around } f_{c_j} (j \neq i)}$$

The bandpass filter around f_{c_i} keeps the bandpass term around f_{c_i} (namely $x_i(t)$) and removes all the other bandpass terms. It is seen that non-overlapping of the spectrums of $x_i(t)$ is needed to avoid distortion. Then each $x_i(t)$ can be demodulated.

Note that for SSB sub-modulation, since the spectrum of $x_i(t)$ contains only one sideband, the required bandpass filter should pass only one sideband as seen in Fig. 11.

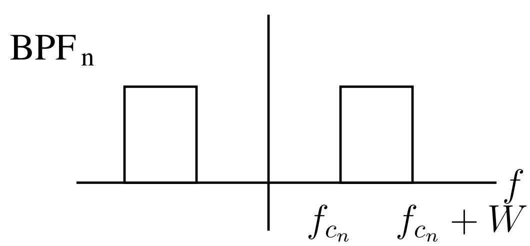


Figure 11: Bandpass filter required for upper sideband SSB

4.4 Angle modulation

With $m(t)$ the message signal, an angle modulated signal is defined as

$$x(t) = A_c \cos\big(\theta(t)\big) = A_c \cos\big(2\pi f_c t + \varphi(t)\big) \tag{13}$$

where $\varphi(t)$ is given by

$$\varphi(t) = K \int_{-\infty}^{t} m(\tau)h(t-\tau)d\tau = 2\pi K_f \int_{-\infty}^{t} m(\tau)h(t-\tau)d\tau$$

where $K = 2\pi K_f$ is the **phase sensitivity** (of the modulator) expressed in rd/Volt, $K_f = \frac{K}{2\pi}$ is the **frequency sensitivity** (of the modulator) expressed in Hz/Volt and $h(t)$ is assumed to be causal. $\varphi(t)$ is the **phase** of $x(t)$ and $\theta(t) = 2\pi f_c t + \varphi(t)$ is the **angle** of $x(t)$. The **instantaneous frequency** of $x(t)$ is defined as

$$f(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

The **maximum phase deviation** of $x(t)$ is

$$\Delta\varphi = \max|\varphi(t)|$$

The **maximum frequency deviation** of $x(t)$ is

$$\Delta f = \max|f(t) - f_c| = \max\left|\frac{1}{2\pi}\frac{d\varphi(t)}{dt}\right|$$

The complex envelope of $x(t)$ is given by

$$\tilde{x}(t) = A_c e^{j\varphi(t)}$$

Phase modulation (PM):

$$\begin{aligned} h(t) &= \delta(t) \quad \Longrightarrow \varphi(t) = Km(t) \\ H(f) &= 1 \end{aligned}$$

The phase of a PM signal is proportional to the message signal.

Frequency modulation (FM):

$$h(t) = u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \quad \Longrightarrow \varphi(t) = K\int_{-\infty}^{t} m(\tau)d\tau$$

$$H(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

The instantaneous frequency of an FM signal is proportional to the message signal up to a carrier frequency shift

$$f(t) = \frac{K}{2\pi}m(t) + f_c = K_f m(t) + f_c$$

Frequency modulation with pre-emphasis/de-emphasis(FM):

When noise analysis is done for FM, it can be shown that the power spectral density of noise at the FM receiver output is proportional to f^2 in the frequency-band of the message, thus the noise power is higher at high frequencies. To increase the overall signal-to-noise ratio, practical systems use a pre-emphasis filter $h_{pe}(t)$ before frequency modulation. The purpose of $h_{pe}(t)$ is to artificially increase the high-frequency components of the message signal to compensate for the high noise level. After FM detection, the recovered "emphasized" message signal is passed through a de-emphasis filter $h_{de}(t)$ which must be ideally the inverse filter of the pre-emphasis filter (i.e. $H_{de}(f) = \frac{1}{H_{pe}(f)}$). If pre-emphasis is used, the "emphasized" message signal is given by

$$m_{pe}(t) = m(t) * h_{pe}(t)$$

and the FM modulated signal is

$$x(t) = \int_{-\infty}^{t} m_{pe}(\tau)d\tau = m_{pe}(t) * u(t) = m(t) * h_{pe}(t) * u(t)$$

