# Asymptotic Receiver Structures for Joint Maximum Likelihood Time Delay Estimation and Channel Identification Using Gaussian Signals

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*Abstract*—**This correspondence addresses the problem of jointly estimating the relative time delay and the impulse response linking two received discrete-time Gaussian signals. Using two different methods, possible structures for the joint maximum likelihood (ML) estimator are proposed, when the observation interval is long compared to both the delay to estimate and the correlation time of the various random processes involved. These structures generalize the cross-correlation method with prefiltering that implements the ML estimation of pure time delays.**

## I. Introduction

In this correspondence, we derive the form of the maximum likelihood estimator that jointly computes the estimates of the time delay and the impulse response that link two observed discrete-time Gaussian signals. The model assumed for these two signals is such that one signal is the delayed and filtered version of the other. It is of the form

$$y_1(n) = s(n) + v_1(n)$$
$$y_2(n) = \mathcal{L}_{D_n, h(n)}[s(n)] + v_2(n) \qquad (1)$$

where $s(n)$ is the transmitted stationary Gaussian signal and $D_n$ is a time-varying delay. The linear operator $\mathcal{L}_{D_n, h(n)}(\cdot)$ is unknown and takes the form of a linear filtering operation, with the filter impulse response $h(n)$, of a delayed by $D_n$ version of the signal $s(n)$. The signals $v_1(n)$ and $v_2(n)$ are zero-mean Gaussian stationary noise processes, assumed uncorrelated with each other, as well as with $s(n)$. All the discrete-time signals defined above are assumed to be the sampled versions, with sampling period $T$, of continuous-time signals that are strictly band limited to the frequency range $-1/2T < f < 1/2T$.

The operator $\mathcal{L}_{D_n, h(n)}[\cdot]$ can correspond to the filtering of a delayed version of $s(n)$ or to a filtering operation followed by a delay. This signal model may be applied when a signal, emanating from a remote source, travels through two different paths and is monitored by two spatially separated sensors in the presence of uncorrelated noise. It can also apply to the cases of noise cancellation (e.g., echo cancellation) or equalization, where one of the two signals is filtered by an estimate of $\mathcal{L}_{D_n, h(n)}(\cdot)$ or its inverse. In this case the two observed signals are used in the estimation of the filtering and delay operation.

In most of the work done on time delay estimation, it is assumed that the filter $h(n)$ is either absent or is represented by a simple gain. The ML estimator has been derived for this particular case, when the delay is constant and the observation interval is both short [1] or large [2] and when the delay is time varying [3]. The simplest form of estimator is obtained for the case of constant delay and long observation interval. The structure of this asymptotic receiver is that of a cross-correlation performed on a prefiltered version of the two signals $y_1(n)$ and $y_2(n)$. Because of its simplicity, this so-called generalized cross-correlation method has been largely accepted, although Champagne *et al.* have shown recently that it can lead to large errors when the actual observation interval is short [1].

The purpose of this correspondence is to present some possible structures for the asymptotic ML receiver, when the filter $h(n)$ is the impulse response of a general linear time-invariant filter and the delay $D_n$ is assumed constant and equal to $D$. Very little work concerning this generalization of the pure time delay estimation problem has appeared in the open literature, despite the fact that an asymptotic receiver structure is easily obtained using the frequency domain and simple matrix theory. Before we present the outline of this derivation, we express the signal model of (1) in vector form and give the likelihood function for the joint estimation of the delay and the filter. Since the form of this function is essentially unchanged from the one derived by Stuller [3] for the ML estimation of a pure delay over short observation intervals, we pay tribute to his work by extending the form of estimator that he derived to our more general case and by specializing it to the asymptotic case. We then show that the same result can be obtained without referring to the general (nonasymptotic) case and give another form of the asymptotic receiver. Note that the goal of our work is to present some asymptotic structures, without making any attempt to study the performance of the respective receivers.

