# Quantizers for the Gamma Distribution and Other Symmetrical Distributions

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*Abstract*—This paper discusses minimum mean-square error quantization for symmetric distributions. If the distribution satisfies a log-concavity condition, the optimal quantizer is itself symmetric. For the gamma distribution often used to model speech signals, the log-concavity condition is not satisfied. It is shown that for this distribution both the uniformly spaced and the nonuniformly spaced optimal quantizers are not symmetrical for even numbers of quantizer levels. New quantization tables giving the optimal levels for quantizers for the gamma distribution are presented. A simple family of symmetric distributions is also examined. This family shows that as the distribution gets concentrated near the point of symmetry, nonsymmetric solutions become optimal.

## I. Introduction

THIS paper focuses on minimum mean-square error scalar quantizers for symmetric distributions. A number of authors have published tables of quantizers for distributions of interest in the processing of speech or visual signals [1]-[7]. These quantizers have been designed for the most part using the iterative methods outlined by Lloyd and Max [1], [2]. It is well known that these design techniques applied to general probability distributions may produce quantizers that are only locally optimal. For symmetrical distributions, as pointed out by Sharma [8], the optimal quantizer need not be symmetrical. Even so, it seems to be widely assumed that for the symmetrical distributions encountered in practice, the resulting quantizers are also symmetrical. It is shown here that nonsymmetric solutions may be optimal for distributions of more than pathological interest.

Fleischer [9] and more recently Trushkin [10] have shown that for the mean-square error criterion, a sufficient condition for uniqueness of the Lloyd–Max solution is log-concavity of the probability density function. The Gaussian and Laplace distributions have associated with them unique (and hence symmetrical) quantizers. However, as shown in this paper, another commonly encountered distribution, the gamma distribution which is often used to model speech signal statistics, does not have a unique minimum. In fact, the optimal quantizer for the gamma distribution is nonsymmetrical if the number of quantizer levels is even. Revised tables for quantizers for the gamma distribution giving both the optimal quantizers and the best symmetric quantizers are presented.

It is also shown that the Laplace distribution occupies a unique place in the continuum of generalized gamma distributions—it sits on the boundary between distributions that have unique optima and those which do not. This indicates that log-concavity is both necessary and sufficient for this family of distributions.

In the last section, a simple family of symmetric distributions is examined. This family has the property that as the distribution gets concentrated near the point of symmetry, nonsymmetric solutions become optimal.

## II. Lloyd-Max Quantizers

Quantization is the process of subdividing the range of a signal into nonoverlapping regions. An output level is then assigned to represent each region. Since this output level is used to represent all of the values in the region, it is usually itself within that region. The quantizer as defined here is a memoryless nonlinearity (see Fig. 1).

Consider an $N$ level quantizer with output levels $y_1, y_2, \cdots, y_N$. The output level $y_k$ is associated with a decision region specified by its boundaries, the decision levels,

$$y_k \Leftrightarrow \{x_{k-1} < x \leqslant x_k\}, \quad i = 1, 2, \cdots, N. \tag{1}$$

For convenience, the $x_i$ are in increasing order and the two extreme decision levels are chosen to be $x_0 = -\infty$ and $x_N = \infty$. The total mean-square error is

$$\overline{e^2} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - y_i)^2 \, p(x) \, dx. \tag{2}$$

Differentiating (2) with respect to $x_k$ and $y_k$ gives

$$x_k = \frac{y_k + y_{k+1}}{2}, \quad \text{for} \quad k = 1, 2, \cdots, N-1, \tag{3}$$

$$y_k = \frac{\int_{x_{k-1}}^{x_k} x p(x) dx}{\int_{x_{k-1}}^{x_k} p(x) dx}, \quad \text{for} \quad k = 1, 2, \cdots, N. \tag{4}$$

These conditions, which must be satisfied by a minimum mean-square error quantizer, can be interpreted to mean that the decision levels should be midway between output levels and that the output levels should be the conditional means of the decision regions.

The conditions (3) and (4) form the basis for an iterative process to determine optimal quantizers. Two versions of the iteration can be used.

