Bandwidth Efficient Transmultiplexers, Part 2: Subband Complements and Performance Aspects

Ravi P. Ramachandran, *Member, IEEE*, and Peter Kabal, *Member, IEEE*

***Abstract*—This paper examines the performance issues relating to the quadrature amplitude modulation (QAM) and vestigial sideband (VSB) transmultiplexers synthesized in [1]. First, an analysis of the limitations of the configured systems regarding intersymbol interference and crosstalk suppression arising from the use of practical filters is made. Based on these observations, a new design technique for an FIR low-pass prototype that takes the practical degradations into account is formulated. The procedure involves the unconstrained optimization of an error function. A performance evaluation reveals that for four of the five systems, the new method is superior to a minimax approach in that lower intersymbol interference and crosstalk distortions are achieved with a smaller number of filter taps. For the other transmultiplexer, the advantage of the optimized design over the minimax design is in the added flexibility of taking crosstalk into account thereby diminishing the crosstalk distortion. The five transmultiplexers can be converted into new subband systems. We show how the optimized design approach formulated for the transmultiplexers carries over to the new subband systems.**

I. Introduction

MULTIRATE digital filter banks have been used in the realization of transmultiplexers and subband systems [2]–[4]. In fact, the two can be viewed as complements of one another. For systems with two bands, quadrature mirror filters (QMF) [5]–[7] that form a low-pass/high-pass pair are often used. For an arbitrary number of bands, three approaches to specify the filter banks are as follows. One method is based on a matrix formalism [3], [4]. Another employs lossless structures [8]. A third approach uses modulated filter banks [1], [9]–[12].

A transmultiplexer structure (multi-input, multi-output) as shown in Fig. 1 is well suited for simultaneous transmission of many data signals across a single channel. At the transmitter, the outputs of the combining filters are multiplexed into one composite signal. At the receiver, the composite signal is passed through a parallel structure of separation filters whose outputs are decimated to recover the original inputs. Note that the decimation and interpolation are performed synchronously at the same rate and in phase with each other.

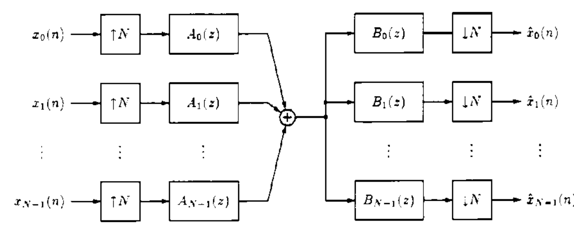

Fig. 1. A transmultiplexer system.

The subband system (single input, single output) as shown in Fig. 2 finds applications in speech processing. The analysis filters split the input signal spectrum into a set of frequency bands. The resultant filtered signals are then decimated and hence, contain aliased components of the input signal. The interpolation step followed by the parallel action of the synthesis filters serves to cancel the aliased components thereby restoring the original signal.

In [1], a number of transmultiplexers are synthesized. These transmultiplexers implement bandwidth efficient modulation schemes in that spectral overlap is accommodated. Both quadrature amplitude modulation (QAM) and vestigial sideband (VSB) schemes are represented. In the first part of this paper, we explore the subband complements to these systems. In the second part of this paper, we investigate the use of practical filters in the transmultiplexers. Filters are designed by minimizing a distortion measure incorporating both intersymbol interference and crosstalk terms.

The general outline of the paper is as follows. Appropriate background material is given in Section II. Then, the systems synthesized in [1] and their subband complements are described in Section III. In examining the performance issues, the first step is to discuss the potential limitations of the transmultiplexers when practical filters are used. This is done in Section IV. Based on the observations in Section IV, a new filter design approach that takes the practical degradations into account is formulated in Section V. Design examples resulting from this new technique are provided in Section VI. The performance of the transmultiplexers with these optimized filters is com-

Manuscript received October 30, 1989; revised December 10, 1990. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

R. P. Ramachandran is with the Speech Research Department, AT&T Bell Laboratories, Murray Hill, NJ 07974.

P. Kabal is with the Department of Electrical Engineering, McGill University, Montreal, Que., H3A 2A7, Canada, and INRS-Telecommunications, Université du Québec, Verdun, Que., H3E 1H6, Canada.

IEEE Log Number 9106578.