Therefore (13) corresponds to an FM signal with pre-emphasis when $h(t) = h_{pe}(t) * u(t)$. Equivalently in the frequency domain

$$H(f) = H_{pe}(f)\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) = \frac{1}{2}H_{pe}(0)\delta(f) + \frac{H_{pe}(f)}{j2\pi f}$$

An example is $H_{pe}(f) = 1 + \frac{jf}{c}$.

a) Narrow-band angle modulation

Narrow-band angle modulation: $\varphi(t) \ll 1$

If $\varphi(t) \ll 1$, then

$$\tilde{x}(t) = A_c e^{j\varphi(t)} \approx A_c\left(1 + j\varphi(t)\right)$$

Hence

$$\begin{aligned} x(t) &= \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} \\ &\approx \qquad\qquad\qquad\qquad \text{(complete this line)} \\ &\approx A_c\cos(2\pi f_c t) - A_c\varphi(t)\sin(2\pi f_c t) \end{aligned}$$

Thus narrow-band angle modulation has similar features to AM.

Bandwidth of narrow-band angle modulation: $2\times$ bandwidth of $\varphi(t)$.

Generation of narrow-band angle modulation:

Exercise: Draw a block diagram that generates a narrow-band angle modulated signal.

Detection of narrow-band angle modulation:

Exercise: Draw a block diagram of a system with input $x(t)$ and output $-\frac{A_c}{2}\varphi(t)$:

b) Wide-band angle modulation

Wide-band angle modulation: $|\varphi(t)| \gg 1$

$$\tilde{x}(t) = A_c e^{j\varphi(t)}$$

Generation of wide-band angle modulation using indirect method (Armstrong's method) :

- Generates a narrow-band angle modulated signal

$$x_0(t) = A_c \cos\left(2\pi f_c t + K \int_{-\infty}^{t} m(\tau)h(t-\tau)d\tau\right)$$

with $\left|K \int_{-\infty}^{t} m(\tau)h(t-\tau)d\tau\right| \ll 1$.

- Pass $x_0(t)$ through a Frequency multiplier by N whose block diagram is illustrated in Fig. 12. The output to the frequency multiplier by N is then

$$x(t) = A_c \cos\left(2\pi N f_c t + NK \int_{-\infty}^{t} h(t-\tau)m(\tau)d\tau\right)$$

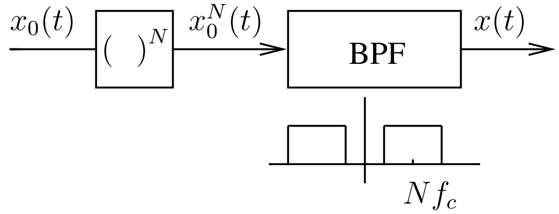


Figure 12: frequency multiplier by N

If $N \gg 1$ we have a wide-band angle-modulated signal around a carrier at Nf_c.
Analysis of a frequency multiplier by N:

$$y(t) = (x_0(t))^N = \frac{1}{2^{N-1}} \cos\left(2\pi N f_c t + NK \int_{-\infty}^{t} m(\tau)h(t-\tau)d\tau\right)$$
$$+ \text{ other terms like } \cos(2\pi n f_c t + \ldots) \text{ with } n < N$$

since

$$(\cos\alpha)^{2n} = \frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1} 2\binom{n}{k}\cos\big(2(n-k)\alpha\big) + \binom{2n}{n}\right]$$
$$(\cos\alpha)^{2n-1} = \frac{1}{2^{2n-2}}\left[\sum_{k=0}^{n-1} \binom{2n-1}{k}\cos\big((2n-2k-1)\alpha\big) + \binom{2n}{n}\right]$$

The other terms are removed by the bandpass filter yielding $x(t)$ as output.