### A. Method I

In the first version, often termed Lloyd's method I, an initial guess is made for the output levels. A set of decision boundaries corresponding to these output levels is determined from

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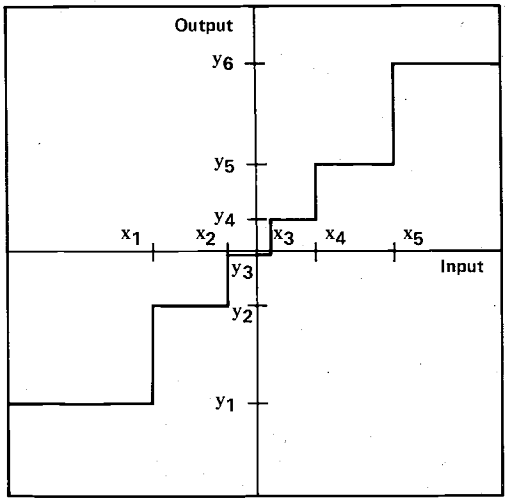


Fig. 1. Quantizer input–output characteristic.

(3). Then (4) can be applied to determine a new set of output levels which is optimal for the decision boundaries just determined. These two steps constitute one iteration. At the end of an iteration, the mean-square error has decreased or remained unchanged.

A variation of this technique used by the author applies both halves of the iteration to each output level in turn. In this way the effect of changing an output level is allowed to propagate to other output levels. This modified version of method I, which uses the same number of integral evaluations as the original technique, often converges faster in practice.

### B. Method II

A variational technique, dubbed method II, proposed by both Lloyd and Max involves a one-dimensional search. An initial guess is made as to the value of the first output level $y_1$. The value of the decision level below this output level, in this case $x_0$, is known. The next decision level can be determined by finding the value of $x_1$ which satisfies (4), in this case for $k = 1$. This step will generally have to be carried out using iterative numerical techniques. The next step is to telescope the process to the next interval. This is done by using (3) to determine $y_2$ from $y_1$ and $x_1$. The process continues interval by interval to determine all of the output levels. The last output level, $y_N$, determined in this manner will generally not be the conditional mean of the last interval. The difference between $y_N$ and the conditional mean of the last interval can be used to determine an update for $y_1$ for the next iteration. The process of determining the output levels continues until sufficient precision has been achieved.

### C. Convergence

The iterative techniques for determining the quantizer levels converge to a *fixed point* which is a stationary point, a minimum, or possibly a saddle point of the mean-square error. For general probability distributions which include discrete probability masses, the decision boundaries found will never coincide with the points of discrete probability [1]. In addition, a practical version of these algorithms can be structured so as to avoid converging to a solution which has zero probability decision regions. The stopping criterion for either algorithm can be based on the change in position or the change in mean-square error. Note that the mean-square error can be expressed in terms of the integrals that are used in (4). This means that no new integrals need be evaluated in order to determine the mean-square error for use as a stopping criterion.

The one-dimensional search of method I generally converges faster than method II. However, the precision to which the last intervals are determined by method I may be poor due to the intervening steps. A practical compromise is to use method I to find an approximate solution and then use method II to refine the solution. This was the strategy adopted to generate the table of optimal quantizers shown later.

## III. Uniqueness

The log-concavity test for a probability density function given by Fleischer [9] is

$$\frac{\partial^2 \log p(x)}{\partial x^2} < 0. \tag{5}$$

A probability density function that satisfies this condition has a unique stationary point in the mean-square error in terms of the $2N-1$ variables—the $N-1$ decision levels and the $N$ output levels. The iterative Lloyd–Max algorithms will find the globally optimal quantizer if the probability density is log-concave. The Gaussian distribution satisfies this condition and hence has a unique stationary point. The Laplace or double-sided exponential distribution is only semi-log-concave, i.e., the inequality in (5) is replaced by an equality. However, Trushkin [10] has shown that the Laplace distribution does have a unique stationary point.

Consider the generalized gamma double-sided probability density function

$$p(x) = \frac{\lambda(\lambda|x|)^{a-1} e^{-\lambda|x|}}{2\Gamma(a)}. \tag{6}$$

The parameter $\lambda$ sets the variance of the distribution

$$\sigma^2 = \frac{a(a+1)}{\lambda^2}. \tag{7}$$

The density function (6) becomes Laplacian for $a = 1$. For modeling the measured distribution of speech signals, the general gamma density with $a = 1/2$ is often used [3]. The log-concavity test for the general gamma density gives

$$\frac{\partial^2 \log p(x)}{\partial x^2} = -\frac{2(a-1)}{x^2}, \quad x \neq 0. \tag{8}$$

For $a \geqslant 1$, a unique stationary point exists—the equality being given by Trushkin's argument.

## IV. Symmetric Distributions

Consider a distribution which is symmetric about its mean. For every quantizer with a given set of output levels, another with the same mean-square error is generated by simply reflecting the levels about the mean. This argument indicates that if a symmetrical distribution is log-concave, the optimal and unique quantizer will have levels symmetrically placed about the mean.