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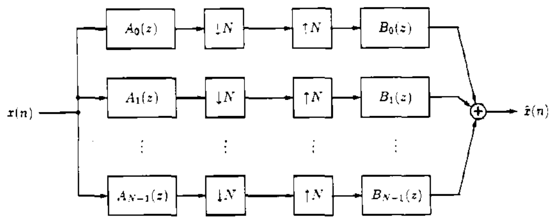


Fig. 2. A subband system.

pared with those designed by a minimax approach in Section VII. Section VIII discusses the conditions under which the new design method carriers over to the corresponding subband systems.

II. Transmultiplexers and Subband Systems

A. Transmultiplexers

For an N-band transmultiplexer with critical sampling (as depicted in Fig. 1), the input-output relations are given by

$$\hat{X}_i(z^N) = \frac{1}{N}\sum_{k=0}^{N-1} X_k(z^N)\sum_{l=0}^{N-1} A_k(zW^{-l})B_i(zW^{-l}) \quad \text{for } 0 \le i \le N-1 \tag{1}$$

where $W = e^{-j(2\pi/N)}$. Each output signal $\hat{X}_i(z^N)$ is related to each input signal $X_k(z^N)$ via a transfer function $(1/N)T_{ki}(z^N)$ where $T_{ki}(z^N) = \Sigma_{l=0}^{N-1} A_k(zW^{-l})B_i(zW^{-l})$. When $k \ne i$, $T_{ki}(z^N)$ is called a crosstalk function and represents the contribution of the input $X_k(z^N)$ to the output $\hat{X}_i(z^N)$. In order to eliminate crosstalk and achieve an identical input-output transfer function $T_{kk}(z^N) = T(z^N)$ for every terminal k, a matrix relationship must be satisfied

$$\boldsymbol{A}(z)\boldsymbol{B}^T(z) = T(z^N)\boldsymbol{I} \tag{2}$$

where

$$\boldsymbol{A}(z) = \begin{bmatrix} A_0(z) & A_0(zW^{-1}) & \cdots & A_0(zW^{-N+1}) \\ A_1(z) & A_1(zW^{-1}) & \cdots & A_1(zW^{-N+1}) \\ \vdots & \vdots & & \vdots \\ A_{N-1}(z) & A_{N-1}(zW^{-1}) & \cdots & A_{N-1}(zW^{-N+1}) \end{bmatrix} \tag{3}$$

$$\boldsymbol{B}(z) = \begin{bmatrix} B_0(z) & B_0(zW^{-1}) & \cdots & B_0(zW^{-N+1}) \\ B_1(z) & B_1(zW^{-1}) & \cdots & B_1(zW^{-N+1}) \\ \vdots & \vdots & & \vdots \\ B_{N-1}(z) & B_{N-1}(zW^{-1}) & \cdots & B_{N-1}(zW^{-N+1}) \end{bmatrix} \tag{4}$$

and $\boldsymbol{I}$ is the identity matrix. If the above matrix equation is satisfied, each of the output signals is given by $\hat{X}_k(z) = (1/N)T(z)X_k(z)$. Intersymbol interference is present if the samples at the output depend on past and future input samples. Intersymbol interference is eliminated if and only if $T(z)$ is of the form cz^{-p}. Then, perfect reconstruction is achieved in that the output samples are scaled and delayed versions of the input samples.

B. Subband System

The input-output description of a subband system (as in Fig. 2) is

$$\hat{X}(z) = \frac{1}{N}\sum_{l=0}^{N-1} X(zW^{-l})\sum_{k=0}^{N-1} A_k(zW^{-l})B_k(z). \tag{5}$$

The output is related to the input and its frequency shifted versions by a system function $(1/N)T_l(z)$ where $T_l(z) = \Sigma_{k=0}^{N-1} A_k(zW^{-l})B_k(z)$. Aliasing is eliminated if $\hat{X}(z)$ does not depend on any of the frequency shifted versions of $X(z)$. Therefore, $T_l(z)$ should be zero for $l \ne 0$. In addition, perfect reconstruction is achieved if and only if $T_0(z) = cz^{-p}$.