## II. The Likelihood Function

We assume that the reference filter $h(n)$ and the reference delay $D$ are estimated, respectively, by the finite impulse response $w(n)$ and by the delay $d$. The mathematical model for the observed waveforms is therefore an extension of the one used by Stuller [3] and its vector form is

$$\boldsymbol{y}(n) = \boldsymbol{s}(n \mid d, w) + \boldsymbol{v}(n) \qquad (2)$$

where the vectors are defined as

$$\boldsymbol{y}(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} \qquad (3)$$

$$\boldsymbol{s}(n \mid d, w) = \begin{bmatrix} s(n) \\ \mathcal{L}_{d, w(n)}[s(n)] \end{bmatrix} \qquad (4)$$

$$\boldsymbol{v}(n) = \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix}. \qquad (5)$$

It is assumed that the zero-mean Gaussian noise processes $v_1(n)$ and $v_2(n)$ are both white with power spectral densities of $N_0/2$

Manuscript received December 18, 1990; revised November 8, 1991. This work was supported by the Canadian Department of Communications and by the Natural Science and Engineering Research Council of Canada.

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IEEE Log Number 9106546.

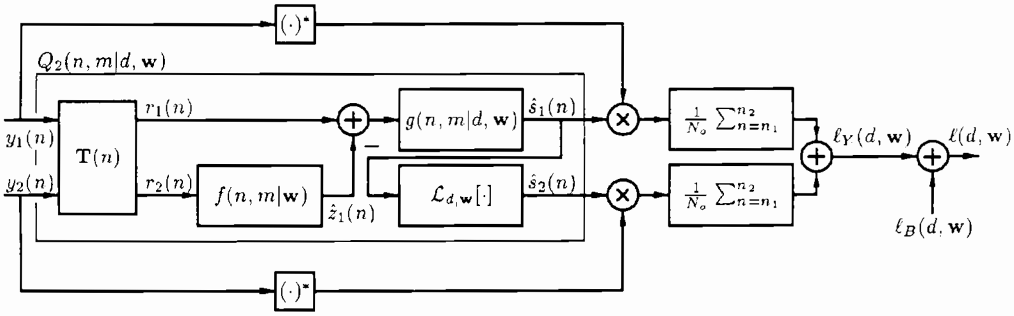


Fig. 1. Block diagram of the noncausal joint maximum likelihood estimator.

$W$/Hz. The vector $w$ is defined as the assumed reference filter weight vector, whose components are the samples of the impulse response $w(n)$. Because of the finite-length nature of $w(n)$, the vector $w$ also has a finite number of dimensions.

The objective is to obtain a processor that computes, over a certain discrete-time interval $[n_1, n_2]$, the probability density function $p_{Y|D,W}(y|d, w)$ (or a function of it) of the observed signals $y_1(n)$ and $y_2(n)$, given the assumed parameters $d$ and $w$. The ML estimates $\hat{D}$ and $\hat{h}(n)$ are the values of $d$ and $w(n)$ that maximize the density or the likelihood function. The likelihood function can be obtained by extending the results in [4] or [3] for the maximum likelihood estimation of the parameters of Gaussian processes in white Gaussian noise. This is performed in detail in [5], using a vector form of the Karhunen-Loève decomposition [6] for discrete-time signals. It is assumed that the transmitted signal covariance matrix $\Phi_{ss}(k|d, w)$, defined as

$$\Phi_{ss}(k|d, w) = E[s(n + k|d, w)s^H(n|d, w)] \tag{6}$$

is a positive definite function of time, which implies that all the eigenvalues are real and strictly positive numbers, and that the set of vector eigenfunctions is a complete orthonormal set over the interval $[n_1, n_2]$ [6]. Therefore, all the information present in $y(n)$ is present in the series coefficients, for a number of terms tending to infinity. In the present case, since we deal with discrete-time signals over a finite time interval of $N = n_2 - n_1 + 1$ samples, $N$ orthogonal vectors are sufficient to represent $y(n)$ [6].