For symmetrical distributions, a solution which satisfies the necessary conditions for optimality can be obtained by considering the density on one side of the mean. If the total number of levels is even, the problem is solved using $N/2$ levels for the density $2p(x)$, $x \geqslant \bar{x}$, where $\bar{x}$ is the mean of the distribution. If the total number of levels is odd, the problem can again be solved with half the number of levels but with one level fixed at the mean. In either case, the solution determined for one side of the distribution can be reflected about the mean to produce a symmetrical solution for the distribution. Thus, every symmetric distribution has a symmetric quantizer which satisfies the necessary conditions for a minimum mean-square error quantizer. However, this solution may represent a local minimum or a saddle point for distributions which are not log-concave.

## V. Gamma Distribution

Consider the general gamma distribution with $a = 1/2$, henceforth referred to simply as the gamma distribution. For the gamma distribution, more than one stationary point may exist.

The optimal one-level quantizer has an output level at the mean and is symmetric. For the two-level quantizer, we can determine the optimal output levels given a decision level $x_1$ using (4) and then combine these using (3) to give a single equation to be solved for $u = \sqrt{3}\,x_1$. Because of the symmetry, consider only $u \geqslant 0$.

$$y_1 = -\frac{Q(\sqrt{u}) + \sqrt{\frac{u}{2\pi}}\, e^{-u/2}}{\sqrt{3}[1 - Q(\sqrt{u})]} \tag{9}$$

$$y_2 = \frac{Q(\sqrt{u}) + \sqrt{\frac{u}{2\pi}}\, e^{-u/2}}{\sqrt{3}Q(\sqrt{u})}$$

$$2u - \left[Q(\sqrt{u}) + \sqrt{\frac{u}{2\pi}}\, e^{-u/2}\right] \frac{1 - 2\,Q(\sqrt{u})}{Q(\sqrt{u})[1 - Q(\sqrt{u})]} = 0 \tag{10}$$

where $Q(x)$ is the integral of the tail of the unit variance Gaussian density function

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt. \tag{11}$$

Equation (10) has a solution $x_1 = 0$, the symmetric solution, as well as solutions at $x_1 = \pm 0.622$. For the symmetric two-level quantizer

$$y_1 = -1/\sqrt{3} \qquad y_2 = 1/\sqrt{3} \tag{12}$$

$$\overline{e^2} = 2/3. \tag{13}$$

The nonsymmetric solutions give a mean-square error which is less than for the symmetric solution. For $x_1 = +0.622$

$$y_1 = -0.266 \qquad y_2 = +1.509 \tag{14}$$

$$\overline{e^2} = 0.599. \tag{15}$$

Fig. 2 shows the signal-to-noise ratio (SNR) as a function of $x_1$ when the output levels are chosen optimally according to (10). As a function of $x_1$ alone, the SNR shows a minimum at the symmetric solution. A more illuminating view of the two-level quantizer is given in Fig. 3, which is a contour plot of the SNR as a function of $y_1$ and $y_2$. In this plot, the decision level is constrained to lie midway between the output levels [see (3)]. The contour plot shows two-fold symmetry, since the quantizers $(a, b)$, $(b, a)$, $(-a, -b)$, and $(-b, -a)$ all have the same mean-square error. A symmetric quantizer is restricted to lie on the diagonal line, $y_1 = -y_2$. The optimal nonsymmetric and symmetric quantizers are shown as crosses on the contour plot. This view shows that the best symmetric quantizer lies at a saddle point in the $y_1 - y_2$ space. This point is also a saddle point in the three-space $y_1 - y_2 - x_1$ since the decision level $x_1$ is chosen optimally in the view shown.

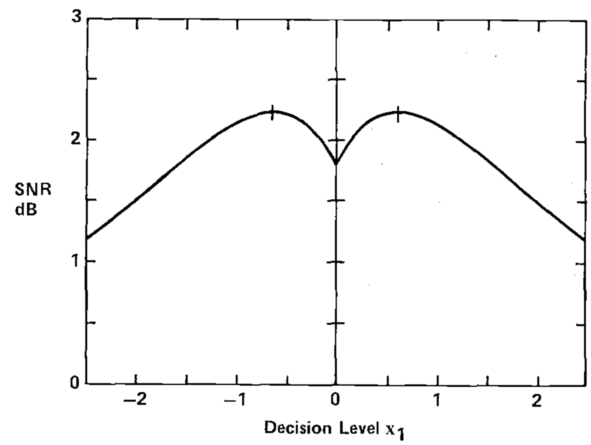


Fig. 2. SNR for a two-level quantizer.