The cancellation of aliasing is equivalent to configuring the analysis and synthesis filters to satisfy the system of equations $\boldsymbol{A}^T(z)[B_0(z)\ \ B_1(z)\ \cdots\ B_{N-1}(z)]^T = [T_0(z)\ \ 0\ \cdots\ 0]^T$. This is equivalent to satisfying the matrix equation

$$\boldsymbol{A}^T(z)\boldsymbol{B}(z) = \begin{bmatrix} T_0(z) & 0 & \cdots & 0 \\ 0 & T_0(zW^{-1}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T_0(zW^{-N-1}) \end{bmatrix}. \tag{6}$$

If the above matrix equation is satisfied, the output signal is given by $\hat{X}(z) = (1/N)T_0(z)X(z)$.

C. Complementary Systems

Transmultiplexers and subband systems are complementary in the sense that both crosstalk cancellation in the former and aliasing cancellation in the latter occur if and only if the product of the $\boldsymbol{A}(z)$ and $\boldsymbol{B}(z)$ matrices (one of them is transposed) is equal to a function of z^N multiplied by the identity matrix [4]. Any transmultiplexer that eliminates crosstalk and achieves an identical input-output transfer function between each pair of terminals can be converted into an alias-free subband system. The two systems will have identical filter banks. The limitation in converting an alias-free subband system into a crosstalk-free transmultiplexer is that the input-output transfer function of the subband system must be a function of z^N.

III. Synthesized Transmultiplexers and Subband Systems

The combining and separation filters of the transmultiplexers synthesized in [1] are given below.[1] The trans-

[1]The reader is referred to [1] for general expressions for $a_k(n)$ and $b_k(n)$.

multiplexers use modulated filter banks in which all of the filters are bandpass versions of a low-pass prototype $h(n)$ or $g(n)$. In the sequel, a filter $H(z)$ is a band-limited low-pass prototype if $H(e^{j\omega})$ is exactly equal to zero in the stopband region $\omega_s \leq \omega \leq \pi$. Under the assumption that the prototype(s) are band limited to no more than 100% in excess of the minimum bandwidth solution (discussed in [1]), the transmultiplexers are crosstalk-free and have an identical input-output transfer function for every pair of corresponding terminals. For systems T1–T3 which accomplish QAM, the minimum bandwidth of the prototype(s) is π/N. In the case of systems T4 and T5 which implement VSB, the minimum bandwidth of the prototype is $\pi/2N$. Note that for all the systems except T4, the number of bands N is constrained to be even.

Since the transmultiplexers T1 through T5 are crosstalk-free and have identical input-output transfer functions, they can be converted into alias-free subband systems S1–S5, respectively. The new subband systems S1–S3 are unusual in that repeated center frequencies are used to establish filter responses in quadrature. The subband systems S4 and S5 have the same distinct center frequencies. System S4 resembles the one in [10] while S5 is an alternative employing delay factors. Note that the transmultiplexers that are synthesized as special two band cases in [1] can also be converted into subband systems.

The input-output transfer function for each system is given as $T(z^N)$. When dealing with a transmultiplexer, the input-output relationship is $\hat{X}_k(z) = (1/N)T(z)X_k(z)$ for $k = 0$ to $N - 1$. For the complementary subband systems, $\hat{X}(z) = (1/N)T(z^N)X(z)$. In the following, $\nu(n)$ is defined to be the inverse z transform of $H^2(z)$ and $w(n)$ is defined to be the inverse z transform of $H(z)G(z)$.

Transmultiplexer T1 and Subband System S1:

$$\begin{aligned}
a_0(n) &= h(n)\cos\frac{\pi}{4}\\
a_1(n) &= h\left(n - \frac{N}{2}\right)\cos\frac{2\pi}{N}n\\
a_2(n) &= h(n)\sin\frac{2\pi}{N}n\\
a_3(n) &= h(n)\cos\frac{4\pi}{N}n\\
a_4(n) &= h\left(n - \frac{N}{2}\right)\sin\frac{4\pi}{N}n\\
&\vdots\\
b_0(n) &= h(n)\cos\frac{\pi}{4}\\
b_1(n) &= h\left(n + \frac{N}{2}\right)\cos\frac{2\pi}{N}n\\
b_2(n) &= -h(n)\sin\frac{2\pi}{N}n\\
b_3(n) &= h(n)\cos\frac{4\pi}{N}n\\
b_4(n) &= -h\left(n + \frac{N}{2}\right)\sin\frac{4\pi}{N}n\\
&\vdots
\end{aligned} \tag{7}$$

The input-output transfer function is

$$\begin{aligned}
T(z^N) = \frac{N}{2}[\cdots &+ \nu(-2N)z^{2N} + \nu(-N)z^N + \nu(0)\\
&+ \nu(N)z^{-N} + \nu(2N)z^{-2N} + \cdots].
\end{aligned} \tag{8}$$

For the special case of $N = 2$, S1 reduces to the classical QMF arrangement.

