

Address Decomposition for the Shaping of Multi-dimensional Signal Constellations

A. K. Khandani¹ and P. Kabal^{1,2}

¹INRS-Telecommunications, 16 Place du Commerce, Verdun, Canada, H3E 1H6

²McGill University, 3480 University, Montreal, Canada, H3A 2A7

Abstract¹: In this work, we introduce an efficient addressing scheme to realize points near to the knee of the tradeoff curves of an optimally shaped constellation. This scheme, called as the address decomposition, is based on decomposing the addressing into a hierarchy of the addressing steps each of a low dimensionality. As the memory size associated with a direct addressing scheme has an exponential growth with the dimensionality, this decomposition of the addressing results in a substantial decrease in the complexity. In this case, by using a memory of a practical size, we can move along a tradeoff curve which is nearly optimum. For example, in a space of dimensionality $N = 32$, we use a block of memory of 2.5 kilo-bytes per N dimensions to achieve a shape gain of $\gamma_s = 0.92$ dB with $CER_s = 1.25$, $PAR = 2.95$. This scheme has no associated computation, is straightforward to implement and is adaptable to the structure of a general coset coding scheme.

1 Introduction

Consider a discrete source producing one of M equiprobable symbols in each signaling interval. We are going to transmit the sequence of the source symbols to a receiver. Our transmission media is a channel which provides us with a given number of dimensions per signaling interval. To achieve the transmission, we select M points over the channel space. Each source symbol is represented by one of these points. This is called a signal constellation.

The constellation points are usually selected as a finite subset of a regular array of points bounded within a shaping region. In selecting the shaping region, the objective is to minimize the average energy of the constellation. In continuous approximation, the distribution of the constellation points is approximated by a continuous uniform density of points within the shaping region. The reduction in the average energy per two dimensions due to the use of the region C as the boundary instead of using a hypercube is called the shaping gain of C and is denoted as, $\gamma_s(C)$. The price to be paid for shaping ($\gamma_s > 1$) involves: (i) an increase in the factor CER_s (Constellation-Expansion-Ratio) which is defined as the ratio of the number of points per two dimensions to the minimum necessary number of points per two dimensions, [1], (ii) an increase in the factor PAR (Peak-to-Average-power-Ratio) which is defined as the ratio of the peak of energy per two dimensions to the average energy per two dimensions, [1], (iii) an increase in the addressing complexity, where addressing is the assignment of the source symbols to the constellation points².

In general, there exists a tradeoff between γ_s and CER_s , and also between γ_s and PAR . An optimum shaping region is the region which optimizes both of these tradeoffs, [2]. The major problem

¹This work was supported by Natural Sciences and Engineering Research Council of Canada (NSERC).

²In a cubic constellation, addressing can be achieved independently along each dimension. This results in a low complexity. However, in a shaped constellation independent addressing is not applicable.

associated with the optimum shaping in a high dimensional space is the addressing complexity.

We use continuous approximation to calculate the performance of our schemes. Usually, this results in a lower bound to the actual performance. To justify this, consider a shaping region which is the union of some unity volume Voronoi regions. The centroid of the Voronoi regions are the constellation points. Using the orthogonality principle, the average second moment of the region with respect to the origin is the sum of two terms, namely, the average second moment of the centroids and a second term which is the second moment of a single Voronoi region. In the discrete analysis, the second term is not present.

In all our discussions, we assume that the dimensionality is even and the constellation points are used with equal probability.

2 A simple example of shaping

In Figs. 1 and 2, we see two examples of a 64-points 2-D signal constellation from the half integer grid. The one shown in Fig. 1 has a cubic shaping region. Assuming a minimum distance of one, the average energy per dimension of this constellation is equal to 5.25. The constellation in Fig. 2 is obtained by replacing the four points of the highest energy in the cubic constellation with another four points of a lower energy. These are the points marked by the circles. As a result, the average energy has reduced to 5.125. This corresponds to the shape gain, $\gamma_s \approx 0.1$ dB.

The cubic constellation employs 8 points per dimension. This is the minimum necessary number of points per dimension to have 64 points in two dimensions. However, in the shaped constellation, we have employed 10 points per dimension. Assuming that CER_s is measured on a one dimensional basis, this corresponds to, $CER_s = 1.25$.

The peak of energy per dimension of the cubic and the shaped constellations are equal to 12.25 and 20.25, respectively. Assuming that PAR is measured on a one dimensional basis, the PAR 's are equal to, 2.33 and 3.95, respectively.

