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# Using a Prefix Code for Addressing the Voronoi Constellations based on Lattices $D_N$ and $D_N^*$

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**Abstract:** Signal constellations for representing data values for transmission benefit from shaping of the constellation boundary. In the Voronoi constellations, the Voronoi region of a lattice, denoted as the shaping lattice, is used as the boundary of the signal constellation. In this work, some properties of the Voronoi constellations based on the shaping lattices $D_N$ and $D_N^*$ are discussed. The induced probability distribution on the two dimensional subspaces is found. A prefix coding scheme as an alternative for the addressing is presented. This code in some sense simulates the effect of the boundary by using the points of the subspaces with nonequal probability.

## 1 Introduction

Shaping concerns the selection of the boundary of a constellation to reduce its average energy. An unshaped signal constellation with dimensionality $N=2n$ is equal to the $n$-fold cartesian product of its two dimensional subconstellations. By employing a higher number of points per two dimensions and forbidding the $N$-dimensional points of high energy, one can reduce the average energy of the set. This is the major benefit of shaping.

Addressing is the assignment of the data bits to the constellation points. In an unshaped constellation, addressing is achieved independently along the two dimensional subspaces. In a shaped constellation, this method is not applicable and a more complex addressing scheme is needed.

In the work of Wei [1], shaping is a side effect of the method employed to transmit a nonintegral number of bits per two dimensions. The addressing of this method is achieved by a table lookup. Forney and Wei generalize this method in [2]. Conway and Sloane in [3] introduced the idea of the Voronoi constellation. The Voronoi constellations are further considered by Forney in [4]. In [5], Calderbank and Ozarow introduced a shaping method which is based on using the points of the subconstellations with nonequal probability. The structure of the optimum shaping regions is introduced in [6]. The idea of the combined shaping and coding over the multitone channel is introduced in [7]. The idea of the unsymmetrical shaping with application to spectral shaping is introduced in [8].

## 2 Voronoi Constellations

A real $N$-dimensional lattice $\Lambda$ is a discrete set of $N$-dimensional vectors in $\mathbf{R}^N$ which form a group under ordinary vector addition. Around each lattice point is its Voronoi region consisting of all points of the space which are closer to that point than to any other point. A sublattice $\Lambda_s$ of a lattice $\Lambda$ is a subset of elements of $\Lambda$ that is itself a lattice. A sublattice $\Lambda_s$ induces a partition of $\Lambda$ into equivalence classes modulo $\Lambda_s$. The order of this partition is shown by $|\Lambda/\Lambda_s|$. An $N$-dimensional binary lattice is an integer lattice (sublattice of $Z^N$, $Z$ denotes the integer lattice) which has $2^m Z^N$ as a sublattice for some integer $m$. Any translate $\Lambda+\mathbf{s}$ of $\Lambda$ is the union of $|\Lambda/\Lambda_s|$ cosets of $\Lambda_s$. A Voronoi constellation is the set of the coset leaders of these cosets. These are a subset points of $\Lambda+\mathbf{s}$ which fall within the Voronoi region around the origin of $\Lambda_s$.

In [4], the constituent two dimensional lattice $\Lambda_2$ of a lattice $\Lambda$ is defined as the projection of $\Lambda$ onto a given pair of dimensions and the constituent two dimensional sublattice $\Lambda_2'$ of a lattice $\Lambda$ is defined as the cross section of $\Lambda$ into a given pair of dimensions. It is shown in [4] that the constituent two dimensional subconstellation of a Voronoi constellation based on the partition $\Lambda/\Lambda_s$ is a two dimensional Voronoi constellation based on the partition $\Lambda_2/\Lambda_{2s}'$.

The problem of ties occurs when some cosets of $\Lambda_s$ in $\Lambda$ have more than one minimum norm element. In this case some points of the lattice $\Lambda$ are located on the boundary of the Voronoi region of $\Lambda_s$. The set of such points is denoted by $[e]$, where $e$ is the energy of the set.