For a three-level quantizer, Fig. 4 gives a contour plot of the signal-to-noise ratio as a function of the two decision levels, $x_1$ and $x_2$. The output levels are again constrained to be the conditional means of the decision regions [see (4)]. This plot is the next higher dimension analog to Fig. 2. In this case, the optimal solution corresponds to a symmetric quantizer.

For higher numbers of levels, the dimensionality of plots corresponding to Figs. 2 or 4 is such that they defy visualization. Instead, a tack suggested by the one-dimensional search algorithm (method II) was adopted. Given an initial output level $y_1$, subsequent output levels up to $y_N$ are found. The difference between $y_N$ and $\hat{y}_N$, the conditional mean of the last decision region, is plotted. When this difference is zero, the entire quantizer satisfies the necessary conditions for a minimum mean-square error quantizer. The minimum of the mean-square error corresponding to the zero crossings of this difference determines the global minimum. Fig. 5 shows such a plot for a six-level quantizer. The plot also shows the SNR as a function of the first output level. Three zero crossings appear. The middle one corresponds to a symmetric solution with $y_1 = -4.773$. The other two correspond to a nonsymmetric solution with $y_1 = -3.111$ or $y_1 = -3.818$.[1] The quantizers corresponding to these last two values of $y_1$ are reflections about zero of each other. Again, a nonsymmetric solution gives the best signal-to-noise ratio.

Previously published tables [3], [7] for the gamma distribution have given only symmetric solutions. Table I compares the best symmetric solution with the optimal solution for selected values of $N$. The three numbers below the quantizer output levels are the mean-square error (for a unit variance distribution), the SNR (in decibels), and the entropy of the

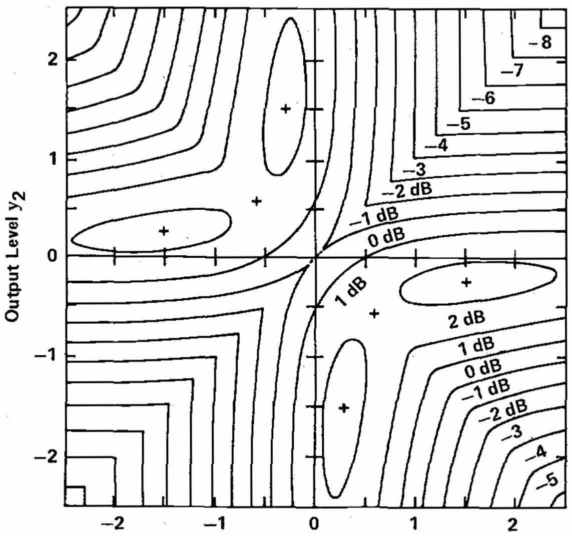


Fig. 3. Contour plot of the SNR for a two-level quantizer.

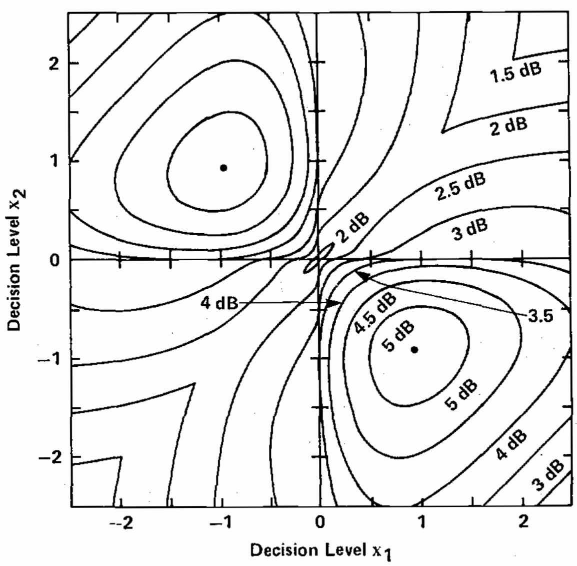


Fig. 4. SNR for a three-level quantizer.

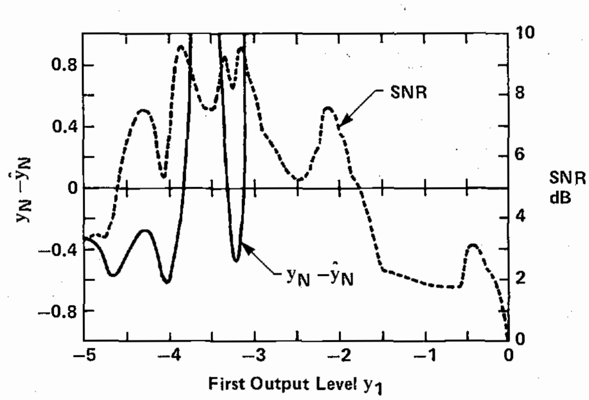


Fig. 5. Last interval difference and SNR for a six-level quantizer.