In the cubic constellation, to map the six bits of data to a constellation point, we can use three bits to select a point along one dimension and another three bits to select a point along the other dimension. However, in the shaped constellation, this method is not applicable. In this simple example, we can use a lookup table with 64 memory locations to achieve the addressing. However, for the same number of bits per dimension, namely 3, in dimensionality 32, we need a lookup table with 2^{96} memory locations which is not practical.

3 Previous relevant works

In the work of Wei, [3], shaping is a side effect of the method employed to transmit a nonintegral number of bits per two dimensions. This method provides moderate shaping gain for low

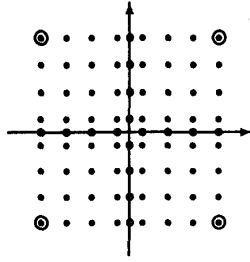


Fig. 1 A 64-points 2-D cubic constellation from half integer grid.

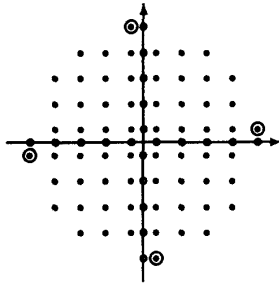


Fig. 2 A 64-points 2-D shaped constellation from half integer grid.

values of CER_s . The addressing of this method is achieved by a lookup table. Forney and Wei generalize this method under the topic of the generalized cross constellation in [1].

Conway and Sloane in [4] introduced the idea of the Voronoi constellation based on using the Voronoi region of a lattice Λ_s as the shaping region. In these constellations, the set of the points form a group under vector addition modulo Λ_s . This property is used to achieve the addressing. The complexity of the addressing is that of a linear mapping plus the decoding of the shaping lattice Λ_s . The Voronoi constellations are further considered by Forney in [5]. The idea of the trellis shaping is introduced in [6]. This is based on using an infinite dimensional Voronoi region, determined by a convolutional code, to shape the constellation.

In [7], Calderbank and Ozarow introduced a shaping method which is directly achieved on the 2-D subspaces. In this method, the 2-D subspaces are partitioned into equal sized shells of increasing average energy. A shaping codes is then used to specify the sequence of the 2-D shells. The shaping code is designed so that the lower energy shells are used more frequently.

Lang and Longstaff in [8] use an addressing scheme which first divides the final constellation into energy shells. Then, a point in a shell is found by successively decomposing the space into lower-dimensional subspaces via generating function techniques. As noted in passing in [8], the same technique can be used for the addressing of a shaped constellation. It should be mentioned that prior to [8], a similar addressing scheme was used by Fischer in [9], to label the points of a vector quantizer with a pyramid quantization region. The addressing scheme of Lang and Longstaff is further discussed and nicely formulated by Kschischang in [10] (see also Kschischang and Pasupathy, [11]). Loria in [12], applies some ideas from a type of structured vector quantizer to the constellation addressing. This results in an addressing scheme similar to that of [9].

In [13], Kschischang and Pasupathy discuss a shaping method which is based on using the 2-D points with nonequal probability (see also [10]). In [14] (see also [15]), Livingston discusses a shaping method in which the 2-D subspaces are partitioned into annular shells of increasing size. In this method, the 2-D shells are used with equal probability inducing a nonuniform distribution on the 2-D points. Recently, Kschischang in [16] discusses the structure of a prefix code which closely approximates the optimum nonuniform probabilities of the 2-D points.

The structure of the region which optimizes the tradeoff between the γ_s and the CER_s , and also between the γ_s and the PAR together with analytical expressions determining the optimum tradeoff is given in [2] (also refer to [17], [18]). Independently, Kschischang in [10] (also refer to Kschischang and Pasupathy, [11]) and Loria in [12] arrive at the same shaping region. References [10] and [11] also give expressions for the optimum tradeoff curves using a different approach than that of [2].

In our work, [19] (also refer to [18]), some practical addressing schemes to achieve (or approximate points) on the optimum tradeoff curves is given. Laroia *et al* in [20] suggest methods to reduce the computational complexity of the addressing scheme based on the same general idea as in, [8], [12].

In this work we introduce an efficient addressing scheme to realize tradeoff points with negligible suboptimality. This is based on decomposing the addressing into a hierarchy of the addressing steps each of a low dimensionality. As the memory size of a direct addressing scheme has an exponential growth with the space dimensionality, this scheme results in a substantial decrease in the complexity. It should be mentioned that this work is being reported in part at [18], [19].

