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Shaping of Multi-dimensional Signal Constellations Using a Lookup Table

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Abstract: Shaping concerns the selection of the boundary of a signal constellation to reduce its average energy. Addressing is the assignment of the data bits to the constellation points. A major concern of the shaping regions is their addressing complexity. In this work, we use a lookup table for addressing. The method is based on partitioning the two-dimensional subconstellations into shaping shells of equal size and increasing average energy. A lookup table is used to select a subset of the cartesian product of the partitions. This partitioning is compatible with a multidimensional trellis coded modulation (TCM) scheme. As part of the calculations, we have found a closed form formula for the weight distribution of the half integer lattice, $Z^N+(1/2)^N$, for dimensionality $N=4,8$.

1 Introduction

In a two-dimensional signal constellation, the points near the boundary are of higher energy. By using these points less frequently, the average energy and the entropy of the set decreases. By appropriate selection of the probabilities, one obtains a lower average energy for a given entropy, [1]. The nonequiprobable use of the signal points reduces the entropy of the set. To have the same rate, more points are needed. This is a price to be paid for the reduction in the average energy and is denoted by Constellation-Expansion-Ratio, CER_s. This will also increase the Peak-to-Average-power-Ratio, PAR, of the set.

In a nonequiprobable signaling scheme, we are potentially faced with variable delay problem. Using an appropriate boundary in an N-dimensional space (N even) and using the N-dimensional points with equal probability is a way to avoid this problem. Another method is given in [1]. The reduction in the average energy due to the use of the region $\mathcal{C}_N$ for shaping is called the shape gain, γ_s, of $\mathcal{C}_N$.

Another issue is the addressing complexity. This is the assignment of the data bits to the constellation points. In an unshaped constellation, which is equal to the cartesian product of its two-dimensional subconstellations, addressing is achieved independently along each two-dimensional subspace. In a shaped constellation, as some of the elements of the cartesian product are not allowed, independent addressing is not applicable and a more complex, generally N-dimensional, addressing scheme is needed.

The factors γ_s, CER_s and PAR are defined as, [2],

$$\gamma_s(C_N)=\frac{|C_N|^{1/n}}{6P_2(C_N)}, \tag{1}$$

$$\mathrm{CER}_s(C_N)=\frac{|C_2|}{|C_N|^{1/n}}, \tag{2}$$

$$\mathrm{PAR}(C_N)=\frac{E_p(C_2)}{P_2(C_N)}, \tag{3}$$

where $n=N/2$, C_2 is the two-dimensional subconstellation of C_N, and $|C|$, $P_2(C)$ and $E_p(C)$ are the cardinality, the energy per two dimensions and the peak energy of the constellation C.

There exists a tradeoff between γ_s and CER_s and also between γ_s and PAR. For a given dimensionality N, a spherical shaping region $\mathcal{S}_N$, results in the highest γ_s but also has large values for CER_s and PAR. As $N\to\infty$, the shape gain of $\mathcal{S}_N$ tends to 1.53 dB, [3]. This is an upper bound for the shape gain of all regions and is achievable at the price of $\mathrm{CER}_s=\infty$ and $\mathrm{PAR}=\infty$.

In the work of Wei, [4], shaping is a side effect of the method employed to transmit a nonintegral number of bits per two dimensions. The addressing of this method is achieved by a lookup table. Forney and Wei generalize this method in [2]. Conway and Sloane in [5] introduced the idea of the Voronoi constellation based on using the Voronoi region of a lattice Λ_s as the shaping region. The Voronoi constellations are further considered by Forney in [6]. In [1], Calderbank and Ozarow introduced a shaping method which is based on using the points of the two-dimensional subconstellations with nonequal probability. The structure of the optimum shaping regions together with the analytical expressions determining the optimum tradeoff are given in [7]. The idea of the combined shaping and coding over a multitone channel is introduced in [8]. The idea of the unsymmetrical shaping, with application to spectral shaping is introduced in [9].