quantizer. For odd values of $N$, the symmetric solution is optimal, although nonoptimal nonsymmetric solutions satisfying (3) and (4) are possible for $N \geqslant 5$. For even values of $N$, both symmetric and nonsymmetric solutions are shown. For the larger values of $N$, several quantizers (apart from those obtained by reflecting the levels about zero) satisfy the necessary conditions of (3) and (4). For example, for $N = 14$ three distinct nonsymmetric and one symmetric configurations can be found. The table shows that each of the optimal quantizers has an output level close to the central portion of the distribution.

TABLE I
OUTPUT LEVELS FOR NONUNIFORM GAMMA QUANTIZERS

| $N=1$ | $N=2$ | | $N=3$ | $N=4$ | |
|---|---|---|---|---|---|
| 0.000 | ±0.577 | −0.266 1.509 | 0.000 | ±0.313 | −1.981 0.899 |
| 1.000 | 0.6667 | 0.5990 | ±1.851 | ±2.223 | −0.108 2.881 |
| 0.00 dB | 1.76 dB | 2.23 dB | 0.2961 | 0.2318 | 0.2127 |
| 0.00 bits | 1.00 bits | 0.61 bits | 5.29 dB | 6.35 dB | 6.72 dB |
| | | | 0.94 bits | 1.58 bits | 1.27 bits |

| $N=5$ | $N=6$ | | $N=7$ | $N=8$ | |
|---|---|---|---|---|---|
| 0.000 | ±0.210 | −3.111 0.635 | 0.000 | ±0.155 | −3.956 0.487 |
| ±1.039 | ±1.291 | −1.110 1.770 | ±0.710 | ±0.899 | −1.901 1.274 |
| ±3.034 | ±3.307 | −0.058 3.818 | ±1.853 | ±2.057 | −0.755 2.457 |
| 0.1395 | 0.1171 | 0.1090 | ±3.905 | ±4.121 | −0.036 4.538 |
| 8.56 dB | 9.32 dB | 9.63 dB | $8.075 \times 10^{-2}$ | $7.047 \times 10^{-2}$ | $6.632 \times 10^{-2}$ |
| 1.50 bits | 1.99 bits | 1.73 bits | 10.93 dB | 11.52 dB | 11.78 dB |
| | | | 1.90 bits | 2.31 bits | 2.08 bits |

| $N=9$ | $N=10$ | | $N=11$ | $N=12$ | |
|---|---|---|---|---|---|
| 0.000 | ±0.122 | −4.631 0.393 | 0.000 | ±0.100 | −5.193 0.327 |
| ±0.534 | ±0.684 | −2.546 0.990 | ±0.424 | ±0.548 | −3.087 0.806 |
| ±1.326 | ±1.488 | −1.359 1.815 | ±1.025 | ±1.159 | −1.877 1.435 |
| ±2.511 | ±2.682 | −0.564 3.023 | ±1.852 | ±1.993 | −1.049 2.281 |
| ±4.595 | ±4.773 | −0.024 5.126 | ±3.061 | ±3.207 | −0.466 3.505 |
| $5.258 \times 10^{-2}$ | $4.704 \times 10^{-2}$ | $4.462 \times 10^{-2}$ | ±5.166 | ±5.317 | −0.017 5.624 |
| 12.79 dB | 13.28 dB | 13.50 dB | $3.694 \times 10^{-2}$ | $3.362 \times 10^{-2}$ | $3.208 \times 10^{-2}$ |
| 2.22 bits | 2.57 bits | 2.36 bits | 14.33 dB | 14.73 dB | 14.94 dB |
| | | | 2.48 bits | 2.79 bits | 2.59 bits |