4 Shell mapping

Assume that CER_s and PAR are measured on a 2-D basis. In an N -D optimally shaped region, \mathcal{A}_N , a 2-D sphere of radius R_2 , $S_2(R_2)$, is the boundary of the 2-D subspaces and an N -D sphere of radius R_N , $S_N(R_N)$, is the boundary of the whole space, i.e.,

$$\begin{aligned} \mathcal{A}_N &= \{X_j, j = 0, \dots, N-1\} \\ &: 0 \leq X_{2p}^2 + X_{2p+1}^2 \leq R_2^2, \quad p = 0, \dots, n-1 \\ &0 \leq \sum_{j=0}^{N-1} X_j^2 \leq R_N^2, \quad 0 \leq R_N^2 \leq nR_2^2. \end{aligned} \quad (1)$$

By applying the change of variable,

$$Y_p = (X_{2p}^2 + X_{2p+1}^2)/R_2^2, \quad p = 0, \dots, n-1, \quad n = N/2, \quad (2)$$

to (1), the region \mathcal{A}_N is mapped to the following n -D solid,

$$\begin{aligned} \mathcal{TC}_n(1, \beta) &= \{Y_p, p = 0, \dots, n-1\} \\ &: 0 \leq Y_p \leq 1 \\ &0 \leq \sum_{p=0}^{n-1} Y_p \leq \beta, \quad 0 \leq \beta = R_N^2/R_2^2 \leq n. \end{aligned} \quad (3)$$

This is a hypercube of edge length one truncated within a simplex of edge length β . We refer to the N -D space as the N -domain and to the n -D space as the n -domain.

The change of variable in (2) is denoted as the shell mapping. This mapping has an important property that: *A uniform density within \mathcal{A}_N results in a uniform density within \mathcal{TC}_n .*

Now, consider an optimum shaping region in an $N = N'n'$ -D space. Without loss of generality, we assume that the shaping (truncation within the N -D sphere) is achieved on the n' -fold cartesian product of an N' -D unshaped region ($\{\mathcal{S}_2\}^{N'/2}$). By the

use of the shell mapping, each $\{S_2\}^{N'/2}$ is mapped to an $\mathcal{TC}_n(1, n)$ where $n = N'/2$. By applying the change of variable,

$$Z_q = \sum_{p=0}^{n-1} Y_p, \quad q = 0, \dots, n'-1, \quad n' = N/N', \quad (4)$$

each \mathcal{TC}_n , or equivalently each $\mathcal{A}_{N'}$, is mapped to one of the dimensions of an $\mathcal{TC}_{n'}$ in a domain denoted as the n' -domain. Fig. (3) shows an example of such a two-level shell mapped constellation for $N = 8$, $N' = 4$ and $K = 4$.

Each of the $\mathcal{A}_{N'}$ regions is partitioned into K energy shells of equal volume. The partitioning of the $\mathcal{A}_{N'}$'s result in equal volume partitions in the n -domain produced by the radial hyperplanes, refer to Fig. (3). The partitions of each $\mathcal{A}_{N'}$ correspond to the set of the points $\{U_i, i = 0, \dots, K\}$ along the corresponding dimension of the $\mathcal{TC}_{n'}$. To achieve the shaping, a subset of the elements in the n' -fold cartesian product of the set of the partitions is selected. The corresponding addressing is achieved using a lookup table.

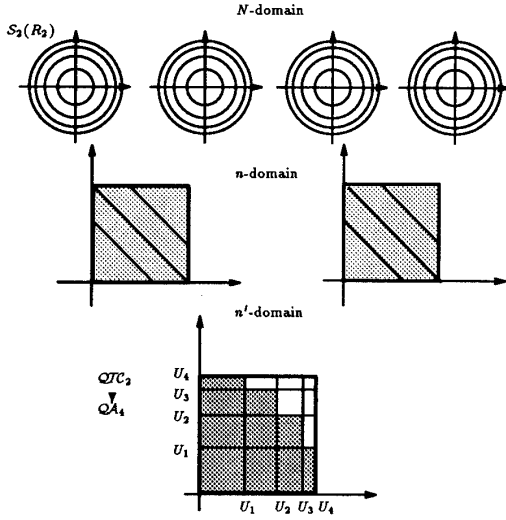


Fig. 3 Example of a two-level shell mapped constellation.