In this work, we use a lookup table for the addressing of a shaped constellation in a TCM scheme. The method is based on partitioning the two-dimensional subconstellations into the shaping shells of equal size and increasing average energy. A lookup table is used to select a subset of the partitions in the cartesian product. As part of the calculations, we have found a closed form formula for the weight distribution of the half integer lattice, $Z^N+(1/2)^N$, for dimensionality $N=4,8$.

2 System model

A general coset coding scheme based on the lattice partition Z^N/Λ, Z^N is the N-dimensional integer lattice and Λ is a sublattice of Z^N, is composed of two different parts. The first part selects a finite number of points from Z^N as the signal constellation. This selection is based on minimizing the average energy of the set for a given number of points and given CER_s. The second part selects a coset of Λ within the signal constellation. In the case that the selected coset has more than one point, a third part is used to address one point within that coset. The first and second parts have to do with shaping and coding, respectively. In continuous approximation, the discrete set of the constellation points is approximated by a continuous uniform density within the shaping region. Assuming continuous approximation, all the parameters concerning shaping like γ_s, CER_s and PAR are determined by the first part and all the parameters concerning channel coding are determined by the second part. The third part scales the number of the constellation points.

Figure (1) shows the block diagram of the coding scheme under consideration. Signal space has $N = 2n$ dimensions and carries Q bits per two dimensions. There is also one bit of coding redundancy per N dimensions. The two-dimensional subcon-

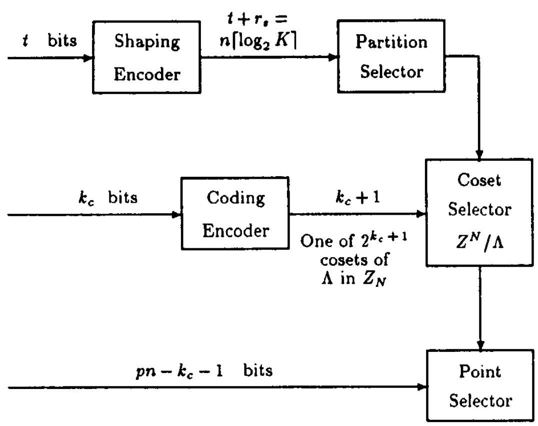

Fig. 1 Block diagram of the coding system.

stellations are selected from the cross constellation, [3]. In the case that we need a nonintegral bit rate per two dimensions, the necessary number of points of the least energy from the larger constellation are added around the existing points. Each two-dimensional point is labeled by a two part label. The first part of the label is determined by the shaping block. The second part of the label is determined by coding block.

For shaping, the two dimensional subconstellation containing M points are partitioned into K shaping shells of equal size and increasing average energy. The shells have four way symmetry. Each shell contains $P = 2^p$ points, $M = K \times P$, and is referenced by $\lceil \log_2 K \rceil$ bits. All the P points within a shaping shell use these bits as their shaping label. We refer to this partitioning/labeling as the shaping partitioning/labeling. Figure (2) shows an example of a 256 points constellation divided into 4 shells. In all cases, a finer partitioning of $2K$ shells can be obtained from a constellation already divided into K shells by subdividing each shell into two subshells.

The two-dimensional shaping shells partition the N-dimensional space into K^n, $n = N/2$, shaping clusters. Shaping is achieved by selecting $T = 2^t$ clusters of the least average energy. The t shaping bits entering the shaping encoder are used to address one of these T clusters. The shaping encoder adds r_s redundant bits to the incoming bits and the $t + r_s = n\lceil \log_2 K \rceil$ bits at its output are used in parallel to address one shaping shell within each two-dimensional subconstellation. The total information rate is equal to $pn + t - 1$ bits.