Transmultiplexer T2 and Subband System S2:

$$\begin{aligned}
a_0(n) &= h(n)\cos\left(\frac{\pi}{N}n + \frac{\pi}{4}\right)\\
a_1(n) &= h\left(n - \frac{N}{2}\right)\cos\left(\frac{\pi}{N}n - \frac{\pi}{4}\right)\\
a_2(n) &= h(n)\cos\left(\frac{3\pi}{N}n - \frac{\pi}{4}\right)\\
a_3(n) &= h\left(n - \frac{N}{2}\right)\cos\left(\frac{3\pi}{N}n + \frac{\pi}{4}\right)\\
&\vdots\\
b_0(n) &= h(n)\cos\left(\frac{\pi}{N}n - \frac{\pi}{4}\right)\\
b_1(n) &= h\left(n + \frac{N}{2}\right)\cos\left(\frac{\pi}{N}n + \frac{\pi}{4}\right)\\
b_2(n) &= h(n)\cos\left(\frac{3\pi}{N}n + \frac{\pi}{4}\right)\\
b_3(n) &= h\left(n + \frac{N}{2}\right)\cos\left(\frac{3\pi}{N}n - \frac{\pi}{4}\right)\\
&\vdots
\end{aligned} \tag{9}$$

The input-output transfer function is

$$\begin{aligned}
T(z^N) = \frac{N}{2}[\cdots &+ \nu(-2N)z^{2N} - \nu(-N)z^N + \nu(0)\\
&- \nu(N)z^{-N} + \nu(2N)z^{-2N} + \cdots].
\end{aligned} \tag{10}$$

Transmultiplexer T3 and Subband System S3:

$$\begin{aligned}
a_0(n) &= h(n)\cos\frac{\pi}{4}\\
a_1(n) &= g\left(n - \frac{N}{2}\right)\cos\frac{2\pi}{N}n
\end{aligned}$$

$$a_2(n) = g(n) \sin \frac{2\pi}{N} n$$
$$a_3(n) = h(n) \cos \frac{4\pi}{N} n$$
$$a_4(n) = h\left(n - \frac{N}{2}\right) \sin \frac{4\pi}{N} n$$
$$\vdots$$
$$b_0(n) = g(n) \cos \frac{\pi}{4}$$
$$b_1(n) = h\left(n + \frac{N}{2}\right) \cos \frac{2\pi}{N} n$$
$$b_2(n) = -h(n) \sin \frac{2\pi}{N} n$$
$$b_3(n) = g(n) \cos \frac{4\pi}{N} n$$
$$b_4(n) = -g\left(n + \frac{N}{2}\right) \sin \frac{4\pi}{N} n$$
$$\vdots \tag{11}$$

The input-output transfer function is

$$T(z^N) = \frac{N}{2}[\cdots + w(-2N)z^{2N} + w(-N)z^N + w(0) + w(N)z^{-N} + w(2N)z^{-2N} + \cdots]. \tag{12}$$

Systems S3 becomes the Smith-Barnwell structure [7] for the case $N = 2$ if $G(z) = H(z^{-1})$. For an arbitrary $H(z)$ and $G(z)$, system S3 degenerates into a general two band two prototype system as proposed in [3], [4].