A point U_i along a dimension of $\mathcal{TC}_{n'}$ corresponds to the region $\mathcal{A}_{N'}$ with $\beta = U_i$. Using the results of [18], the volume of this region is equal to,

$$V(\mathcal{A}_{N'}) = (\pi R_2^2)^n \sum_{k=0}^{\lfloor U_i \rfloor} (-1)^k C_n^k \frac{(U_i - k)^n}{n!}, \quad (5)$$

where $n = N'/2$. To obtain partitions of equal volume, the points U_i should satisfy,

$$\sum_{k=0}^{\lfloor U_i \rfloor} (-1)^k C_n^k \frac{(U_i - k)^n}{n!} = \frac{i}{K}. \quad (6)$$

Equations (6) can be used to calculate the points U_i (for example using a bisection algorithm).

The partitioning of the N' -D subspaces results in $K^{n'}$ equal volume partitions (clusters) in the N -domain. Each of these clusters corresponds to a parallelepiped in the n' -domain. A parallelepiped located at point $(U_{I_0}, \dots, U_{I_{n'-1}})$, $I_j \in \{0, \dots, n'-1\} \in \{0, \dots, K\}$ is

shown by,

$$\begin{aligned} \mathcal{P}(U_{I_0}, \dots, U_{I_{n'-1}}) &= \{Z_q, q = 0, \dots, n'-1\} \\ &: U_{I_0} \leq Z_0 \leq U_{I_0+1}, \\ &\dots \\ &U_{I_{n'-1}} \leq Z_{n'-1} \leq U_{I_{n'-1}+1}. \end{aligned} \quad (7)$$

Shaping is achieved by selecting T of the N -D clusters with the least second moment. In the example of Fig. (3), we have $T = 10$. Considering that the first moment of $\mathcal{TC}_{n'}$ is proportional to the second moment of the N -domain, the selected subset corresponds to the parallelepipeds with the least average first moment. This procedure in fact uses a quantized version of $\mathcal{TC}_{n'}$, denoted by $\mathcal{QTC}_{n'}$, as the shaping region in the n' -domain. The final region is denoted by $\mathcal{QA}_{N'}^{N'}(K, T)$.

In a parallelepiped the average first moment is equal to the sum of the average first moments along different dimensions. Using this fact and some other results from [18], it can be shown that,

$$\begin{aligned} F_m(\mathcal{P}(U_{I_0}, \dots, U_{I_{n'-1}})) &= \left(\frac{1}{K}\right)^{n'-1} \times \\ &\sum_{q=0}^{n'-1} \left\{ \sum_{k=0}^{\lfloor U_{I_q+1} \rfloor} (-1)^k C_n^k \frac{n(\beta - k)^{n+1} + k(n+1)(\beta - k)^n}{(n+1)!} - \right. \\ &\left. \sum_{k=0}^{\lfloor U_{I_q} \rfloor} (-1)^k C_n^k \frac{n(\beta - k)^{n+1} + k(n+1)(\beta - k)^n}{(n+1)!} \right\}. \end{aligned} \quad (8)$$

This is used to calculate the average first moment of the selected subset of the parallelepipeds, $F_m(\mathcal{QTC}_{n'})$. The average energy per 2-D of the N -domain is equal to,

$$P_2(\mathcal{QA}_{N'}^{N'}(K, T)) = \frac{2R_2^2}{N} \frac{K^{n'}}{T} F_m(\mathcal{QTC}_{n'}). \quad (9)$$

It is easy to show that the volume of $\mathcal{QA}_{N'}^{N'}$ is equal to,

$$V(\mathcal{QA}_{N'}^{N'}(K, T)) = (\pi R_2^2)^{N/2} \frac{T}{K^{n'}}. \quad (10)$$

Equations (9) and (10) can be used to calculate the tradeoff. The final results for $n' = 2$ is shown in Fig. (4). The optimum tradeoff curves ($K = \infty$) are extracted from [2].