| $N=13$ | $N=14$ | | $N=15$ | $N=16$ | |
|---|---|---|---|---|---|
| 0.000 | ±0.084 | −5.675 0.280 | 0.000 | ±0.073 | −6.100 0.243 |
| ±0.350 | ±0.455 | −3.554 0.677 | ±0.297 | ±0.387 | −3.968 0.582 |
| ±0.831 | ±0.945 | −2.329 1.183 | ±0.696 | ±0.795 | −2.730 1.004 |
| ±1.462 | ±1.582 | −1.481 2.690 | ±1.203 | ±1.307 | −1.869 1.525 |
| ±2.308 | ±2.433 | −0.850 3.925 | ±1.851 | ±1.959 | −1.219 2.184 |
| ±3.533 | ±3.662 | −0.367 6.056 | ±2.711 | ±2.822 | −0.711 3.053 |
| ±5.653 | ±5.785 | 0.013 6.373 | ±3.947 | ±4.061 | −0.310 4.298 |
| $2.736 \times 10^{-2}$ | $2.521 \times 10^{-2}$ | $2.418 \times 10^{-2}$ | ±6.078 | ±6.195 | −0.010 6.437 |
| 15.63 dB | 15.98 dB | 16.16 dB | $2.107 \times 10^{-2}$ | $1.961 \times 10^{-2}$ | $1.888 \times 10^{-2}$ |
| 2.70 bits | 2.97 bits | 2.79 bits | 16.76 dB | 17.08 dB | 17.24 dB |
| | | | 2.89 bits | 3.14 bits | 2.97 bits |

For uniformly spaced quantizers, the optimal quantizers are not necessarily symmetrically placed with respect to the mean. This is clear from the two-level example above, for in this case the uniform and nonuniform quantizers are the same. Table II compares symmetric and nonsymmetric uniformly spaced quantizers. The table entries are the interval between levels, $\Delta$ and the offset of the quantizer relative to a symmetrical quantizer, $\epsilon$. Specifically, the output levels are given by

$$y_i = \left(i - \frac{N+1}{2}\right) \Delta + \epsilon, \qquad i = 1, 2, \cdots, N. \tag{16}$$

The step size and offset were calculated using a two-dimensional minimization with the mean-square error as the objective function. The three numbers at the bottom of each entry in the table are the mean-square error (for a unit variance distribution), the SNR (in decibels), and the entropy of the quantizer. This table shows that for $N$ even, the offset for nonsymmetric quantizers is nearly equal to one half of the step

[1]The example chosen for Fig. 1 is the optimal six-level quantizer for the gamma distribution.

TABLE II
STEP SIZE AND OFFSET FOR UNIFORM GAMMA QUANTIZERS

| $N = 1$ | $N = 2$ | | $N = 3$ | $N = 4$ | |
|---|---|---|---|---|---|
| – | 1.155 | 1.775 | 1.851 | 1.066 | 1.560 |
| 0.000 | 0.000 | ±0.622 | 0.000 | 0.000 | ±0.710 |
| 1.000 | 0.6667 | 0.5990 | 0.2961 | 0.3200 | 0.2330 |
| 0.00 dB | 1.76 dB | 2.23 dB | 5.29 dB | 4.95 dB | 6.33 dB |
| 0.00 bits | 1.00 bits | 0.61 bits | 0.94 bits | 1.67 bits | 1.14 bits |
| $N = 5$ | $N = 6$ | | $N = 7$ | $N = 8$ | |
| 1.342 | 0.912 | 1.208 | 1.079 | 0.796 | 0.998 |
| 0.000 | 0.000 | ±0.571 | 0.000 | 0.000 | ±0.480 |
| 0.1597 | 0.1934 | 0.1346 | 0.1045 | 0.1323 | $9.130 \times 10^{-2}$ |
| 7.97 dB | 7.14 dB | 8.71 dB | 9.81 dB | 8.78 dB | 10.40 dB |
| 1.35 bits | 1.94 bits | 1.48 bits | 1.62 bits | 2.13 bits | 1.72 bits |
| $N = 9$ | $N = 10$ | | $N = 11$ | $N = 12$ | |
| 0.913 | 0.708 | 0.858 | 0.798 | 0.640 | 0.757 |
| 0.000 | 0.000 | ±0.416 | 0.000 | 0.000 | ±0.369 |
| $7.551 \times 10^{-2}$ | $9.756 \times 10^{-2}$ | $6.747 \times 10^{-2}$ | $5.795 \times 10^{-2}$ | $7.560 \times 10^{-2}$ | $5.260 \times 10^{-2}$ |
| 11.22 dB | 10.11 dB | 11.71 dB | 12.37 dB | 11.21 dB | 12.79 dB |
| 1.83 bits | 2.28 bits | 1.91 bits | 2.00 bits | 2.40 bits | 2.07 bits |
| $N = 13$ | $N = 14$ | | $N = 15$ | $N = 16$ | |
| 0.712 | 0.585 | 0.681 | 0.645 | 0.540 | 0.620 |
| 0.000 | 0.000 | ±0.334 | 0.000 | 0.000 | ±0.305 |
| $4.633 \times 10^{-2}$ | $6.071 \times 10^{-2}$ | $4.255 \times 10^{-2}$ | $3.816 \times 10^{-2}$ | $5.008 \times 10^{-2}$ | $3.536 \times 10^{-2}$ |
| 13.34 dB | 12.17 dB | 13.71 dB | 14.18 dB | 13.00 dB | 14.51 dB |
| 2.15 bits | 2.52 bits | 2.21 bits | 2.27 bits | 2.62 bits | 2.32 bits |

size. Note also that for the symmetric case, adding an additional output level to a quantizer with an odd number of levels actually increases the mean-square error.