In the following, we first talk about the complexity of using a lookup table for the addressing of a Voronoi constellation. In the partition $Z^N/2^k\Lambda_s/2^{k+1}Z^N$, the Voronoi region of $2^k\Lambda_s$ is a subset of the Voronoi region of $2^{k+1}Z^N$. As a result, the Voronoi constellation based on the partition $Z^N/2^k\Lambda_s$ is a subset of the Voronoi constellation based on the partition $Z^N/2^{k+1}Z^N$. The Voronoi constellation based on the partition $Z^N/2^{k+1}Z^N$ is the $N$-fold cartesian product of the Voronoi constellation based on the partition $Z/2^{k+1}Z$. In this constellation, addressing can be achieved independently along each dimension and has a trivial complexity. To address the original constellation, we need a means to specify the desired $2^{kN+J}$, $2^J=|Z^N/\Lambda_s|$, points of the constellation based on the partition $Z^N/2^k\Lambda_s$ among the total $2^{kN+N}$ points of the constellation based on the partition $Z^N/2^{k+1}Z^N$. This task can be achieved by using a lookup table with $kN+J$ input line and $kN+N$ output lines.

## 2.1 Voronoi constellations based on the lattices $D_N$ and $D_N^*$

Lattice $D_N$ is a modulo-2 binary lattice defined as, [10],

$$D_N = \{(X_0, X_1, ..., X_{N-1}) \in Z^N;\ X_0 + ... + X_{N-1} \text{ even}\}. \quad (1)$$

The set of the $2N(N-1)$ nearest neighbors in this lattice are located at points $\left[(\pm 1)^2, (0)^{N-2}\right]$. The Voronoi region is determined by the set of the nearest neighbors. The constituent two dimensional sublattice of $D_N$ is equal to $\Re Z^2$. We have $|Z^N/D_N| = 2$.

Lattice $D_N^*$ is a modulo-2 binary lattice defined as, [10],

$$D_N^* = \left\{(2Z)^N\right\} \cup \left\{(2Z)^N + (1)^N\right\}. \quad (2)$$

The set of the closest points to the origin in the first set consist of $2N$ points of the form $\left[(\pm 2), (0)^{N-1}\right]$ and the closest points in the second set consist of $2^N$ points of the form $\left[(\pm 1)^N\right]$. The Voronoi region is the intersection of the Voronoi regions determined by these two sets. We have $|Z^N/D_N^*| = 2^{N-1}$ and $|D_N/D_N^*| = 2^{N-2}$.

The Voronoi constellation obtained from the partition $Z^N/2^kD_N$ consist of $2^{kN+1}$ points and is appropriate in a TCM carring $k$ bits per dimension with one bit coding redundancy per $N$-dimensions. Its constituent two dimensional subconstellation is a Voronoi constellation based on the partition $Z^2/2^k\Re Z^2$. The Voronoi constellation obtained from the partition $Z^N/2^kD_N^*$ consist of $2^{(k+1)N-1}$ points. Its constituent two dimensional subconstellation is a Voronoi constellation based on the partition $Z^2/2^{k+1}Z^2$.

In the following, we compute the induced probability distribution on the points of the two dimensional subconstellation in a Voronoi constellation based on the partition $Z^N/\Lambda_s$. The space dimensions are labeled by $X_i, i = 0, \ldots, N-1$. We draw from every point of the $(C_2)^{N-2}$, $C_2$ is the two dimensional subconstellation and $(C_2)^{N-2}$ is its $(N-2)$-fold cartesian product, a two dimensional plane parallel to a given two dimensional subspace say $(X_0, X_1)$. We find the part of this plane which is located inside of the Voronoi region of $\Lambda_s$. The intersection of such a plane with the $Z^N$ lattice is a $Z^2$ lattice. The points of this lattice which are inside the Voronoi region of $\Lambda_s$ will be mapped to the corresponding points of the $(X_0, X_1)$ subspace. By counting the number of times that a given point on the $(X_0, X_1)$ subspace is used the desired probability distribution will be calculated.