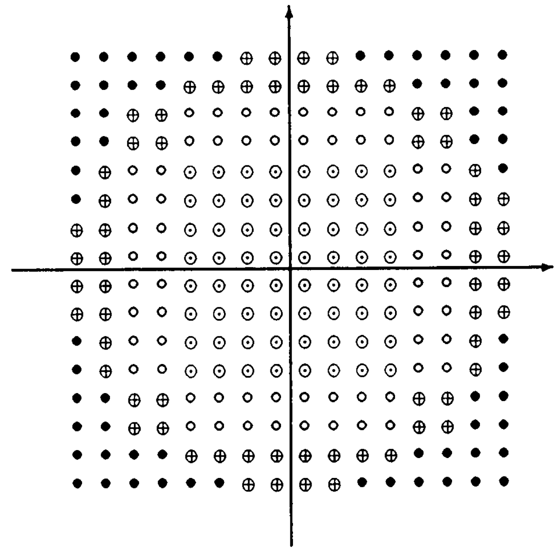

Fig. 2 Example of the two-dimensional shaping shells.

For coding, the signal constellation is partitioned into the cosets of Λ in Z^N. Assuming that Λ is a binary lattice each coset of Λ is labeled by $\log_2 |Z^N/\Lambda| = k_c + 1$ bits. The points within each coset have its label as their coding label. This is denoted as the coding partitioning/labeling. In the case that Λ is one of the Barnes Wall lattices, [10], coding partitioning/labeling is achieved by applying the Ungerboeck partitioning/labeling rules to the two dimensional subconstellations, [4]. Most of the interesting lattices, like D_4, Schläfli lattice, and E_8, Gosset lattice, belong to this group, [10]. In the coding part, k_c bits enter the encoder. After adding one bit redundancy, the $k_c + 1$ bits at the encoder output are used to select one of the 2^{k_c+1} cosets of Λ in the shaping cluster already selected by the shaping part. For this selection to be possible, each N-dimensional cluster should have an equal number of points from each coset of Λ. If Λ is one of the Barnes Wall lattices, this condition is satisfied if in the shaping partitioning of the two-dimensional subconstellations, each shaping shell contains an equal number of points from each partition of the Ungerboeck partition chain, [4]. This is the point where the shaping and coding potentially interfere with each other.

To transmit Q bits per two dimensions with one bit redundancy, we should have, $pn + t = nQ + 1$, or, $t = n(Q - p) + 1$. For the coding partitioning to be possible, we should have $2^{k_c+1} \leq 2^{pn}$ or $p \geq (k_c + 1)/n$. To transmit the total (coded) rate of $nQ + 1$, we should also have, $n(p + \log_2 K) \geq nQ + 1$, or, $\log_2 K \geq Q - p + (1/n)$.

An example of this partitioning with $M = 24$ and $K = 6$ ($P = 4$) is shown in Fig. (3). The coding partitions are denoted by A$\leftrightarrow$ 00, B$\leftrightarrow$ 10, C$\leftrightarrow$ 11, D$\leftrightarrow$ 01. The shaping shells are denoted by the $\bullet \leftrightarrow$ 000, $\otimes \leftrightarrow$ 001, $\ominus \leftrightarrow$ 010, $\oslash \leftrightarrow$ 011, $\odot \leftrightarrow$ 100, and $\circ \leftrightarrow$ 101. The first two bits of the label of each point is the coding label and the last three bits is the shaping label. Considering the condition $2^{k_c+1} \leq 2^{pn}$, this constellation can

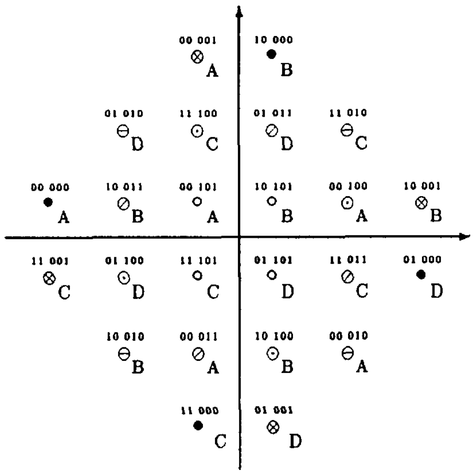

Fig. 3 An $M = 24$ point constellation divided into $K = 6$ shells.

be used as long as $|Z^N/\Lambda| < 2^N$, for example when $\Lambda = E_8$, $|Z^8/E_8| = 2^4$, or when $\Lambda = D_4$, $|Z^4/D_4| = 2^3$.