Another issue of interest is the convexity of the mean-square error as a function of the number of bits, $\log_2 N$. The bit assignment procedures used for the optimal allocation of a quota of available bits to the components of a vector source assume convexity of the distortion function [11]. For an integral number of bits, the mean-square error for the nonsymmetric quantizers (both uniformly and nonuniformly spaced) is convex while for the symmetric quantizers it is not. However, even for the nonsymmetric quantizers, the mean-square error is locally not convex as a function of $\log_2 N$, when only $N$ is required to be an integer. This can be seen from the fact that if the number of output levels is odd, adding one more level results in a relatively small decrease in mean-square error, but adding yet another output level to give an odd number of levels results in a relatively larger decrease in mean-square error.

Nonconvexity of the mean-square error can have interesting consequences. For instance, consider coding a gamma distributed signal with 1 bit per sample. For symmetric quantizers, a lower *average* mean-square error is obtained if samples are coded alternately using a 2 bit and a 0 bit quantizer, than if a 1 bit quantizer is used for every sample.

## VI. GENERAL GAMMA DISTRIBUTION

Plots of the difference between $y_N$ and $\hat{y}_N$, the conditional mean of the last decision region, were also generated for the generalized gamma distribution. Fig. 6 shows such a plot for a two-level quantizer for a density with parameter $a = 0.9$. The plot shows that a nonsymmetric solution is optimal for this case. As the parameter $a$ approaches unity, the three zero crossings evident in the plot coalesce to give a single unique solution for the Laplace density. For values of the parameter $a$ below unity, a nonsymmetric solution is optimal. This then

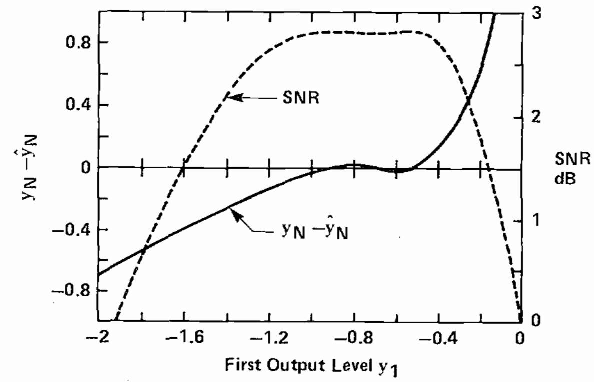


Fig. 6. Last interval difference and SNR for a general gamma distribution ($a = 0.9$).

indicates that the Laplace distribution occupies a unique place amongst the family of general gamma distributions—on the boundary separating those distributions which have unique minima and those which do not. For the general gamma distribution, log-concavity seems to be both a necessary and sufficient condition for uniqueness.

## VII. A FAMILY OF SYMMETRIC DISTRIBUTIONS

In order to better understand the phenomena which account for the multiple stationary points, a simple distribution was contrived. This distribution consists of two superimposed uniform densities (Fig. 7). The underlying density extends between −1 and +1. The superimposed density is parameterized by $r$, the fraction of the probability in the superimposed density and by $b$, the extent of the superimposed density. For $b = 1$, $r = 0$, or $r = 1$, the overall density degenerates into a simple uniform density with no nontrivial stationary points.

For a two-level quantizer, we can adopt a procedure analogous to (9) and (10) and solve for $x_1$. Because of the symmetry, consider only $x_1 \geqslant 0$. For $0 \leqslant x_1 \leqslant b$, a symmetric and possibly a nonsymmetric solution appear:

$$x_1 = 0$$

$$x_1^2 = \frac{b(b - r(1-b)^2(1-r))}{(b + r(1-b))^2}. \tag{16}$$

The nonsymmetric solution appears for $b' \leqslant b \leqslant \hat{b}$ where

$$b' = 1 - \frac{\sqrt{1 + 4r(1-r)} - 1}{2r(1-r)}$$

$$\hat{b} = \frac{\sqrt{8r+1} - 1 - 2r}{2(1-r)}. \tag{17}$$

The nonsymmetric solution given in (16) corresponds to a local maximum of the mean-square error with respect to changes in $x_1$. It is a saddle point in the $y_1 - y_2 - x_1$ space. For $b \leqslant x_1 \leqslant 1$

$$x_1 = \hat{b} \tag{18}$$

gives a local minimum of the mean-square error, corresponding to a nonsymmetrical solution. The situation is summarized in Fig. 8 which plots the various regions in the $b$–$r$ plane. The