The final form of the likelihood function is found to be the sum of a noncausal term $l_Y(d, w)$ and a bias term $l_B(d, w)$ [3] and [4, pp. 10–14]. Therefore,

$$l(d, w) = l_Y(d, w) + l_B(d, w) \tag{7}$$

where

$$l_Y(d, w) = 1/N_0 \sum_{n=n_1}^{n_2} \sum_{m=n_1}^{n_2} y^H(n) Q_2(n, m|d, w) y(m) \tag{8}$$

and

$$l_B(d, w) = -\frac{1}{2} \sum_{i=1}^{N} \ln\left[1 + \frac{2\lambda_i(d, w)}{N_o}\right]. \tag{9}$$

In (8), $H$ denotes complex conjugate tranpose, $Q_2(n, m|d, w)$ is the matrix impulse response of the noncausal linear minimum mean square error (MMSE) point estimator of $s(n|d, w)$, from the received vector $y(n)$, given the parameters $d$ and $w$ [6]. In (9), $\lambda_i(d, w)$ is the $i$th eigenvalue of the covariance matrix $\Phi_{ss}(k|d, w)$. The matrix impulse response $Q_2(n, m|d, w)$ is given by the solution of the "normal" equation

$$\frac{N_o}{2} Q_2(n, m|d, w) + \sum_{k=n_1}^{n_2} Q_2(n, k|d, w) \Phi_{ss}(k - m|d, w) = \Phi_{ss}(n - m|d, w) \tag{10}$$

for $n_1 \le n \le n_2$, $n_1 \le m \le n_2$.

In order to compute the likelihood function, this form of integral equation must be solved for $Q_2(n, k|d, w)$, and the bias term must be computed. The solution of (10) has been derived in [3] for a pure delay by using the reversibility theorem [6] and a "constructive derivation" method. This method is extended in [5] to the general case (1) for a finite observation interval. The received vector $y(n)$ is first processed by a linear invertible matrix operator $T(n)$, which output $r(n)$ is defined as

$$r(n) = \begin{bmatrix} r_1(n) \\ r_2(n) \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathcal{L}_{d,w}^{-1}[y_2(n)] \\ 0 \end{bmatrix}, & n_1 - \lfloor d/T \rfloor \le n < n_1 \\ \frac{1}{2}\begin{bmatrix} y_1(n) + \mathcal{L}_{d,w}^{-1}[y_2(n)] \\ y_1(n) - \mathcal{L}_{d,w}^{-1}[y_2(n)] \end{bmatrix}, & n_1 \le n < n_2 - \lfloor d/T \rfloor \\ \begin{bmatrix} y_1(n) \\ 0 \end{bmatrix}, & n_2 - \lfloor d/T \rfloor \le n \le n_2, \end{cases} \tag{11}$$

where $\lfloor x \rfloor$ denotes the integer part of $x$ and $\mathcal{L}_{d,w}^{-1}[\cdot]$ is the inverse of $\mathcal{L}_{d,w}[\cdot]$. The form of estimator obtained with Stuller's method is given in Fig. 1, where $f(n, m|w)$ is the impulse response of the noncausal linear MMSE point estimator of $z_1(n)$ (the noisy part of $r_1(n)$) from $z_2(n)$ (the noisy part of $r_2(n)$), and $g(n, m|d, w)$ is the impulse response of the noncausal linear MMSE point estimator of $s(n)$ from $s(n) + z_1(n) - \hat{z}_1(n)$. The signals $\hat{s}_1(n)$ and $\hat{s}_2(n)$ are the components of the vector $\hat{s}(n)$, resulting from the application of $Q_2(n, m|d, w)$ on $y(n)$.

