7 Digital transmission through bandlimited channels

A baseband linear transmission system is illustrated in Fig. 35. The transmitted and received

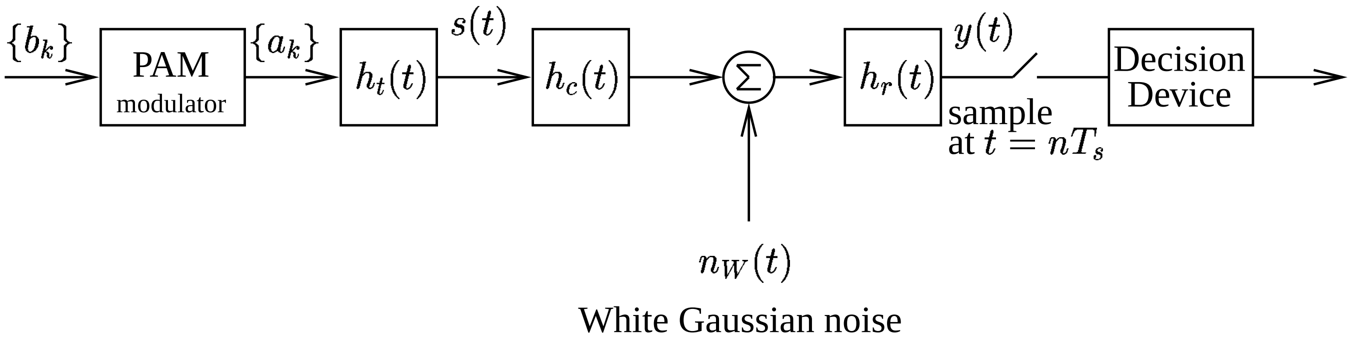

Figure 35: Baseband data transmission system

signals are given by

$$\begin{aligned}
s(t) &= h_t(t) * \sum_k a_k \delta(t - kT_s) = \sum_k a_k h_t(t - kT_s) \\
y(t) &= [s(t) * h_c(t) + n_W(t)] * h_r(t) \\
&= \sum_k a_k \delta(t - kT_s) * h_t(t) * h_c(t) * h_r(t) + n_W(t) * h_r(t) \\
&= \sum_k a_k h(t - kT_s) + n(t)
\end{aligned}$$

where

$$h(t) = h_t(t) * h_c(t) * h_r(t)$$

and

$$n(t) = n_W(t) * h_r(t)$$

Based on $y(nT_s)$ we want to make a decision on which symbol a_n was transmitted. Sampling $y(t)$ at $t = nT_s$ yields

$$\begin{aligned}
y(nT_s) &= \sum_k a_k h(nT_s - kT_s) + n(nT_s) \\
&= \underbrace{a_n h(0)}_{\text{data to be decoded}} + \underbrace{\sum_{k \neq n} a_k h(nT_s - kT_s)}_{\text{ISI}} + \underbrace{n(nT_s)}_{\text{noise term}}
\end{aligned}$$

$$a_n h(0) : \text{contribution of the } n^{th} \text{ transmitted symbol}$$

$$\sum_{k \neq n} a_k h(nT_s - kT_s) : \text{Residual effect of all the other transmitted symbols on the decoding of the } n^{th} \text{ transmitted symbol (ISI term)}$$

$$n(nT_s) : \text{noise term}$$

Due to the fact that the channel is bandlimited, its impulse response has an infinite duration (dispersive channel), thus it may create Intersymbol Interference (ISI) represented by the term $\sum_{k \neq n} a_k h(nT_s - kT_s)$. The objective of a designer is to determine $h_t(t)$ and $h_r(t)$ in order to minimize the effect of ISI and noise.

7.1 Intersymbol Interference (ISI)

$$\text{ISI} = \sum_{k \neq n} a_k h\left((n-k)T_s\right) = I(nT_s)$$

If $h((n-k)T_s) = 0$ for $k \neq n$ then $\text{ISI} = 0 = I(nT_s)$, and in the absence of noise $y(nT_s) = a_n h(0)$ resulting in perfect reception.

Nyquist First criterion (time domain):

$$h(iT_s) = 0 \quad i \neq 0 \quad \text{and} \quad h(0) \neq 0$$

Poisson sum formula:

$$\frac{1}{T_s} \sum_{k=-\infty}^{\infty} H\left(f - \frac{k}{T_s}\right) = \sum_{k=-\infty}^{\infty} h(kT_s) e^{-j2\pi k f T_s}$$

Proof. Let

$$h_\delta(t) = h(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \tag{22}$$

$$= \sum_{k=-\infty}^{\infty} h(kT_s) \delta(t - kT_s) \tag{23}$$

From (22)

$$H_\delta(f) = \mathcal{F}\left\{h_\delta(t)\right\} = H(f) * \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t - kT_s)\right\} = H(f) * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H\left(f - \frac{k}{T_s}\right)$$

From (23)

$$H_\delta(f) = \sum_{k=-\infty}^{\infty} h(kT_b)e^{-j2\pi k f T_b}$$

Q.E.D

Therefore from Poisson's sum formula, an equivalent form of the Nyquist first criterion is obtained.

Nyquist first criterion (frequency domain):

$$\sum_{k=-\infty}^{\infty} H\left(f - \frac{k}{T_s}\right) = constant$$

where $R_s = \frac{1}{T_s}$ is the symbol rate.