The CER_s of this scheme is equal to,

$$\text{CER}_s = M \times 2^{-Q-(1/n)}. \qquad (4)$$

In the receiver, we first do the channel decoding and decide which point is transmitted along each two dimensional subspace. After that, shaping labels of these points are concatenated and are passed through a system which inverts the effect of the shaping encoder to recover the original shaping bits.

3 Numerical results

The parameters of some signal constellations obtained by employing this shaping method are shown in Tables (1) and (2). The constellation points belong to the half integer lattice, $Z^N + (1/2)^N$. The entries marked by the $\surd$ sign are of special interest. They achieve a good shape gain with low CER_s, PAR and use a small lookup table. In general, a lookup table with t input and $n\lceil\log_2 K\rceil$ output bits can be implemented by a block of memory composed of 2^t words of $n\lceil\log_2 K\rceil$ bits.

By employing a better shaping region (comparing to that of the cross constellation) over the two-dimensional subconstellations, one can (possibly) improve the overall shape gain. This is achieved at the price of more points per dimension which, although not affecting the CER_s, is important in practical considerations.

For a fixed number of two dimensional shells (K), by changing the number of points per two dimension (M), we obtain constellations with different total rate but with *essentially* the same shaping performance and size of the lookup table. Indeed, assuming continuous approximation, this argument is exactly true. This is due to the fact that for a continuous approximation, the shaping performance and complexity are determined by dimensionless quantities.

For a given M, to obtain the highest shape gain, the energy of the points in each two-dimensional shell should be equal. Such a partitioning can be achieved by grouping the points of the

Q	M	K	CER_s	PAR	γ_s (dB)	Encoder
4	24	3	1.061	1.857	0.324	3→4 $\surd$
4	24	6	1.061	1.857	0.324	5→6
4	28	7	1.237	2.472	0.402	5 →6
4	32	2	1.414	2.267	0.025	1→2
4	32	4	1.414	2.429	0.324	3→4
4	32	8	1.414	2.519	0.482	5→6
5	48	3	1.061	2.071	0.324	3→4 $\surd$
5	48	6	1.061	2.071	0.324	5→6
5	64	4	1.414	3.500	0.324	3→4
5	64	8	1.414	3.580	0.422	5→6
6	96	3	1.061	2.169	0.305	3→4 $\surd$
6	96	6	1.061	2.173	0.312	5→6
6	112	7	1.237	2.380	0.432	5→6
6	128	4	1.414	3.022	0.305	3→4
6	128	8	1.414	3.112	0.432	5→6
7	192	3	1.061	2.156	0.314	3→4 $\surd$
7	192	6	1.061	2.156	0.314	5→6
7	224	7	1.237	2.642	0.412	5→6
7	256	4	1.414	4.009	0.314	3→4
7	256	8	1.414	4.100	0.412	5→6

Table 1 Parameters of the signal constellations obtained by using a lookup table over dimensionality $N = 4$, Q is the number of bits per two dimensions, M is the number of points per two dimensions and K is the number of two-dimensional shells.

Q	M	K	CER_s	PAR	γ_s (dB)	Encoder
4	20	5	1.051	2.194	0.296	9→12
4	24	3	1.261	2.311	0.521	5→8 $\surd$
4	24	6	1.261	2.358	0.609	9→12
4	28	7	1.472	3.119	0.659	9→12
4	32	2	1.682	2.720	0.064	1→4
4	32	4	1.682	3.030	0.533	5→8
4	32	8	1.682	3.156	0.710	9→12
5	40	5	1.051	2.130	0.337	9→12
5	48	3	1.261	2.622	0.595	5→8 $\surd$
5	48	6	1.261	2.636	0.618	9→12
5	64	4	1.682	4.451	0.616	5→8
5	64	8	1.682	4.511	0.674	9→12
6	80	5	1.051	2.120	0.404	9→12
6	96	3	1.261	2.732	0.555	5→8 $\surd$
6	96	6	1.261	2.769	0.613	9→12
6	112	7	1.472	2.996	0.680	9→12
6	128	4	1.682	3.838	0.590	5→8
6	128	8	1.682	3.932	0.695	9→12
7	160	5	1.051	2.163	0.361	9→12
7	192	3	1.261	2.707	0.550	5→8 $\surd$
7	192	6	1.261	2.739	0.602	9→12
7	224	7	1.472	3.326	0.660	9→12
7	256	4	1.682	5.056	0.570	5→8
7	256	8	1.682	5.165	0.662	9→12