Generation of any FM modulation using direct method; use of Voltage-Controlled-Oscillator (VCO):

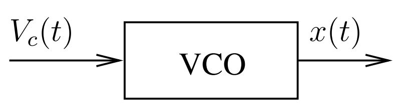


Figure 13: Voltage-Controlled Oscillator (VCO)

If the input of a VCO is a voltage $V_c(t)$ then the output of a VCO with unmodulated frequency of oscillation f_c has an instantaneous frequency given by

$$f(t) = f_c + K_f V_c(t)$$

Thus its output is

$$x(t) = A_c \cos\left(2\pi f_c t + K \int_{-\infty}^{t} V_c(\tau) d\tau\right)$$

see for example Hartley oscillator .

d) Tone modulation

Tone modulation corresponds to a sinusoidal message signal. Let $m(t) = A_m \cos(2\pi f_c t)$ ($A_m \geq 0$) applied at some time t_0 such that $t_0 \ll t$. Let us calculate the steady state expression of $\varphi(t)$ corresponding to a general angle modulated signal.

$$\begin{aligned}
\varphi(t) &= K \int_{t_0}^{t} A_m \cos(2\pi f_m \tau) h(t-\tau) d\tau \\
&\approx K \int_{-\infty}^{t} A_m \cos(2\pi f_m \tau) h(t-\tau) d\tau \quad \text{(steady state)} \\
&= K \int_{-\infty}^{\infty} A_m \cos(2\pi f_m \tau) h(t-\tau) d\tau \quad \text{($h(t)$ is causal)} \\
&= K \int_{-\infty}^{\infty} \mathcal{F}_\tau \left\{A_m \cos(2\pi f_m \tau)\right\} \mathcal{F}_\tau \left\{h(t-\tau)\right\}^* df \quad \text{Generalization of Parseval theorem (t fixed)} \\
&= \frac{KA_m}{2} \int_{-\infty}^{\infty} \left[\delta(f-f_m) + \delta(f+f_m)\right] \left[H(-f) e^{-j2\pi f t}\right]^* df \\
&= \frac{KA_m}{2} \left\{H^*(-f_m) e^{j2\pi f_m t} + H^*(f_m) e^{-j2\pi f_m t}\right\} \\
&= KA_m \Re\left\{H(f_m) e^{j2\pi f_m t}\right\} \quad \text{($h(t)$ is real, thus $H^*(-f) = H(f)$)} \\
&= KA_m |H(f_m)| \cos\big(2\pi f_m t + \arg\left[H(f_m)\right]\big) \quad \text{(steady state)}
\end{aligned} \tag{14}$$

Show that for PM and FM $\varphi(t)$ is given by

$$\varphi(t) = \begin{cases} KA_m \cos(2\pi f_m t) & \text{PM} \\ \frac{KA_m}{2\pi f_m} \sin(2\pi f_m t) & \text{FM} \end{cases}$$

The **modulation index** is defined as the maximum phase deviation, or equivalently the maximum deviation of the angle $\theta(t)$ from $2\pi f_c t$.

$$\beta = \max|\varphi(t)| = \max|\theta(t) - 2\pi f_c t|$$

$$= KA_m\,|H(f_m)| \quad \text{assuming } K, A_m \geq 0$$

Hence from (14) the steady state expression of $\varphi(t)$ for tone modulation is also given by

$$\begin{aligned}\varphi(t) &= \beta\cos\bigl(2\pi f_m t + \arg\left[H(f_m)\right]\bigr)\\ &= \beta\sin\left(2\pi f_m t + \theta\right)\end{aligned}$$

where $\theta = \arg\left[H(f_m)\right] + \frac{\pi}{2}$.