The complexity of this receiver resides in the evaluation of $f(n, m|w)$, $g(n, m|d, w)$ and $l_B(d, w)$ that must be done for a finite observation interval. An approximation of these functions can be obtained by assuming that the observation interval is large in comparison to both the delay to estimate and the correlation time of the various random processes involved. In this case, the computation of the likelihood function is done by using time-invariant filters and frequency domain relationships. The form of the estimator obtained using this method is of practical importance because if the observation time is long compared with the time necessary for the system transients to die out, the estimator performance is nearly optimum [4]. The assumption of a long interval can be used only to solve the integral equations of the form of (10), while the resulting receivers can still be used over the interval $[n_1, n_2]$ (although, as pointed out in [1], the performance can be poor if the interval is short). The result of this approximation is given in the next subsection.

Before we leave the subject of ML estimation over a short observation interval, it is important to mention the recent work by Champagne *et al.* In [7], the authors have considered the factorization properties of ML space-time processors in nonstationary environments which, when specialized to the case of pure delay estimation, lead directly to Stuller's results. In [1], the same authors derive another form for the ML pure delay estimator by applying a dimensionality reduction technique that generalizes Stuller's method. The extension of this work to the case of joint delay and impulse response estimation remains a potential area of research.

### A. The Likelihood Function for a Long Observation Interval

In this subsection, we present briefly the form of the critical components of the asymptotic likelihood function. The impulse responses $f(n, m \mid w)$ and $g(n, m \mid d, w)$ are first given, followed by the asymptotic form of the eigenvalues $\lambda_i$ used in the computation of $l_B(d, w)$.

*1) The Asymptotic Function $l_Y(d, w)$:* When the observation interval is long, it can be shown that both the impulse responses $f(n, m \mid w)$ and $g(n, m \mid d, w)$ become time invariant and independent of $d$ [5]. They are given by

$$f(n \mid w) = F^{-1}\left[\frac{|W(e^{j\omega})|^2 - 1}{|W(e^{j\omega})|^2 + 1}\right] \tag{12}$$

and

$$g(n \mid w) = F^{-1}\left[\frac{\Phi_{ss}(e^{j\omega})(|W(e^{j\omega})|^2 + 1)}{\Phi_{ss}(e^{j\omega})(|W(e^{j\omega})|^2 + 1) + N_o/2}\right] \tag{13}$$

where $W(e^{j\omega}) = F[w(n)]$, $F[\cdot]$ is the Fourier transform operator and $\Phi_{ss}(e^{j\omega})$ is the power spectral density of the transmitted scalar signal $s(n)$.

Under the asymptotic conditions, the quantity $\lfloor d/T \rfloor$ becomes negligible compared to the length of the observation interval and the matrix operator $T(n)$ is given by the middle equation of (11). The resulting form of $Q_2(n \mid d, w)$, as shown in Fig. 1, is a noncausal processor, and a causal form can be obtained by delaying the matrix impulse response and the input vector by a suitable value.

*2) The Asymptotic Function $l_B(d, w)$:* Some care must be exercised in the evaluation of the bias term given in (9). As $N$ goes to infinity, this term becomes infinite itself because it can be expressed as the integral of a function of the input signal power spectral density, times the length of the observation interval [3], [6, p. 207]. Proceeding as in [6, pp. 206-207], it can be shown that, for $N \gg (TF_{\max})^{-1}$, where $F_{\max}$ is the maximum frequency contained in the analog signal $s(t)$, the asymptotic form of the eigenvalues is independent of $d$ and is given by

$$\lambda_i(w) \approx \Phi_{ss}(e^{j(2\pi/N)i})(1 + |W(e^{j(2\pi/N)i})|^2). \tag{14}$$

The corresponding vector eigenfunction is

$$f_i(n) = e^{j(2\pi/N)ni}\begin{bmatrix} 1 \\ e^{-j(2\pi/N)d/T}W(e^{j(2\pi/N)i}) \end{bmatrix}.$$

The eigenvalues are therefore function of $w(n)$ only and can be computed using (14). The asymptotic bias term is consequently also independent of $d$. By using (9), (12)-(14) in Fig. 1, we obtain a form for the asymptotic ML receiver.