Table 2 Parameters of the signal constellations obtained by using a lookup table, over dimensionality $N = 8$, Q is the number of bits per two dimensions, M is the number of points per two dimensions and K is the number of two dimensional shells.

Q	M	K	CER_s	PAR	γ_s (dB)	Encoder
4	24	6	1.061	1.857	0.324	5→6
4	28	7	1.237	2.472	0.402	5→6
4	32	8	1.414	2.519	0.482	5→6
5	48	12	1.061	2.071	0.324	7→8
5	64	16	1.414	3.596	0.442	7→8
6	96	24	1.061	2.179	0.325	9→10
6	112	28	1.237	2.388	0.447	9→10
6	128	32	1.414	3.130	0.457	9→10
7	192	48	1.061	2.164	0.331	11→12
7	224	56	1.237	2.661	0.443	11→12
7	256	64	1.414	4.132	0.446	11→12

Table 3 Parameters of the constellations given in Table (1) for $P=4$, Q is the number of bits per two dimensions, M is the number of points per two dimensions and K is the number of two dimensional shells.

Q	M	K	CER_s	PAR	γ_s (dB)	Encoder
4	20	5	1.051	2.194	0.296	9→12
4	24	6	1.261	2.358	0.609	9→12
4	28	7	1.472	3.119	0.659	9→12
4	32	8	1.682	3.156	0.710	9→12
5	40	10	1.051	2.131	0.340	13→16
5	48	12	1.261	2.643	0.630	13→16
5	64	16	1.682	4.565	0.725	13→16
6	80	20	1.051	2.125	0.415	17→20
6	96	24	1.261	2.787	0.642	17→20
6	112	28	1.472	3.016	0.708	17→20
6	128	32	1.682	3.956	0.722	17→20
7	160	40	1.051	2.187	0.409	21→24
7	192	48	1.261	2.767	0.646	21→24
7	224	56	1.472	3.363	0.708	21→24
7	256	64	1.682	5.226	0.713	21→24

Table 4 Parameters of the constellations given in Table (2) for $P=4$, Q is the number of bits per two dimensions, M is the number of points per two dimensions and K is the number of two dimensional shells.

same shape into the shaping shells where the shape of an N-tuple is defined as the set of the magnitudes of its elements. This partitioning applied to the constellations based of the half integer lattice results in the shells with 4, 8, 12 or 16 points. In general, the addressing of this scheme can be achieved by using a prefix code. To avoid the problems associated with prefix coding, we further subpartition the shells with more than 4 points into finer subshells of four points. The parameters of the signal constellations tabulated in Tables (1) and (2) when partitioned in this way are shown in Tables (3) and (4).

4 Spherical constellations based on Z^N and $Z^N+(1/2)^N$

A spherical constellation has the least possible average energy (highest shape gain) for a given number of points from the corresponding coding lattice. We study the coding lattices Z^N and $Z^N+(1/2)^N$. To calculate the parameters of the spherical constellations, we need the weight distribution function of these lattices.