Phase modulation (PM):

$$\beta_{\text{PM}} = KA_m = \Delta\varphi \qquad \Delta\varphi\text{: maximum phase deviation}$$

Frequency modulation (FM):

$$\beta_{\text{FM}} = \frac{KA_m}{2\pi f_m} = \frac{K_f A_m}{f_m} = \frac{\Delta f}{f_m} \qquad \Delta f = K_f A_m\text{: maximum frequency deviation}$$

Transmission bandwidth of an angle modulated signal with tone modulation:

For convenience, we define $\theta = \arg\left[H(f_m)\right]$ such that $\theta_{\text{FM}} = 0$ and $\theta_{\text{PM}} = \frac{\pi}{2}$ and we use

$$\varphi(t) = \beta\sin\left(2\pi f_m t + \theta\right)$$

The complex envelope of the angle-modulated signal is

$$\tilde{x}(t) = A_c e^{j\varphi(t)} = A_c \exp\left\{j\beta\sin(2\pi f_m t + \theta)\right\}$$

Unlike the original signal $x(t)$, the complex envelope $\tilde{x}(t)$ is periodic with period $T_m = \frac{1}{f_m}$, therefore $\tilde{x}(t)$ admits a Fourier series representation

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

where the Fourier coefficients are given by

$$\begin{aligned}c_n &\triangleq \frac{1}{T_m}\int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} \tilde{x}(t) e^{-j2\pi n f_m t} dt\\ &= A_c f_m \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} \tilde{x}(t) e^{j[\beta\sin(2\pi f_m t + \theta) - 2\pi n f_m t]} dt\\ &= \frac{A_c f_m}{2\pi f_m}\int_{\theta-\pi}^{\theta+\pi} e^{j[\beta\sin(u) - nu + n\theta]} du \quad (u = 2\pi f_m t + \theta)\\ &= A_c e^{jn\theta}\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} e^{j[\beta\sin(u) - nu]} du\end{aligned}$$

$$= A_c e^{jn\theta} J_n(\beta)$$

where $J_n(\cdot)$ is the n^{th} Bessel function of the first kind defined as

$$J_n(\beta) = \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} e^{j[\beta\sin(u)-nu]}du \quad \text{(integral independent of } \theta \text{ [1])}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{n+2k}}{k!(k+n)!}$$

Therefore the modulated signal $x(t)$ with tone modulation is given by

$$x(t) = \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\}$$

$$= \Re\left\{\sum_{n=-\infty}^{\infty} A_c e^{jn\theta} J_n(\beta) e^{j[2\pi f_c t + 2\pi n f_m t]}\right\} \;\; = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos\big(2\pi(f_c + nf_m)t + n\theta\big)$$

and has a Fourier transform given by

$$X(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left\{\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]\cos(n\theta)\right.$$
$$\left. + j\left[\delta(f - f_c - nf_m) - \delta(f + f_c + nf_m)\right]\sin(n\theta)\right\}$$

Thus it is seen that angle-modulated signals have an infinite bandwidth.

$$X(f) = \begin{cases} \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right] & \text{FM} \\ \frac{A_c}{2}\sum_{p=-\infty}^{\infty}\left\{J_{2p}(\beta)\left[\delta(f - f_c - 2pf_m) + \delta(f + f_c + 2pf_m)\right](-1)^p\right. & \\ \quad \left. + \, jJ_{2p+1}(\beta)\left[\delta(f - f_c - (2p+1)f_m) - \delta(f + f_c + (2p+1)f_m)\right]\right\} & \text{PM} \end{cases}$$

Using $J_{-n}(\beta) = (-1)^{-n}J_n(\beta)$ and mathematical tables, the Fourier transform $X(f)$ for FM is illustrated for example in Fig. 14 and Fig. 15. Note that usually only the magnitude of $X(f)$ is drawn. It is obtained by reversing the negative peaks to become positive peaks.

[1] Expand the integral as the sum of the three integrals $\int_{\theta-\pi}^{-\pi} + \int_{-\pi}^{\pi} + \int_{\pi}^{\theta+\pi}$ and makes the change of variable $v = u + 2\pi$ in the first one.