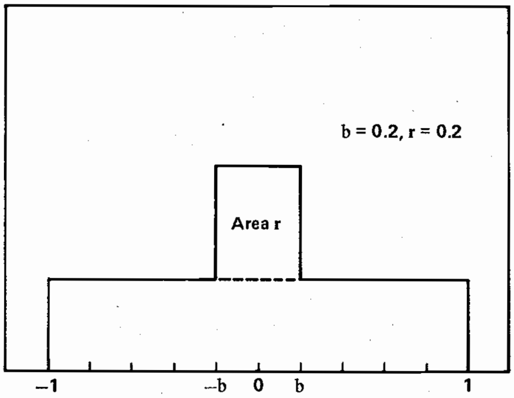


Fig. 7. A symmetric density function.

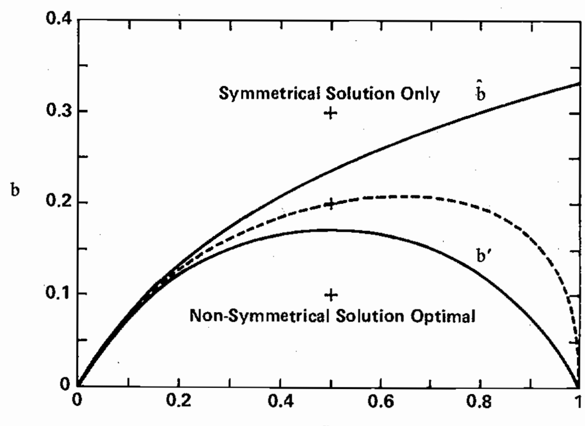


Fig. 8. Quantizer solutions in the $b$–$r$ plane.

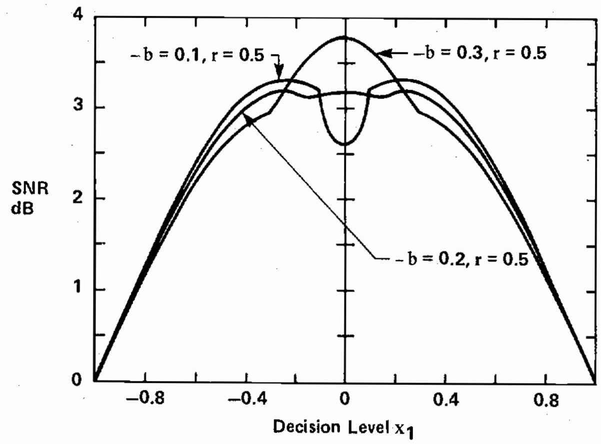


Fig. 9. SNR for a symmetric density function.

region above $\hat{b}$ gives only a single symmetrical solution. The region below $\hat{b}$ admits two or three solutions. Between $\hat{b}$ and $b'$ there are two minima in the mean-square error ($x_1 = 0$ and $x_1 = \hat{b}$) separated by a local maximum [whose location is given by (16)]. An additional dashed line is shown in Fig. 8. For $\hat{b}$ below the dashed line, the nonsymmetrical solution has a lower mean-square error than the symmetrical solution. Fig. 9 shows the SNR in decibels as a function of $x_1$ for the three points shown as crosses in Fig. 8 to illustrate three different regions. This example shows that as the probability tends to get concentrated near the origin, specifically in the area of the $b$-$r$ plane below the dashed line of Fig. 8, nonsymmetrical solutions become optimal. This particular example has a probability concentration near the point of symmetry such that optimal quantizers with even numbers of levels will be nonsymmetrical while those with odd numbers of levels will be symmetrical. Other examples with probability concentrations symmetrically placed about the mean would lead to the situation in which optimal quantizers with odd numbers of levels would be nonsymmetric while those with even numbers of levels would be symmetric.

## VIII. Summary

We have shown that symmetrical distributions of practical interest can have nonsymmetrical optimal minimum mean-square error quantizers. New revised quantizer tables have been given for the gamma density function. Generalized gamma densities with $a < 1$ also admit nonsymmetrical optimal quantizers. A simple family of probability density functions has been studied to examine the conditions under which local stationary points appear.

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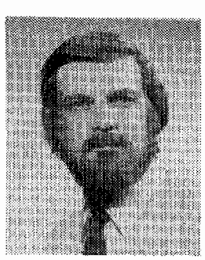

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