Mathematically, the weight distribution function of a lattice Λ is defined as, [11],

$$\Theta_\Lambda(q) = \sum_{u\in\Lambda} q^{\|u\|^2} = \sum_n N(n)q^n , \quad (5)$$

where $\|u\|$ denotes the norm of the vector associated with point u. The weight distribution function of the lattices Z^2 and $Z^2+(1/2)^2$ are equal to,

$$\Theta_{Z^2}(q) = \Theta_3(q) = \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} , \quad (6)$$

$$\Theta_{Z^2+(1/2)^2}(q) = \Theta_2(q) = \sum_{n=-\infty}^{+\infty} q^{n^2} , \quad (7)$$

respectively. If a lattice is equal to the cartesian product of some lower dimensional lattices, its weight distribution will be equal to the product of the weight distributions of those lattices. As a result, the weight distributions of the lattices Z^N and $Z^N+(1/2)^N$ are equal to,

$$\Theta_{Z^N}(q) = [\Theta_3(q)]^{N/2} , \quad (8)$$

$$\Theta_{Z^N+(1/2)^N}(q) = [\Theta_2(q)]^{N/2} . \quad (9)$$

Define,

$$\delta(n) = d_1(n) - d_3(n) , \quad (10)$$

where $d_1(n)$ is the number of divisors of n congruent to 1 modulo 4 and $d_3(n)$ is the number of divisors of n congruent to 3 modulo 4. Also define,

$$\sigma'(n) = \sum_{d|n,\ d\neq 0\ (\mathrm{mod}\ 4)} d , \quad (11)$$

where $d|n$ means that d is a divisor of n. Using these notations, one can show, [12],

$$\Theta_{Z^2}(q) = \Theta_3(q) = 1 + 4\sum_{n=1}^{\infty} \delta(n)q^n . \quad (12)$$

$$\Theta_{Z^4}(q) = [\Theta_3(q)]^2 = 1 + 8\sum_{n=1}^{\infty} \sigma'(n)q^n , \quad (13)$$

$$\Theta_{Z^8}(q) = [\Theta_3(q)]^4 = 1 + 16\sum_{n=1}^{\infty}\left[\sum_{d|n}(-1)^{n+d}d^3\right] q^n . \quad (14)$$

For lattice $Z^N+(1/2)^N$, $N=2,4,8$, Eq. (9) is simplified to,

$$\Theta_{Z^2+(1/2)^2}(q) = \Theta_2(q) = 4q^{-1/2}\sum_{n=1}^{\infty} \delta(2n-1)q^n , \quad (15)$$

$$\Theta_{Z^4+(1/2)^4}(q) = [\Theta_2(q)]^2 = 16\sum_{n=1}^{\infty} \sigma'(2n-1)q^{2n-1} , \quad (16)$$

$$\Theta_{Z^8+(1/2)^8}(q) = [\Theta_2(q)]^4 = 256\times(8)^w\sum_{n=1}^{\infty}\left[\sum_{d|n'} d^3\right] q^{2n} , \quad (17)$$

where,

$$n = 2^w \times n', \quad n'\mathrm{odd} . \quad (18)$$

Recently, Fortier in [13], has independently found the weight distribution of $Z^4+(1/2)^4$ given in Eq. (16). But, to our knowledge, the closed form for the weight distribution of $Z^8+(1/2)^8$ given in Eq.(17) has not appeared before in the literature.

Tables (5) and (6) show the parameters of the spherical signal constellations obtained from lattices Z^N and $Z^N+(1/2)^N$,

Q	N	CER_s	PAR	γ_s (dB)
4	4	1.282	2.625	0.414
5	4	1.525	2.944	0.454
6	4	1.425	2.945	0.455
7	4	1.398	2.982	0.456
4	8	1.945	3.731	0.731
5	8	2.339	4.846	0.727
6	8	2.326	4.943	0.730
7	8	2.240	4.942	0.729

Table 5 Parameters of the spherical constellations based on lattice Z^N, $N = 4, 8$.

Q	N	CER_s	PAR	γ_s (dB)
4	4	1.414	2.519	0.482
5	4	1.326	2.715	0.442
6	4	1.326	2.688	0.457
7	4	1.414	2.889	0.455
4	8	1.682	3.156	0.710
5	8	1.997	4.197	0.730
6	8	2.155	4.706	0.727
7	8	2.181	4.779	0.729

Table 6 Parameters of the spherical constellations based on lattice $Z^N + (1/2)^N$, $N = 4, 8$.