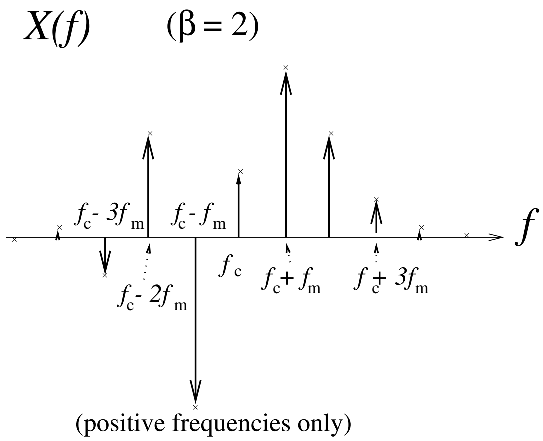


Figure 14: Fourier transform of $X(f)$ ($\beta = 2$) with FM modulation

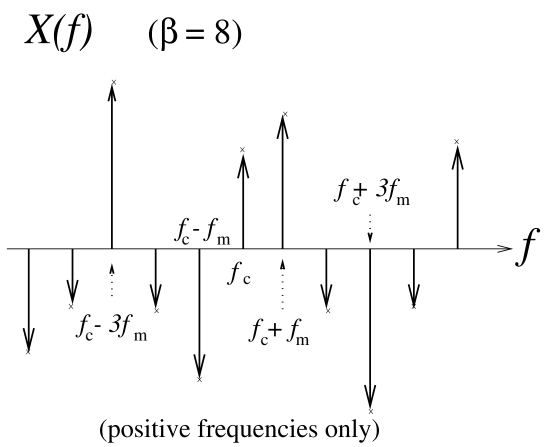


Figure 15: Fourier transform of $X(f)$ ($\beta = 8$) with FM modulation

Some properties of Bessel functions:

$$J_{-n}(\beta) = (-1)^{-n} J_n(\beta)$$

When $\beta \ll 1$,

$$J_n(\beta) \approx \left(\frac{\beta}{2}\right)^n \frac{1}{n!} \qquad \Longrightarrow \qquad \begin{cases} J_0(\beta) & \approx 1 \\ J_1(\beta) & \approx \frac{\beta}{2} \\ J_n(\beta) & \approx 0, n \geq 2 \end{cases}$$

$$\lim_{\beta \to 0} J_n(\beta) = \begin{cases} 1 & n = 0, \\ 0 & \text{else.} \end{cases}$$

To obtain a definition of an effective (or essential) bandwidth, that is a bandwidth that contains most of the total power (usually 98% or 99%), let us consider the average power of $x(t)$. The average power of $x(t)$ is given by

$$
\begin{aligned}
P_x &\triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt \\
&= \lim_{T\to\infty} \frac{A_c^2}{2T}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} J_n(\beta)J_p(\beta)\int_{-T}^{T}\cos\bigl(2\pi(f_c+nf_m)t+n\theta\bigr)\cos\bigl(2\pi(f_c+pf_m)t+p\theta\bigr)dt \\
&= \frac{A_c^2}{2}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} J_n(\beta)J_p(\beta)\lim_{T\to\infty}\frac{1}{2T}\biggl[\int_{-T}^{T}\cos\bigl(2\pi(2f_c+(n+p)f_m)t+(n+p)\theta\bigr)dt \\
&\quad+\int_{-T}^{T}\cos\bigl(2\pi(n-p)f_m t+(n-p)\theta\bigr)dt\biggr] \\
&= \frac{A_c^2}{2}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} J_n(\beta)J_p(\beta)\left[\frac{1}{2T_{n,p}}\int_{-T_{n,p}}^{T_{n,p}}\cos\left(\frac{2\pi t}{T_{n,p}}+(n+p)\theta\right)dt\right. \\
&\quad\left.+\frac{1}{2T'_{n,p}}\int_{-T'_{n,p}}^{T'_{n,p}}\cos\left(\frac{2\pi t}{T'_{n,p}}+(n-p)\theta\right)dt\right]
\end{aligned}
$$