## III. Solving the Asymptotic Integral Equation in the Frequency Domain

In the asymptotic case, the solution of the integral equation (10) can be done directly, without having to rely on the estimator form for arbitrary observation intervals. This implies that the form of estimator based on the impulse responses of (12) and (13) can be obtained in a more straightforward manner. We present this result as an alternative to the method given above. Note that the asymptotic form of $l_B(w)$ does not change.

When the asymptotic condition $N \gg (TF_{\max})^{-1}$ is satisfied, we can set $n_1 \to -\infty$ and $n_2 \to \infty$ and write (10) as

$$\sum_{k=-\infty}^{\infty} Q_2(k \mid d, w)\Phi_{yy}(n - k \mid d, w) = \Phi_{ss}(n \mid d, w) \tag{15}$$

where $\Phi_{yy}(k \mid d, w)$ is defined as

$$\begin{aligned}\Phi_{yy}(k \mid d, w) &= E[y(n + k)y^H(n) \mid d, w]\\ &= E[s(n + k \mid d, w)s^H(n \mid d, w)] + E[v(n + k)v^H(n)]\\ &= \Phi_{ss}(k \mid d, w) + \frac{N_o}{2} I\delta(k)\end{aligned} \tag{16}$$

with $I$ representing the $2 \times 2$ identity matrix and $\delta(k)$ the unit sample function.

Taking the Fourier transform and solving, the frequency domain solution is the matrix transfer function of the Wiener filter, given by

$$Q_2(e^{j\omega} \mid d, w) = \Phi_{ss}(e^{j\omega} \mid d, w)\Phi_{yy}^{-1}(e^{j\omega} \mid d, w). \tag{17}$$

Solving the above equation and using the result in (8) gives, after some manipulations

$$\begin{aligned} l_Y(d, w) = {} & 1/2N_o \sum_n [\tilde{w}(-n \mid w) \otimes y_1^*(nT - d)]y_2(n)\\ & + 1/2N_o \sum_n [\tilde{w}^*(-n \mid w) \otimes y_1(nT - d)]y_2^*(n)\\ & + 1/2N_o \sum_n [\tilde{w}^*(-n \mid w) \otimes w^{-1}(n) \otimes y_1(n)]y_1^*(n)\\ & + 1/2N_o \sum_n [\tilde{w}(n \mid w) \otimes w(n) \otimes y_2(n)]y_2^*(n) \end{aligned} \tag{18}$$

where

$$\tilde{w}(n \mid w) = F^{-1}\left[\frac{2G(e^{j\omega})W^*(e^{j\omega})}{|W(e^{j\omega})|^2 + 1}\right] \tag{19}$$

$$G(e^{j\omega}) = \frac{\Phi_{ss}(e^{j\omega})(|W(e^{j\omega})|^2 + 1)}{\Phi_{ss}(e^{j\omega})(|W(e^{j\omega})|^2 + 1) + N_o/2} \tag{20}$$

and $\otimes$ is the convolution operator.

The realization of the asymptotic estimator, based on (18), is illustrated in Fig. 2. It is essentially identical to the one of Fig. 1 with the definitions of (12) and (13) (and some block diagram manipulations).

Note that if the reference filter is absent, we only have to estimate the delay $d$. In this case, the two bottom branches of the estimator and the bias term are not functions of the delay and can be removed from the computation of the likelihood function. This leaves only $l_Y(d)$ to compute with the remaining upper part of Fig. 2. This structure is the generalized cross correlator implementing ML delay estimation when complex signals are used. In the case