Transmultiplexer T4 and Subband System S4:

$$a_0(n) = h(n) \cos\left(\frac{\pi}{2N} n - \frac{\pi}{4}\right)$$
$$a_1(n) = h(n) \cos\left(\frac{3\pi}{2N} n + \frac{\pi}{4}\right)$$
$$a_2(n) = h(n) \cos\left(\frac{5\pi}{2N} n - \frac{\pi}{4}\right)$$
$$\vdots$$
$$b_0(n) = h(n) \cos\left(\frac{\pi}{2N} n + \frac{\pi}{4}\right)$$
$$b_1(n) = h(n) \cos\left(\frac{3\pi}{2N} n - \frac{\pi}{4}\right)$$
$$b_2(n) = h(n) \cos\left(\frac{5\pi}{2N} n + \frac{\pi}{4}\right)$$
$$\vdots \tag{13}$$

The input-output transfer function is

$$T(z^N) = \frac{N}{2}[\cdots + \nu(-4N)z^{4N} - \nu(-2N)z^{2N} + \nu(0) - \nu(2N)z^{-2N} + \nu(4N)z^{-4N} + \cdots]. \tag{14}$$

Transmultiplexer T5 and Subband System S5:

$$a_0(n) = h(n) \cos\left(\frac{\pi}{2N} n - \frac{\pi}{4}\right)$$
$$a_1(n) = h\left(n - \frac{N}{2}\right) \cos \frac{3\pi}{2N} n$$
$$a_2(n) = h(n) \cos\left(\frac{5\pi}{2N} n + \frac{\pi}{4}\right)$$
$$a_3(n) = h\left(n - \frac{N}{2}\right) \sin \frac{7\pi}{2N} n$$
$$\vdots$$
$$b_0(n) = h(n) \cos\left(\frac{\pi}{2N} n + \frac{\pi}{4}\right)$$
$$b_1(n) = h\left(n + \frac{N}{2}\right) \cos \frac{3\pi}{2N} n$$
$$b_2(n) = h(n) \cos\left(\frac{5\pi}{2N} n - \frac{\pi}{4}\right)$$
$$b_3(n) = -h\left(n + \frac{N}{2}\right) \sin \frac{7\pi}{2N} n$$
$$\vdots \tag{15}$$

The input-output transfer function for T5 is the same as that for T4 (see (14)).

A. Nyquist Criterion and Perfect Reconstruction

A filter is said to satisfy the Nyquist criterion if its impulse response has regular zero crossings except for a reference coefficient. These Nyquist filters can be used to eliminate intersymbol interference in transmultiplexers and subband systems. For systems T1, S1, T2, and S2, $H^2(z)$ should be a Nyquist filter with zero crossings every Nth sample (except for a reference sample). The same is true for $H(z)G(z)$ in systems T3 and S3. For transmultiplexers T4 and T5 and their subband complements, the Nyquist filter $H^2(z)$ must have zero crossings every $2N$th sample (except for a reference sample).

In the general case, for achieving perfect reconstruction in the above transmultiplexers and subband systems, the prototypes must be band limited (up to 100% above the minimum bandwidth, as discussed earlier) and the Nyquist criterion must be satisfied (as stated above). When $G(z) = H(z^{-1})$ in systems T3 and S3, the two conditions of band limitedness and the Nyquist characteristic lead to perfect reconstruction with $B_k(z) = A_k(z^{-1})$. Therefore, both systems are lossless [8] under the same two conditions. For systems T1–T3, S1–S3 under the special case of $N = 2$, the band-limitedness condition can be dropped for assuring perfect reconstruction. The Smith–Barnwell structure (system S3 with $N = 2$) is lossless only if $H(z)H(z^{-1})$ satisfies the Nyquist criterion as no further band-limitedness condition is required.

IV. Practical Considerations

Transmultiplexers T1 through T5 have each been configured with band-limited filters such that 1) the input–output transfer function is the same for every pair of corresponding terminals and 2) crosstalk is cancelled. In addition, satisfying the Nyquist criterion eliminates intersymbol interference and hence, achieves perfect reconstruction. Since band-limited filters (stopband response is exactly zero) cannot be designed, a natural question concerns how the design of a practical low-pass prototype can be performed. A practical low-pass prototype is distinguished from a band-limited prototype in that the frequency response of the practical filter only approximates zero in the stopband. In particular, the practical prototype has a stopband response which is small but not exactly zero (stopband attenuation is high but not infinite). With practical filters, the input–output transfer function may not be the same for all pairs of terminals. In addition, the design procedure may give filters such that the Nyquist criterion is not exactly satisfied. Therefore, intersymbol interference need not be eliminated at each output terminal. Moreover, the use of practical filters may lead to residual crosstalk which would otherwise be cancelled with a band-limited prototype. In this section, we further analyze each transmultiplexer in terms of the possible limitation of not achieving perfect reconstruction due to the use of practical filters. In Section V, filter design strategies are formulated with the aim of suppressing intersymbol interference and crosstalk distortions.