where $T_{n,p}=[2f_c+(n+p)f_m]^{-1}$ and $T'_{n,p}=(n-p)^{-1}f_m^{-1}$. Since the two trigonometric integrals are zero unless $n=p$ and are equal to 1 when $n=p$, the average power of $x(t)$ is given by

$$
\begin{aligned}
P_x &= \frac{A_c^2}{2}\sum_{n=-\infty}^{\infty} J_n^2(\beta) = P_c + \sum_{n=-\infty}^{\infty} P_n \\
P_c &= \frac{A_c^2}{2}J_0^2(\beta) \\
P_n &= \frac{A_c^2}{2}J_n^2(\beta) \quad n^{th} \text{ side-band power} \\
P_{-n} &= \frac{A_c^2}{2}J_{-n}^2(\beta) = \frac{A_c^2}{2}J_n^2(\beta) = P_n
\end{aligned}
$$

Note that from $x(t)=A_c\cos\left(2\pi f_c t+\varphi(t)\right)$, the average power of $x(t)$ is also given by

$$P_x = \frac{A_c^2}{2}$$

therefore we can deduce that

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

An effective (essential) bandwidth can be defined as

$$B_T = 2n_{\max} f_m$$

where $n_{\max}$ is the largest n such that $|J_n(\beta)| \geq 0.01$ and $\forall\, k > n |J_k(\beta)| < 0.01$. It turns out that $n_{\max}$ depends on β. In terms of power, it is equivalent to neglect side-bands that contribute to less than 0.01% of the total power. Another rule for the effective bandwidth is obtained by approximating $n_{\max}$ by a linear curve,

$$B_T = 2f_m(\beta + c) \qquad 1 \leq c \leq 2$$

When $c = 1$, the classical Carson's rule is obtained

$$B_T^{\text{Cars}} = 2f_m(\beta + 1)$$

example: Show that Carson's rule for FM is

$$B_T^{\text{FM}} = 2\left(f_m + \Delta f\right)$$

d) General modulating signal with bandwidth W

If $m(t)$ is periodic, $m(t)$ should be expressed in terms of its Fourier series representation. If $m(t)$ is a general non-periodic deterministic signal, we can use $e^{j\varphi(t)} = \sum_{k=0}^{\infty} \frac{[\varphi(t)]^k}{k!}$.

Generalization of Carson's rule to a general modulating signal

Worst-case tone approach. Let $m(t)$ be a message signal with bandwidth W and maximum amplitude $\max|m(t)|$. Assume we model $m(t)$ as an infinite number of tones of frequency f_{m_k} and maximum amplitude A_{m_k}, then its effective bandwidth would be

$$\begin{aligned} B_T &= \max_k \left\{2f_{m_k}\left[\beta_k + 1\right]\right\} \\ &= \max_k 2f_{m_k} + \max_k K A_{m_k} f_{m_k} |H(f_{m_k})| \\ &\leq \max_k 2f_{m_k} + K \max_k A_{m_k} \max_k \left[f_{m_k}|H(f_{m_k})|\right] \\ &= W + K \max|m(t)| \max_k \left[f_{m_k}|H(f_{m_k})|\right] \end{aligned} \tag{15}$$

where to get (15) we have applied the worse case tone approach. The worse case tone approach consists of evaluating the bandwidth obtained by considering a tone at the highest possible frequency and the highest possible amplitude, thus maximizing the product of the second term in B_T by maximizing its two terms A_{m_k} and $f_{m_k}|H(f_{m_k})|$ separately. Let us consider now the term $\max_k \left[f_{m_k}|H(f_{m_k})|\right]$ for the special case of FM and PM modulation. For FM modulation

$$\max_k \left[f_{m_k}|H(f_{m_k})|\right] = \max_k \left[\frac{f_{m_k}}{2\pi f_{m_k}}\right] = \max_k 1 = 1 = W|H(W)|$$

consistent with the worse tone at W. Note that for FM maximizing A_{m_k} is equivalent to maximize the frequency deviation Δf_k. For PM modulation

$$\max_k \left[f_{m_k}|H(f_{m_k})|\right] = \max_k f_{m_k} = W = W|H(W)|$$

Hence we obtain

$$B_T = 2W + K\max(|m(t)|)W|H(W)| = 2W(\beta+1)$$

where β called for general modulating signal the **deviation ratio** is defined as

$$\beta = K\max(|m(t)|)|H(W)| = \begin{cases} \frac{\Delta f}{W} = \frac{K_f \max|m(t)|}{W} & \text{FM} \\ \Delta\varphi = K\max|m(t)| & \text{PM} \end{cases}$$