$N = 4, 8$. In comparing the Tables (1), (2) and (3), (4) with Table (6), one should keep in mind that in some cases by employing a circular constellation (instead of a cross constellation) over the two-dimensional subspaces the shape gain of the schemes given in Tables (1), (2) and (3), (4) can be improved. But, in most of the cases, a cross constellation and a spherical constellation represents the same discrete set of points.

Table (7) shows the parameters of the corresponding spherical constellation obtained by using continuous approximation, [6].

N	CER_s	PAR	γ_s (dB)
4	1.414	3	0.46
8	2.213	5	0.73

Table 7 Parameters of the spherical constellations, continuous approximation.

5 Summary and conclusions

We have discussed the structure of a TCM scheme focusing on the constellation shaping. The shaping is achieved by a lookup table. Numerical results for different rates and dimensionalities $N = 4, 8$ are presented. The results show several cases which may be of practical interest. We presented a closed form formula for the weight distribution of the half integer lattice, $Z^N + (1/2)^N$, for dimensionality $N = 4, 8$. These are used to calculate the parameters of the spherical constellations. It is seen that the over a given dimensionality, an appreciable amount of the maximum shape gain, corresponding to a spherical constellation, can be obtained with much lower values for CER_s and PAR than that of the spherical case and with a lookup table of low complexity.

References

[1] A. R. Calderbank and L. H. Ozarow, "Nonequiprobable signaling on the Gaussian channel," *IEEE Trans. Inform. Theory*, vol. IT-36, pp. 726–740, July 1990.

[2] G. D. Forney, Jr. and L. F. Wei, "Multidimensional constellations—Part I: Introduction, Figures of merit, and Generalized Cross Constellations," *IEEE J. Select. Areas Commun.*, vol. SAC-7, pp. 877–892, August 1989.

[3] G. D. Forney, Jr., R. G. Gallager, G. R. Lang, F. M. Longstaff, and S. U. Quereshi, "Efficient modulation for bandlimited channels," *IEEE J. Select. Areas Commun.*, vol. SAC-2, pp. 632–647, Sept 1984.

[4] L. F. Wei, "Trellis coded modulation with multidimensional constellations," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 483–501, July 1987.

[5] J. H. Conway and N. J. A. Sloane, "A fast encoding method for lattice codes and quantizers," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 820–824, Nov 1983.

[6] G. D. Forney, Jr., "Multidimensional constellations—Part II: Voronoi constellations," *IEEE J. Select. Areas Commun.*, vol. SAC-7, pp. 941–958, Aug 1989.

[7] A. K. Khandani and P. Kabal, "Shaping multidimensional signal constellations," *Proc. IEEE Int. Symp. Inform. Theory* (Budapest, Hungary) p. 4, June 1990.

[8] A. K. Khandani, P. Kabal and H. Leib, "Combined coding and shaping over a multitone channel," *Proc. IEEE Global Telecommunications Conference* (Phoenix, Arizona) pp. 1182–1186, Dec 1991,

[9] A. K. Khandani and P. Kabal "Unsymmetrical boundary shaping in multi-dimensional spaces," *Proc. Twenty-ninth Annual Allerton Conference on Communications, Control, and Computing* (Allerton, Illinois) pp. 798–808, Oct 1991.

[10] G. D. Forney, "Coset codes—Part II: Binary lattices and related codes," *IEEE Trans. Inform. Theory*, vol. IT-34, pp. 1152–1187, Sept 1988.

[11] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. Springer-Verlag, 1988.

[12] E. Grosswald, *Representation of Integers as Sums of Squares*. Springer-Verlag, 1985.

[13] P. Fortier "On the number of points on shells for shifted Z^{4n} lattices," *IEEE Trans. Inform. Theory*, vol. IT-37, pp. 391-393, March 1991.