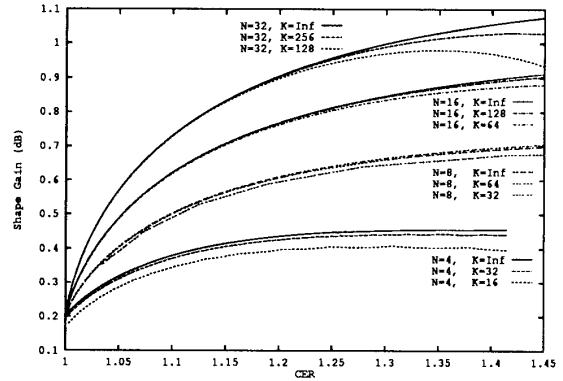


Fig. 4 Tradeoff between CER_s and γ_s using a finite number of the energy shells in the $N/2$ -D subspaces.

5 Address decomposition

Referring to Fig. (4), for moderate values of CER_s in an N -D space, one can closely approximate the optimum tradeoff using a small number of equi-volume energy shells per $N/2$ -D subspaces. A similar effect over dimensionality two was observed for the first time in [7]. This property allows us to decompose the addressing of a constellation into some intermediate steps achieved on the 2-fold cartesian product of a set with low cardinality. We call this method as the *address decomposition*.

For a dimensionality $N = 2^u$, this results in $u - 1$ addressing steps. The i 'th step, $i \in [1, u - 1]$, is achieved on the 2^i -D subspaces and results in dimensionality 2^{i+1} . We assume that the subspaces involved in the i 'th step are partitioned into $K_i = 2^{k_i}$ shells. Referring to Fig. (4), we select $\{k_i, i = 1, \dots\} = \{6, 6, 7, 8, \dots\}$. The i 'th addressing step requires a memory with $2k_i \times 2^{2k_i}$ bits. The last step requires $2k_u \times 2^{2k_u - r_s}$ bits, where $r_s = (N/2) \log_2 CER_s$. An upper bound to total memory size, M_t , is obtained by setting $r_s = 0$, ($CER_s = 1$).

Figure (5) shows the final tradeoff curves. The optimum tradeoff curves are extracted from [2]. It is seen that the overall suboptimality is negligible.

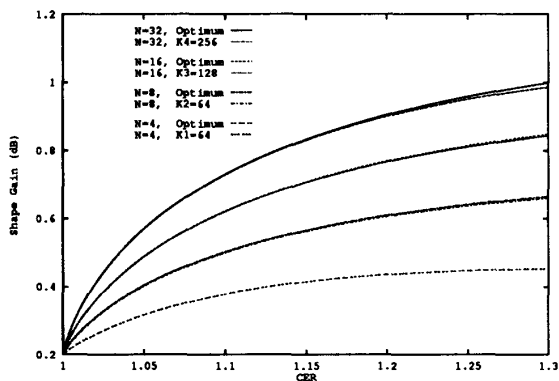


Fig. 5 Tradeoff between CER_s and γ_s using the address decomposition method.

The introduced addressing scheme has some similarities with the scheme of [8]. The key point to the scheme of [8] is that the weight distribution of the integer lattice (or more generally a lattice which is equal to the cartesian product of some lower dimensional sublattices) is equal to the convolution of the weight distributions in its subspaces. This fact is used in [8] to successively decompose the addressing into lower-dimensional subspaces via generating function techniques. The major difference is the here, by aggregating the energy shells in higher dimensional spaces, we have been able to avoid the computation of the convolutions. This substantially reduces the complexity while the suboptimality is negligible.

It should be mentioned that by selecting smaller values for the k_i 's, one can further decrease the size of the memory at the price of some performance degradation. It is specifically appropriate to use a small number of partitions in the first $u - 2$ addressing stages, and a large number at the last stage. This is due to the facts that: (i) shaping is achieved at the last stage and, consequently, a higher resolution at this stage has a more important effect on the overall performance, (ii) even for a large k_{u-1} , due to the subtraction of r_s , the addressing of the last stage does not need a large lookup table. Table (1) shows some examples of the performance and complexity of the proposed scheme.

N	CER_s	PAR	γ_s dB	M_t	$\{k_i, i = 1, \dots\}$
16	1.25	2.86	0.79 (0.81)	1.5 k	{4, 4, 6}
16	1.35	3.12	0.84 (0.87)	1.0 k	{4, 4, 6}
16	1.50	3.52	0.90 (0.92)	2.6 k	{4, 5, 7}
32	1.25	2.95	0.92 (0.96)	2.5 k	{4, 4, 5, 7}
32	1.35	3.21	0.95 (1.03)	2.0 k	{4, 4, 5, 7}
32	1.35	3.25	1.00 (1.03)	12 k	{5, 5, 6, 8}

Table 1 Parameters of the point achieved using the addressing decomposition method. The values inside parenthesis are the optimum γ_s . Column M_t is memory size in bytes (8 bits)/ N -D (no computation).