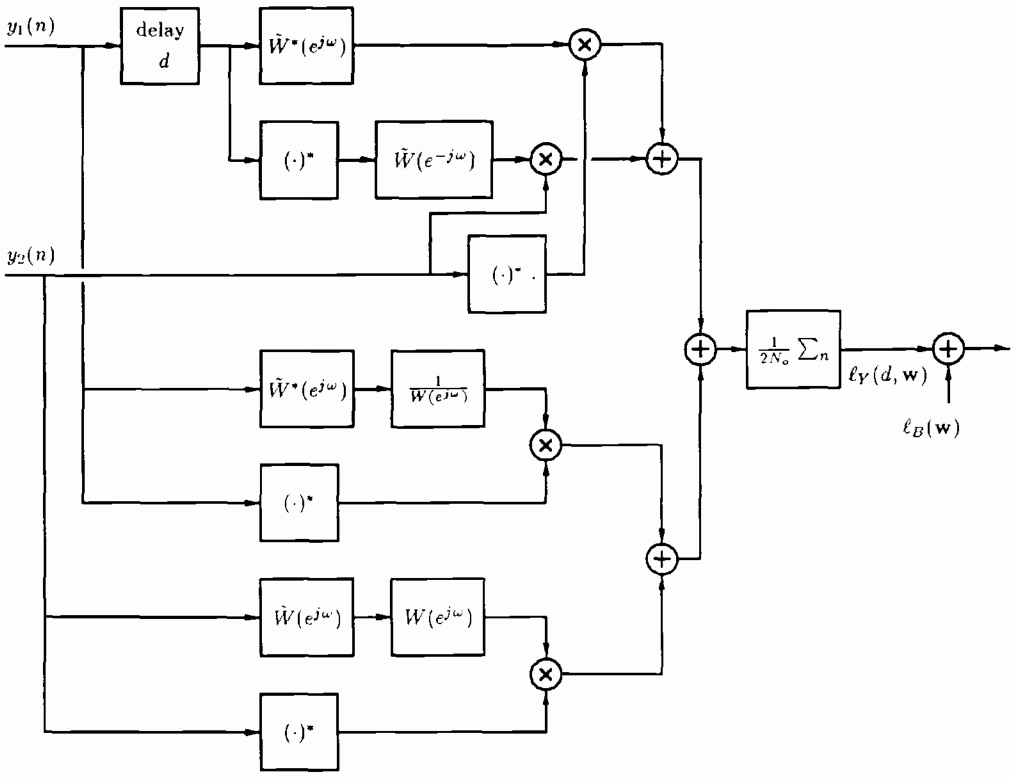


Fig. 2. Block diagram of an approximate noncausal joint maximum likelihood receiver.

of real signals and $h(n) = \delta(n)$, the structure simplifies further to the exact generalized cross correlator of Knapp and Carter with the likelihood function given by

$$l_Y(d) = \frac{1}{N_o} \sum_n [g(n \mid w) \otimes y_1(nT - d)] y_2(n). \tag{21}$$

The structure of Fig. 2 has some advantages over that of Fig. 1. First of all it clearly identifies the delay estimator as a single delay element. It is also given in terms of three different correlation branches that could be identified as: 1) a generalized cross-correlation branch (the upper one) involving $y_1(n)$ and $y_2(n)$ and 2) two generalized autocorrelation branches involving the two received signals separately. In term of complexity, the two structures are more or less equivalent since one can be obtained from the other by simple block diagram manipulations.

## IV. Conclusion

We have presented in this correspondence two asymptotic forms for the joint maximum likelihood estimator, for time-invariant delay and filter. The two structures can be obtained by specializing the general nonasymptotic form to the long observation case, or by solving directly the asymptotic normal equation in the frequency domain. The form presented in Fig. 1 is compatible with that of the ML estimator derived by Stuller. The estimator of Fig. 2 retains the generalized cross-correlator form defined by Knapp and Carter. The structures discussed in this publication can be of practical interest in evaluating other forms of suboptimum joint estimators for the signal model of (1). They are also of academic interest since they solve the generalization of the pure time delay estimation problem.

## Acknowledgment

The computing equipment of Bell Northern Research, of l'Institut National de la Recherche Scientifique, and of the Communications Research Centre are greatly acknowledged. The comments from the various reviewers are appreciated.

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