To completely eliminate ISI, we need to know $h_c(t)$ the channel impulse response. In practice, there will be residual distortion due to imprecision in estimating $h_c(t)$, and also due to inaccuracies in the physical implementation of the transmit and receive filters. Hence we need equalizers to compensate for such distortion.

Exercise: Give two examples of pulses satisfying Nyquist first criterion.

7.2 Noise effect: matched filter

Recall that

$$y(t) = \sum_k a_k h(t - kT_s) + n(t)$$

The receiver should detect the information a_k given the received signal $y(t)$, as relable as possible. To achieve this $h(t)$ should be such that the contribution of the $\sum_k a_k h(t - kT_s)$ to the receiver output becomes considerably larger than the output noise component.

To study the effect of noise not necessarily in the presence of ISI, let us consider the receiver model illustrated in Fig. 36, where $n_W(t)$ is a White noise (not necessarily Gaussian) with power spectral density $\frac{N_0}{2}$ (i.e. $E\left[n_W(t)n_W(u)\right] = \frac{N_0}{2}\delta(t-u)$).

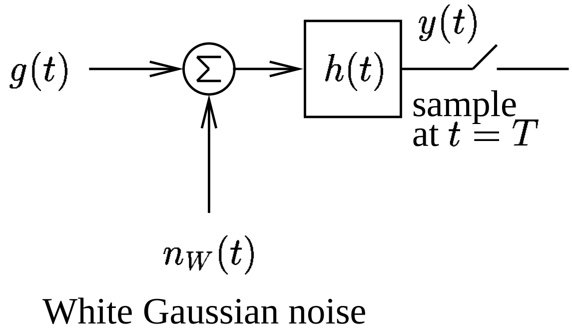

Figure 36: Matched filter receiver model

The received signal is given by

$$x(t) = g(t) + n_W(t) \quad 0 \leq t \leq T$$

where

$$g(t) = 0 \quad t \notin [0, T]$$

The output of the receiver is given by

$$y(t) = g_o(t) + n(t)$$

where

$$g_o(t) = g(t) * h(t)$$
$$n(t) = n_W(t) * h(t)$$

Making the output signal component $g_o(t)$ much larger than the output noise component $n(t)$ at a particulat time instant can be expressed mathematically as maximizing the output Signal-to-Noise Ratio SNR_o given by

$$\mathrm{SNR}_o = \frac{|g_o(T)|^2}{E\left[n^2(T)\right]}$$

where $|g_o(T)|^2$ is the instantaneous output-signal power at the decision time T.

(Time domain) derivation of the filter $h(t)$ that maximizes SNR_o:

In this derivation, we will assume for simplicity that $h(t) = 0, \forall\, t \notin [0, T]$. Note that a similar derivation without this assumption can be done by replacing the finite limits 0 and T by $-\infty$ and

∞ in the integrals.

$$\begin{aligned}
E\left[n^2(T)\right] &= E\left[\left(\int_0^T n_W(T-\tau)h(\tau)d\tau\right)^2\right] \\
&= E\left[\int_0^T n_W(T-\tau)h(\tau)d\tau \int_0^T n_W(T-u)h(u)du\right] \\
&= \frac{N_0}{2}\int_0^T\int_0^T \delta(\tau-u)h(\tau)h(u)d\tau du \qquad \text{interchanging } E \text{ and } \int \\
&= \frac{N_0}{2}\int_0^T h^2(\tau)d\tau
\end{aligned}$$

Therefore the output-signal-to-noise ratio is given by

$$\mathrm{SNR}_o = \frac{\left(\int_0^T g(\tau)h(T-\tau)d\tau\right)^2}{\frac{N_0}{2}\int_0^T h^2(\tau)d\tau}$$

From Cauchy's Schwarz inequality SNR_o is maximized if

$$h(T-t) = Cg(t) \quad \text{or equivalently} \quad h(t) = Cg(T-t) \tag{24}$$

The filter with impulse response given by (24) is the filter **matched to** $g(t)$ and thus it is called the **matched filter**. Note that if $g(t)$ is time-limited to $[0,T]$, $h(t)$ is also time-limited to $[0,T]$ consistent with the initial assumption.

The maximum output signal-to-noise ratio is

$$\mathrm{SNR}_{o\max} = \frac{E_g}{\frac{N_0}{2}}$$

where E_g is the energy of $g(t)$ given by

$$E_g = \int_{-\infty}^{\infty}|g(t)|^2dt = \int_0^T |g(t)|^2dt$$

(Frequency domain) derivation of the filter $h(t)$ that maximizes SNR_o*:*

In this derivation, $h(t)$ is arbitrary. The output noise component, $n_o(t)$, is obtained by passing the white noise $n_W(t)$ through the filter $h(t)$, and its power spectral density is given by

$$S_n(f) = \frac{N_0}{2}|H(f)|^2.$$

Therefore

$$E\left[n^2(t)\right] = \int_{-\infty}^{\infty} \frac{N_0}{2}|H(f)|^2 df$$

$$g_o(t) = \mathcal{F}^{-1}\left\{G_o(f)\right\} = \mathcal{F}^{-1}\left\{H(f)G(f)\right\} = \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft} df$$

$$g_o(T) = \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df$$

Hence

$$\mathrm{SNR}_o = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df\right|^2}{\int_{-\infty}^{\infty} \frac{N_0}{2}|H(f)|^2 df}$$

Using Cauchy's Schwarz inequality, SNR_o is maximized if

$$H(f) = C\left[G(f)e^{j2\pi fT}\right]^* = CG^*(f)e^{-j2\pi fT}$$

which is equivalent to (24).