6 Comparison with other techniques

In the following, we compare our addressing schemes with the schemes of [5], [6] [7], and [20].

As we are essentially able to closely approximate any point up to the knee of the optimum tradeoff curves, in Table (2), we have compared some of the values obtained in [5] and [7] with the optimum values calculated in [2].

N	γ_s	VC, [5]		γ_s	C/O, [7]	
		CER_s	PAR		CER_s	PAR
4	0.37	1.41 (1.09)	4.62 (2.27)			
8	0.65	2.00 (1.26)	6.98 (2.81)	0.60	1.27 (1.19)	2.80 (2.61)
12	0.75	3.00 (1.26)	8.24 (2.86)	0.63	1.55 (1.13)	3.42 (2.51)
16	0.81	1.54 (1.24)	5.58 (2.88)	0.69	1.45 (1.14)	3.02 (2.55)
24	1.03	5.20 (1.50)	15.2 (3.67)	0.80	1.50 (1.16)	3.46 (2.67)
32	0.85	1.35 (1.16)	4.96 (2.70)	0.86	1.46 (1.17)	3.40 (2.72)

Table 2 Comparison between the the Voronoi constellations (VC) and the Calderbank, Ozarow method (C/O) with the optimum constellations, the values in the parenthesis are the optimum values of CER_s , PAR for the given γ_s .

It should be mentioned that by applying the *peak constraint* technique, it is possible to modify the Voronoi constellation in such a way that the 2-D points outside a circle of selected radius are not allowed, [5]. This constraint can be applied to the minimum distance decoder, [21], of the lattice. Such a modification, to some extent, improves the performance of a Voronoi constellation. For example, our simulation results show that for the lattice E_8 , one can achieve almost all the shape gain given in Table (2) but with $CER_s = 1.7$ and $PAR = 4$ instead of $CER_s = 2$ and $PAR = 6.98$. It should be mentioned that most probably for the higher dimensional lattices (like Λ_{24}), the improvement will be more pronounced.

As a more detailed comparison, a four state trellis diagram of [6] (in conjunction with the peak constraint technique) achieves $\gamma_s = 0.97$ dB, $CER_s = 1.5$, $PAR = 3.75$. For $N = 32$, a two-level shaping code of [7] achieves, $\gamma_s = 0.86$ dB, $CER_s = 1.46$, $PAR = 3.40$. In [20], an example for $N = 64$ is given which needs 15 multiply-adds per 2-D, together with a total memory of 1.5 kilo-bytes/ N -D, to achieve a tradeoff point with $CER_s = 1.5$ near to the optimum curve (the optimum γ_s for $N = 64$, $CER_s = 1.5$ is equal to, 1.2 dB).

Table (1) contains some examples of the performance of our scheme. It seems that our method, which has no associated computation, is easier to implement.

7 Application to lattice vector quantization

The ideas presented here for the addressing of the TC_n/A_N regions can be naturally used in conjunction with a lattice vector quantizer with a pyramid/spherical shaping region. Generalization to the case of a region with nonequal values of power along different dimensions (for example in the case of the transform coding) and/or to the case that the quantization region is composed of a set of nested regions with nonequal densities in different layers, [22], [23], is straightforward.

8 Summary and conclusions

We discussed an addressing scheme based on partitioning the constellation into clusters of equal volume and then selecting a subset of the clusters with the lowest average energy. The addressing of this subset is achieved by a lookup table. By using the address decomposition method, we have substantially decreased the size of the lookup table while the suboptimality is negligible. The key point is that for moderate values of CER, in an N -D space, one can closely approximate the optimum tradeoff using a small number of equi-volume energy shells per $N/2$ -D subspaces. This method uses a modest amount of memory, has no associated computation, and can be easily used in conjunction with a general coset coding scheme. It is also possible to achieve the shaping in higher dimensional spaces and use its lower dimensional subspaces for the channel coding. As an example of performance, in an $N = 32$ -D space, by increasing the number of the 2-D points by 25% and using 2.5 kilo-bytes of memory, we realize a shaping gain of 0.92 dB. It seems that after the initial coding gain associated the Ungerboeck's schemes, [24], or with the Wei's schemes, [3], this is the easiest way to obtain higher gains.

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