Fairness in Network Optimal Flow Control: Optimality of Product Forms

Consider a *cooperative game* of $N$ players (users). Let each individual player $i$ have an objective function $f_i(x) : X \rightarrow \mathbb{R}$ where $X$ is a convex, closed, and bounded subset of $\mathbb{R}^N$. Let $X$ denote the space of throughputs. Let $u^* = [u_1^*, u_2^*, \ldots, u_N^*]$ where $u_i^* = f_i(x^*)$ for some $x^* \in X$ denote a common agreement point which all players agree to as a starting point for the game. Let $[U, u^*]$ denote the game defined on $X$ with the initial agreement point $u^*$ where $U$ denotes the image of the set $X$ under $f(\cdot)$, i.e., $f(X) = U$. Let $F[\cdot, u^*] : U \rightarrow U$ be an arbitration strategy.

**Cooperative Bargaining Games and Arbitration Strategies**
\[ X = \{ x : f(x) = 0 \} \]

\[ \mathbb{U} = \{ u = f(x) : J(x) = 0 \} \]

\[ u = f(x) \text{ or } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \]
Then $F$ is a Nash arbitration strategy if it satisfies the following four axioms:

1) Let $\phi(u) = u'$ where $u' = a_i u_i + b_i$ for $i = 1, 2, \cdots, N$ and $a_i > 0$, $b$ are arbitrary constants. Then

$$F[\phi(U), \phi(u^*)] = \phi(F[U, u^*]).$$

In words this states that the N.A.S. or desired operating point as determined by the set of throughputs is invariant under affine utility transformations.

2) The arbitration scheme must satisfy

$$(F[U, u^*])_i \geq u^*_i \text{ for } i = 1, 2, \cdots, N$$

and there exists no $u \in f(X)$ such that $u_i \geq (F[U, u^*])_i$ for all $i = 1, 2, \cdots, N$.

This assures Pareto optimality of the arbitrated solution.
3) Let \([U_1,u^*]\) and \([U_2,u^*]\), be two games with the same initial agreement point such that:

i) \(U_1 \subseteq U_2\)

ii) \(F[U_2,u^*] \in U_1\)

Then \(F[U_1,u^*] = F[U_2,u^*]\).

This is called the axiom of independence of irrelevant alternatives.

4) Let \(U\) be symmetrical with respect to a subset \(J \subseteq \{1,2,3,\cdots,N\}\) of indices (i.e., let \(i, j \in J\) and \(i < j\), then \(\{u_1,u_2,\cdots,u_{i-1},u_j,u_{i+1},\cdots,u_{j-1},u_i,u_{j+1},\cdots,u_N\} \in U\).

If \(u^*_i = u^*_j\), then \((F[U,u^*])_i = (F[U,u^*])_j\) for \(i, j, \in J\).

This is the axiom of symmetry which states that if the set of utilities is symmetric then for any two players, if the initial agreement point corresponds to equal performance then their arbitrated values are equal.
Stefanescu and Stefanescu's Theorem: (THE NASH ARBITRATION STRATEGY)

Let $f_i(x) : X \rightarrow \mathbb{R}$ be concave, upper bounded, functions defined on $X$ a convex, closed, and bounded subset of $\mathbb{R}^N$. Let $U = \{u \in \mathbb{R}^N : \exists x \in X \text{ such that } u \leq f(x)\}$ and $X(u) = \{x : u \leq f(x)\}$ and $X_0 = X(u^*)$.

Then the Nash arbitration strategy is given by the point which maximizes the unique function

$$V(x) = \prod_{i=1}^{N} (f_i(x) - u_i^*) \quad \text{over } X_0$$

if $X_0$ contains vectors $x$ which results in the user objectives strictly superior to $u^*$. If the vectors in $X_0$ have the property that there exist $x \in X_0$ such that only $k$ of the individual objectives are superior to the corresponding elements of $u^*$, then the unique function is taken as the product of the individual objectives for which there exist superior solutions. The remaining $(n - k)$ components of $u^*$ are the user objectives at the Nash arbitration point.
Product of Powers Criterion

Let $S = [S_1, S_2, \ldots, S_N]^T$ denote the mean throughputs for the $N$ players in the network.

Let $\Delta = [\Delta_1, \Delta_2, \ldots, \Delta_N]^T$ denote the vector of corresponding user delays.

Let $P_i = \frac{S_i}{\Delta_i}$ for $i = 1, 2, \ldots, N$ denote the Power function of user $i$, which is defined on the set of admissible throughputs

$$U = \{S \geq 0 : 0 \leq \gamma_l < c_l; \quad l = 1, 2, \ldots, L\}$$

where $\gamma_l$ denotes the total throughput on link $l$.

**Lemma 1:** For a Jackson Network the inverse of the power of user $i$, $i = 1, 2, \ldots, N$ defined by $P_i^{-1} = \frac{\Delta_i}{S_i}$ is convex in the space of throughputs.

The proof given in the paper uses the fact that mean delay is additive over the links in the used path and the Hessian matrix $H_{ii} = \left[ \frac{\partial^2 P_i^{-1}}{\partial S_j \partial S_k} \right]$ is PSD for a Jackson Network.
**Theorem**: The flow control scheme which maximizes the product of user powers corresponding to the unique operating point given by

\[ S^* = \arg \max \prod_{i=1}^{N} P_i(S) \]

is a Nash Arbitration Strategy for the user's negative inverse power, \(-P_i^{-1}\) by application of Stefenescu's Theorem.

By taking an initial agreement point \(u_i^* = -aP_i^{-1}(S^*)\) \(i = 1, 2, ..., N\) for \(a > 1\) and sufficiently large, then \(u^*\) is a valid initial agreement point for the game played by \(N\) players with \(-P_i^{-1}(S)\) as individual objective functions. By taking the limit as \(a \to \infty\), this corresponds to an initial agreement point \([0,0,...,0]\) in the space of throughputs \(S\).