The Voronoi region of the lattice $2^kD_N$ is the region bounded by the hyperplanes,

$$\pm X_i \pm Xj = 2^k, \quad (i, j = 0, ..., N-1). \quad (3)$$

Based on the general approach, we draw a two dimensional plane from the point $(X_2, ..., X_{N-1}) \in (C_2)^{N-2}$ parallel to

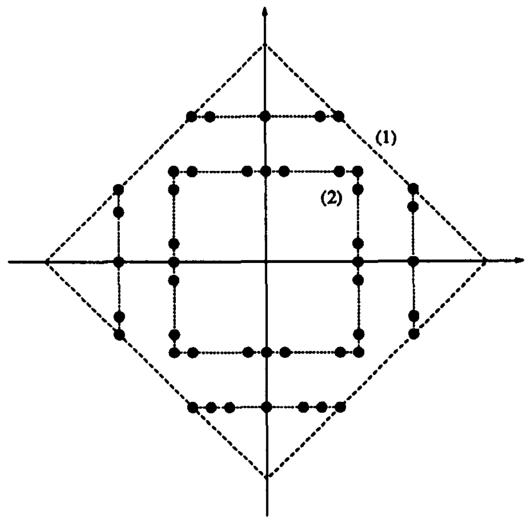


**Fig. 1** The shells of the equiprobable points, case (1) corresponds to $a > 2^{k-1}$ and case (2) corresponds to $a < 2^{k-1}$.

the $(X_0, X_1)$ subspace. The projection of the part of this plane which is interior to the Voronoi region of $2^kD_N$ onto the $(X_0, X_1)$ subspace is the interior part of the region bounded by the lines,

$$\begin{aligned} \pm X_0 &= \min_{i\in\{2,...,N-1\}} \left(2^k - |X_i|\right) \\ \pm X_1 &= \min_{i\in\{2,...,N-1\}} \left(2^k - |X_i|\right) \\ \pm X_0 &\pm X_1 = 2^k. \end{aligned} \quad (4)$$

Define,

$$a = \min_{i\in\{2,...,N-1\}} \left(2^k - |X_i|\right). \quad (5)$$

As the point $(X_2, ..., X_{N-1}) \in (C_2)^{N-2}$ spans the set $(C_2)^{N-2}$, $a$ takes value from the set $\{0, 1, ..., 2^k\}$. This can be verified considering that, $|X_i| \leq 2^k$. Corresponding to each value of $a$, the points in the $(X_0, X_1)$ subspace which are inside or on the sides of the rectangle $X_0 = \pm a$, $X_1 = \pm a$ are used one more time. This is used to calculate the induced probability distribution on the two dimensional subspaces. This also means that the points of the constituent two dimensional subconstellations which are located on the sides of the rectangles,

$$\begin{aligned} \pm X_0 &= a,\ a \in \{0, 1, ..., 2^k\} \\ \pm X_1 &= a,\ a \in \{0, 1, ..., 2^k\} \end{aligned}, \quad (6)$$

are used with equal probability. Figure (1) shows the equiprobable points for the two cases of $a < 2^{k-1}$ and $a > 2^{k-1}$.

This provides us with a method to subdivide the points of the two dimensional subconstellations into the addressing partitions such that the Voronoi constellation can be built up by forbidding some elements of the product space ($N/2$-fold cartesian product of the set of two dimensional

| shape | $\|e\|^2$ | $\Pi(e)$ | $\|[e]\|$ | $\Pi(e)/\|[e]\|$ |
|---|---|---|---|---|
| 0000 | 0 | 1 | 1 | 1 |
| 1000 | 1 | 8 | 1 | 8 |
| 1100 | 2 | 24 | 1 | 24 |
| 1110 | 3 | 32 | 1 | 32 |
| 1111 | 4 | 16 | 1 | 16 |
| 2000 | 4 | 8 | 1 | 8 |
| 2100 | 5 | 48 | 1 | 48 |
| 2110 | 6 | 96 | 1 | 96 |
| 2111 | 7 | 64 | 1 | 64 |
| 2200 | 8 | 24 | 2 | 12 |
| 2210 | 9 | 96 | 2 | 48 |
| 3000 | 9 | 8 | 1 | 8 |
| 2211 | 10 | 96 | 2 | 48 |
| 3100 | 10 | 48 | 2 | 24 |
| 3110 | 11 | 96 | 3 | 32 |
| 2222 | 12 | 32 | 4 | 8 |
| 3111 | 12 | 64 | 4 | 16 |
| 2221 | 13 | 64 | 4 | 16 |
| 2222 | 16 | 16 | 8 | 2 |
| 4000 | 16 | 8 | 8 | 1 |
| | | 849 | | 512 |