A. The Input–Output Transfer Function

In analyzing the transmultiplexers, we return to the synthesis procedure in [1] to see where the band limitedness property was used in getting a common input–output transfer function. The band limitedness of the low-pass prototype was invoked in the analysis in [1] to cancel term(s) in the general expression for the input–output transfer function as a first step in making it the same for all pairs of corresponding terminals. However, this term(s) is naturally cancelled for all terminals in T2, T4, and T5 and for the terminals in T1 and T3 operating at the center frequencies of 0 and π.

The preceding analysis reveals that the input–output transfer function is indeed the same for all pairs of terminals in each of the systems T2, T4, and T5. Moreover, this property holds for any practical prototype $H(z)$. Therefore, for any $H(z)$, the common input–output transfer function $T(z^N)$ is given by (10) for system T2 and by (14) for T4 and T5. Now, consider systems T1 and T3. The common input–output transfer function $T(z^N)$ as given in (8) (system T1) and in (12) (system T3) holds only for the terminals specified by center frequencies of 0 and π. Again, this is true for practical prototypes. The input–output transfer functions for the other terminals of T1 and T3 are different from those given by (8) and (12) when practical filters are used. These differences are due to the fact that the prototypes are not band limited.

The next step is to identify the sources of intersymbol interference in each of the transmultiplexers. In systems T2, T4, and T5, intersymbol interference is cancelled at all terminals given any $H(z)$ if $H^2(z)$ satisfies the Nyquist criterion. The only potential source of intersymbol interference is due to the limitation of the design procedure in giving $H(z)$ such that $H^2(z)$ does not exactly satisfy the Nyquist criterion.

When dealing with systems T1 and T3, two cases must be considered. First, consider the terminals operating at center frequencies of 0 and π. At these terminals, the only source of intersymbol interference is due to the design procedure in giving filters such that the Nyquist criterion is not exactly satisfied. At the other terminals, an additional source of intersymbol interference arises since the filters are not band limited.

B. Crosstalk Functions

Here, we wish to determine the sources of crosstalk that arises with practical filters. From the synthesis procedure in [1], crosstalk cancellation with band-limited prototypes occurs in two ways. First, the crosstalk terms (which comprise the crosstalk function $T_{kl}(z^N)$) that involve either partial or complete spectral overlap are cancelled by choosing the center frequencies, delays, and phases. This cancellation depends only on the center frequencies, delays, and phases and is independent of any particular form of $H(z)$ and $G(z)$. Therefore, these terms continue to be cancelled with practical filters. Second, terms in the crosstalk function that do not involve spectral overlap are zero due to the band limitedness of the prototypes. However, these crosstalk terms are not zero with practical filters. This will lead to residual crosstalk. Summarizing, we note that all the crosstalk terms in $T_{kl}(z^N)$ that involve spectral overlap with band-limited filters continue to be cancelled with practical filters.

Note that with practical filters, although the terms in $T_{kl}(z^N)$ that involve spectral overlap are cancelled (as discussed above), this does not generally imply that $T_{kl}(z^N) = 0$. We further analyze each of the transmultiplexers to determine the number of crosstalk functions that are exactly zero with practical filters (also referred to in the sequel as exact crosstalk cancellation). Exact crosstalk cancellation depends only on the center frequencies, delays, and phases and occurs independently of the prototypes $H(z)$ and $G(z)$. For a particular output terminal, there are

$N - 1$ crosstalk functions. For each of the transmultiplexers, a certain number of these $N - 1$ functions may be exactly zero. We proceed to enumerate the number of exact crosstalk cancellations.