Exercise:

Calculate using Carson's rule the bandwidth of commercial FM broadcasting characterized by a maximum allowed maximal frequency deviation of 75kHz and a maximum audio signal of 15kHz.

e) Detection of FM signals

Basic structure: using a differentiator and an envelope detector

Show that for large f_c, if $x(t)$ is applied to a differentiator followed by an envelope detector, the resulting output is proportional to the message signal:

$$x(t) = A_c \cos\left(2\pi f_c t + K\int_{-\infty}^{t} m(\tau)d\tau\right)$$

$$\frac{dx(t)}{dt} = \qquad\qquad \text{complete}$$

$$z(t) = \qquad\qquad \text{Output of envelope detector (complete)}$$

Frequency domain differentiation (slope demodulator)

Draw the transfer function of a real filter that can implement the differentiation.

Note that the differentiation operation has to be implemented only in the bandwidth of the modulated signal.

Time domain differentiation

Based on the following approximation

$$\frac{dx}{dt} \approx \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

draw the block diagram of a time domain differentiator.

Quadrature detector for an angle-modulated signal

Using a differentiator and a $\pi/2$ phase shifter, draw the block diagram of a demodulator for an angle modulated signal, that is based on similar principles as coherent detection for AM.

Show that the output of the quadrature detector is given by $-\frac{A_c^2}{2}\frac{d\varphi(t)}{dt}$.

f) Form of $H(f)$ for FM

The phase of an FM modulated signal is given by

$$\varphi(t) = K\int_{-\infty}^{\infty} h(t-\tau)m(\tau)d\tau = K\int_{-\infty}^{t} m(\tau)d\tau$$

Its Fourier transform is given by

$$\mathcal{F}\left\{\varphi(t)\right\} = KH(f)M(f) = K\left[\frac{1}{j2\pi f}M(f) + \frac{M(0)}{2}\delta(f)\right]$$

Hence in theory

$$H(f) = \frac{1}{j2\pi f} + \frac{\delta(f)}{2}$$

Let $H^{'}(f) = \frac{1}{j2\pi f}$, then the corresponding phase $\varphi^{'}(t)$ is given by

$$\varphi^{'}(t) = \mathcal{F}^{-1}\left[KH^{'}(f)M(f)\right] = \mathcal{F}^{-1}\left[K\frac{1}{j2\pi f}M(f)\right]$$

and the original phase is

$$\begin{aligned}\varphi(t) &= \mathcal{F}^{-1}\left[KH(f)M(f)\right] = \mathcal{F}^{-1}\left[K\frac{1}{j2\pi f}M(f)\right] + \mathcal{F}^{-1}\left[\frac{K}{2}\delta(f)M(0)\right]\\ &= \mathcal{F}^{-1}\left[K\frac{1}{j2\pi f}M(f)\right] + \frac{K}{2}M(0)\end{aligned}$$

Hence the difference between the two phases is only a constant phase shift which is equivalent to a change of the time of origin. Furthermore the instantaneous frequency given by

$$f(t) = f_c + \frac{1}{2\pi}\frac{d\varphi(t)}{dt}$$

is independent of the constant $\frac{KM(0)}{2}$. Finally the two possible expressions of $H(f)$

$$1. \quad H(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \qquad\qquad 2. \quad H(f) = \frac{1}{j2\pi f}$$

yield the same result for the deviation ratio β. Hence in the derivation of results

$$H(f) = \frac{1}{j2\pi f}$$

can be considered instead of $H(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$.