**Table 1** Points of the Voronoi constellation based on the partition $Z^4/4D^4$, [4].

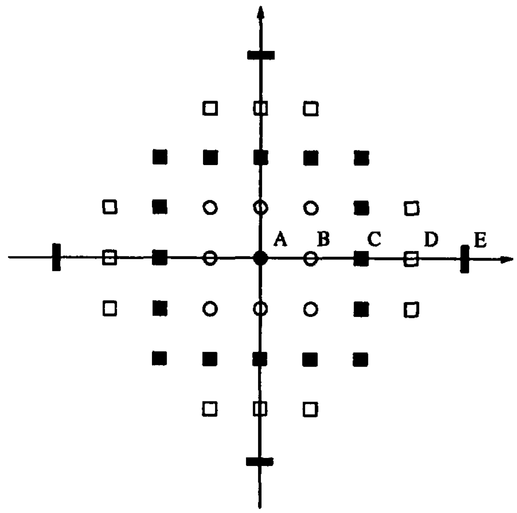


**Fig. 2** The two dimensional subconstellation of the Voronoi constellation based on the partition $Z^4/4D^4$.

| Shell | $f$ | $n$ | $P$ |
|---|---|---|---|
| $A$ | 41 | 1 | 0.363 |
| $B$ | 37 | 8 | 0.327 |
| $C$ | 25 | 16 | 0.221 |
| $D$ | 9 | 12 | 0.080 |
| $E$ | 1 | 4 | 0.009 |

**Table 2** Induced probability distribution on the two dimensional subspace of the Voronoi constellation based on the partition $Z^4/4D^4$.

partitions with itself). In the following, this method is further explained by the use of an example.

The Voronoi constellation based on the partition $Z^4/4D_4$ consists of the points shown in table (1), [4]. The shape of a 4-tuple is defined as the set of the magnitudes of its elements. The listing is by shape in order of the increasing norm. For each shape, there is also listed its norm $\|e\|^2$, the total number $\Pi(e)$ of points in $Z^4$ of that shape that can be obtained by permutation and sign changes of the coordinates, the number $|[e]|$ of nearest neighbors in $4D_4$ to any 4-tuple of that shape, and finally the number $\Pi(e)/|[e]|$ of 4–tuples of that shape included in a constellation with resolved ties. The two dimensional subconstellation is a Voronoi constellation based on $Z^2/4\Re Z^2$, Fig. (2). To find the induced probability distribution on the points of the $(X_0, X_1)$ subspace, we draw a two dimensional plane from every point of the $(X_2, X_3)$ subspace parallel to the $(X_0, X_1)$ subspace. The general shape of the part of this plane which is located inside of the Voronoi region is shown in Fig. (1). Figure (2) shows the shells of the equiprobable points, denoted as $A, B, C, D$ and $E$.

In this case, (5) reduces to,

$$a = \min_{i \in \{2,3\}} (4 - |X_i|) . \qquad (7)$$

Using (7), it is easy to verify that for the planes drawn from the points of the shell $A$ in ($X_2$, $X_3$) subspace, we have $a = 4$, similarly for the shell $B$, we have $a = 3$, for $C$, $a = 2$, for $D$, $a = 1$ and for the shell $E$, $a = 0$. The total number of times that shell $A$ is used is equal to the total number of planes with $0 \le a \le 4$. Similarly, shell $B$ is used when $1 \le a \le 4$, $C$ is used when $2 \le a \le 4$, $D$ is used when $3 \le a \le 4$ and $E$ is used when $a = 4$. This can be used to find the frequency of the two dimensional points. Table (2) shows the result of these calculations where $f$ is the number of times that a shell is used, $n$ is the number of points in the shell and $P$ is the probability of using a shell, $P(i) = f(i)/\sum_j f(j)$. It is seen that $\sum_{A,B,C,D,E} f(i)n(i) = 849$ which is equal to the total number of points in the original constellation, table (1). The obtained probability distribution is in agreement with table (1).