In system T1, the crosstalk is exactly zero between two signals sent at the same center frequency, at center frequencies separated by an odd multiple of $2\pi/N$ and at center frequencies separated by an even multiple of $2\pi/N$ if the difference in the delay factors is an odd multiple of $N/2$. In system T2, exact crosstalk cancellation occurs between any two signals as long as the difference in the delay factors of the associated combining and separation filtes is an odd multiple of $N/2$. System T3, like T1, has crosstalk functions involving one prototype for signals sent at center frequencies separated by an odd multiple of $2\pi/N$. For these cases, the crosstalk function is exactly zero. When two prototypes are involved in the crosstalk function, exact crosstalk cancellation only occurs between two signals sent with a center frequency of $\pi/2$ (this center frequency appears when N is a multiple of 4). For transmultiplexer T4, none of the crosstalk functions is exactly zero. In T5, the crosstalk function $T_{kl}(z^N)$ is exactly zero if $k + l = N - 1$ for N not a multiple of 4. If N is a multiple of 4, $T_{kl}(z^N)$ is never exactly zero in T5.

Given the preceding discussion, all the cases were examined in detail and the number of exact crosstalk cancellations enumerated for each output terminal. Table I summarizes the results. The Appendix gives the derivation of one case for system T1, namely, for output terminals operating at center frequencies that are even multiples of $2\pi/N$ when N is a multiple of 4. We see that for the case explored in the Appendix, the number of exact crosstalk cancellations is different for the two terminals at each of these center frequencies. At one of the terminals, there are $(3N - 4)/4$ exact crosstalk cancellations. At the other terminal, $(3N + 4)/4$ exact crosstalk cancellations occur. A similar situation in T1 develops when N is not a multiple of 4 and the center frequencies are either even or odd multiples of $2\pi/N$. In this case, the two terminals at these frequencies will show a different number of crosstalk functions that are exactly zero. The number of exact crosstalk cancellations is approximately $3N/4$ for all the terminals.

Transmultiplexer T3 has approximately $N/2$ exact crosstalk cancellations at each output terminal, the actual number depending on whether a center frequency of $\pi/2$ is used. Transmultiplexers T2 and T4 have $N/2$ and 0 exact crosstalk cancellations at each output terminal, respectively. In system T5, one crosstalk function is exactly zero for each output terminal when N is not a multiple of 4. When N is a multiple of 4, none of the crosstalk functions is exactly zero in T5.

Of the transmultiplexers, T1 achieves the most number of exact crosstalk cancellations (about 3/4 of the total number of crosstalk functions). In systems T2 and T3, about half of the crosstalk functions are exactly zero. The table shows that for reasonably large N, the QAM schemes (T1-T3) achieve many more exact crosstalk cancellations than their VSB counterparts (T4 and T5).

TABLE I
NUMBER OF EXACT CROSSTALK CANCELLATIONS FOR EACH OUTPUT TERMINAL

Transmultiplexer	Number of Cancellations
T1	$\approx \frac{3N}{4}$
T2	$\frac{N}{2}$
T3	$\approx \frac{N}{2}$
T4	0
T5	0 or 1

V. Filter Design

This section examines the design of FIR low-pass prototypes for the transmultiplexers. First, a minimax method is discussed. Then, a new design procedure based on the minimization of an error function that takes the practical degradations into account is proposed. The filters that result are compared to those obtained by a minimax design in terms of their relative performance with respect to suppression of intersymbol interference and crosstalk.

A linear phase $H(z)$ is designed for T1, T2, T4, and T5. For T3, we design a nonlinear phase $H(z)$ and set $G(z) = H(z^{-1})$. In the designs, the stopband edge frequency is $\omega_s = (1 + \beta)\omega_{\min}$ where $\omega_{\min}$ is the minimum bandwidth of the low-pass prototype and $0 \leq \beta \leq 1$. Recall that $\omega_{\min} = \pi/N$ for T1, T2, and T3 and $\omega_{\min} = \pi/2N$ for T4 and T5. The parameter β is the roll-off factor that controls the bandwidth in excess of $\omega_{\min}$.

A. Minimax Approach

The minimax approaches are based on the input–output transfer functions given in Section III (assumed to be the same for all terminals). Therefore, any differences in the input–output transfer functions of the various systems that can arise with practical filters are not considered. Also, the crosstalk is not explicitly considered. For the transmultiplexers T1, T2, T4, and T5 a low-pass $H(z)$ should be designed such that $H^2(z)$ is a Nyquist filter. In the case of T3 with $G(z) = H(z^{-1})$, we need a low-pass $H(z)$ such that $H(z)H(z^{-1})$ is a Nyquist filter.