In the following, we specify the signal constellation as a subset of the two fold cartesian product of the two dimensional subconstellation. To do this, the two dimensional subconstellation is partitioned into the set of the equiprobable shells, Fig. (2). Then, certain elements in the cartesian product of the set of the partitions are forbidden. Table (3) shows the allowed combinations with the points that they represent. The points are obtained from the shown 4-tuples by the following rules: (i) Permutation and sign changes of the elements of the first two tuple and the second two tuple

| Combination | Points | $N$ | $E$ |
|---|---|---|---|
| $AA$ | 0000 | 1 | 0 |
| $AB, BA$ | 1000 | 8 | 1 |
| | 1100 | 8 | 2 |
| $AC, CA$ | 2000 | 8 | 4 |
| | 2100 | 16 | 5 |
| | 2200 | 8 | 8 |
| $AD, DA$ | 3000 | 8 | 9 |
| | 3100 | 16 | 10 |
| $AE, EA$ | 4000 | 8 | 16 |
| $BB$ | 1010 | 16 | 2 |
| | 1111 | 16 | 4 |
| | 1011 | 32 | 3 |
| $BC, CB$ | 1020 | 32 | 5 |
| | 1021 | 64 | 6 |
| | 1022 | 32 | 9 |
| | 1120 | 32 | 6 |
| | 1121 | 64 | 7 |
| | 1122 | 32 | 10 |
| $BD, DB$ | 1030 | 32 | 10 |
| | 1031 | 64 | 11 |
| | 1130 | 32 | 11 |
| | 1131 | 64 | 12 |
| $CC$ | 2020 | 16 | 8 |
| | 2021 | 64 | 9 |
| | 2022 | 32 | 12 |
| | 2121 | 64 | 10 |
| | 2122 | 64 | 13 |
| | 2222 | 16 | 16 |

**Table 3** Four dimensional combinations of the Voronoi constellation based on the partition $Z^4/D_4$.

| shape | $\|e\|^2$ | $\Pi(e)$ | $\|[e]\|$ | $\Pi(e)/\|[e]\|$ |
|---|---|---|---|---|
| 1111 | 1 | 16 | 1 | 16 |
| 1113 | 3 | 64 | 1 | 64 |
| 1133 | 5 | 96 | 1 | 96 |
| 1333 | 7 | 64 | 1 | 64 |
| 3333 | 9 | 16 | 1 | 16 |
| 1115 | 7 | 64 | 1 | 64 |
| 1135 | 9 | 192 | 2 | 96 |
| 1335 | 11 | 192 | 3 | 64 |
| 3335 | 13 | 64 | 4 | 16 |
| 1117 | 13 | 64 | 4 | 16 |
| | | | | 512 |

**Table 4** Points of the Voronoi constellation based on the partition $Z^4/4D^4$, $\mathbf{s}=(1/2)^4$, [4].

| Combination | Points | $N$ | $E$ |
|---|---|---|---|
| $AA$ | 1111 | 16 | 4 |
| $AB_1, B_1A$ | 1113 | 64 | 12 |
| $AB_2, B_2A$ | 1133 | 32 | 20 |
| $AC_1, C_1A$ | 1115 | 64 | 28 |
| $AC_2, C_2A$ | 1135 | 64 | 36 |
| $B_1B_1$ | 1313 | 64 | 20 |
| $B_1B_2, B_2B_1$ | 1333 | 64 | 28 |
| $B_2B_2$ | 3333 | 16 | 36 |
| $B_1C_1, C_1B_1$ | 1315 | 128 | 36 |
| | | 512 | |

**Table 5** Four dimensional combinations of the Voronoi constellation based on the partition $Z^4/D_4$, $\mathbf{s}=(1/2)^4$.

are allowed. (ii) Permutation of the first two tuple with the second two tuple is also allowed. Column $N$ shows the number of points obtained from each 4-tuple and column $E$ shows the average energy of that 4-tuple. This table includes all the points of table (1).

In practice we are faced with two problems. First, what to do with the problem of ties to obtain a constellation with an integral bit rate. Second, how to map the data bits to the signal points. To solve these problems, we subdivide each equiprobable shell into subpartitions each carring an integral bit rate and forbid certain elements in the cartesian product. This subpartitioning is achieved with such a resolution that makes it possible to obtain the desired number of points (512 in the constellation based on the partition $Z^4/4D_4$). Here, we are faced with a tradeoff between the complexity of the resulting lookup table and the average energy of the constellation. A finer subpartitioning reduces the average energy but requires an encoder with a higher complexity.

In the following, we consider an example of the signal constellation based on the partition $Z^4/4D_4$ when $Z^4$ is shifted to the point $\mathbf{s}=(1/2)^4$. The points of this constellation with the shape multiplied by two are listed in table (4), [4]. The constituent two dimensional subconstellation consists of the points of $Z^2+(1/2)^2$ bounded within the Voronoi region of $4\Re Z^2$. Figure (3) shows this constellation divided into the equiprobable shells, namely $A$, $B_1$, $B_2$, $C_1$, $C_2$ and $D$. We also notice that each subpartition has four way symmetry and contains an equal number of points from each partition of the Ungerboeck partition chain. These are practical considerations in using the constellation in a coset coding scheme, [1]. Table (5) shows a 512 points signal constellation obtained from the original constellation by discarding some combinations of the highest energy. In this table $E$ is the energy of the combination and $N$ is the number of points in the combination. The average energy of this constellation is equal to 3.375 per two dimensions. It is also seen that the shell $D$ is not. This reduces the $\mathrm{CER}_s$.

To map the data bits to the signal points we employ the prefix coding scheme shown in table (6). The first bits select the combination and the following bits indicated by '×' select a point of that combination. It is seen that the lookup table has many "don't care" (×) entries. This facilitates the hardware realization of the scheme.

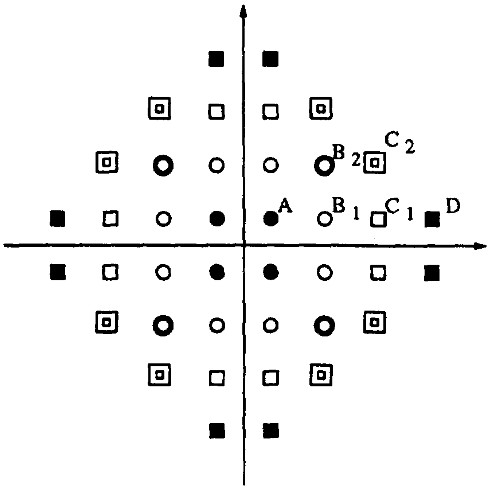


**Fig. 3** The two dimensional subconstellation of the Voronoi constellation based on the partition $Z^4/4D^4$, $s=(1/2)^4$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $B_1B_1$ | 0 | 0 | 0 | × | × | × | × | × | × |
| $B_1C_1$ | 0 | 0 | 1 | × | × | × | × | × | × |
| $C_1B_1$ | 0 | 1 | 0 | × | × | × | × | × | × |
| $AB_1$ | 0 | 1 | 1 | 0 | × | × | × | × | × |
| $B_1A$ | 0 | 1 | 1 | 1 | × | × | × | × | × |
| $AC_1$ | 1 | 0 | 0 | 0 | × | × | × | × | × |
| $C_1A$ | 1 | 0 | 0 | 1 | × | × | × | × | × |
| $AC_2$ | 1 | 0 | 1 | 0 | × | × | × | × | × |
| $C_2A$ | 1 | 0 | 1 | 1 | × | × | × | × | × |
| $B_1B_2$ | 1 | 1 | 0 | 0 | × | × | × | × | × |
| $B_2B_1$ | 1 | 1 | 0 | 1 | × | × | × | × | × |
| $AA$ | 1 | 1 | 1 | 0 | 0 | × | × | × | × |
| $AB_2$ | 1 | 1 | 1 | 0 | 1 | × | × | × | × |
| $B_2A$ | 1 | 1 | 1 | 1 | 0 | × | × | × | × |
| $B_2B_2$ | 1 | 1 | 1 | 1 | 1 | × | × | × | × |

**Table 6** A prefix code for the direct addressing of the two dimensional subconstellation of the Voronoi constellation based on the partition $Z^4/4D^4$, $s=(1/2)^4$.

## 3 Summary and conclusions

In this work, we studied the Voronoi constellations based on the shaping lattices $D_N$ and $D_N^*$. The probability distribution induced on the two dimensional subspaces is found. This is used to determine the structure of a prefix coding scheme for the addressing. An example of such a coding scheme is presented. The corresponding lookup table contains many 'don't' care entries and is relatively easy to implement.

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