The attempt to ensure exact zero crossings in the response of $H^2(z)$ leads to many nonlinear constraints on the coefficients of $H(z)$ which compromise the desired low-pass nature of $H(z)$. Therefore, there is an inherent limitation in the design procedure in not achieving exact zero crossings. Our strategy is to obtain a low-pass $H(z)$ such that the time domain constraints on $H^2(z)$ are approximately satisfied. A linear phase $H(z)$ is designed by the McClellan–Parks algorithm [13] to approximate the square root of a raised cosine spectrum in a minimiax sense. For transmultiplexer T3, exact zero crossings in the response of a low-pass $H(z)H(z^{-1})$ can be assured [14]. The technique in [14] (known as the factorable minimax method) is used to design a linear phase Nyquist filter with Chebyshev stopband behavior that can be split into a minimum phase $H(z)$ and a maximum phase $H(z^{-1})$. The two components $H(z)$ and $H(z^{-1})$ form the prototypes for T3. The minimax approaches allow for both equiripple and nonequiripple low-pass prototypes.

Consider systems T1 and T2. For these two systems, N must be even. If a linear phase $H(z)$ has an odd number of taps, an appropriate choice of filter delay results in the center coefficient of $H^2(z)$ emerging at a time index which is a multiple of N. If $H(z)$ has an even number of taps, there is no choice of delay that allows the center coefficient of $H^2(z)$ to emerge at a time index which is a multiple of N. For an even number of taps, the center coefficient of $H^2(z)$ never shows up in the computation of $T(z^N)$. For systems T4 and T5, it can also be shown that a linear phase $H(z)$ must have an odd number of taps. For system T3, the linear phase Nyquist filter $H(z)H(z^{-1})$ must have an odd number of taps to allow it to be split into min/max phase parts. However, there is no constraint on the number of coefficients of the min/max phase components. To ensure causal combining and separation banks, additional appropriately chosen delay factors may be required.[2]

B. Optimized Design—Error Function Formulation

Note that the minimax approaches do not explicitly take the crosstalk distortion into account. Therefore, we attempt to include both the practical degradations of intersymbol interference and crosstalk in the design of the prototype. The new optimized FIR design is based on the minimization of an error function. Upon minimization of the error function, the resulting low-pass prototype should have a good stopband behavior and in addition, the intersymbol interference and crosstalk distortions should be small. Note that the passband characteristic is not explicitly controlled since an approximately zero stopband response and a low intersymbol interference distortion ensure an approximately constant passband response if $\beta < 1$.

For notational convenience, we assume throughout that $h(n)$ is in zero-phase form with nonzero samples from $n = -L$ to L. A nonlinear phase $h(n)$ with $L + 1$ taps from $n = 0$ to L is designed for transmultiplexer T3 with $G(z) = H(z^{-1})$. Hence, the reference coefficient of $H^2(z)$ and $H(z)H(z^{-1})$ is at $n = 0$. The error function is a weighted linear combination of various factors, each of which is discussed below.

1) Stopband: The factor in the error function representing the stopband characteristic is denoted by E_{sb} where

$$\sqrt{E_{sb}} = \frac{1}{2\pi}\int_S |H(e^{j\omega})|^2\, d\omega, \tag{16}$$

$S = [-\pi, -\omega_s] \cup [\omega_s, \pi]$ and ω_s is the stopband edge. Therefore, E_{sb} is the square of the energy in the stopband. This function has been used in [16] as part of a general least squares linear phase FIR design. For a zero-phase $H(z)$ with an odd number of taps (designed for T1, T2, T4, and T5), the frequency response can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{L} b(n)\cos\omega n \tag{17}$$

where $b(0) = h(0)$ and $b(n) = 2h(n)$ for $n \neq 0$. The quantity $\sqrt{E_{sb}}$ can be expressed as $\boldsymbol{b}^T\boldsymbol{P}\boldsymbol{b}$ where $\boldsymbol{b} = [b(0)b(1) \cdots b(L)]^T$ and $\boldsymbol{P}$ is a positive definite symmetric matrix whose entries are given by

$$P(r, s) = \frac{1}{\pi}\int_{\omega_s}^{\pi} \cos r\omega \cos s\omega\, d\omega \tag{18}$$

for $0 \leq r, s \leq L$.

Since $G(z) = H(z^{-1})$ in system T3, the stopband energies of both filters are the same. For a nonlinear phase $H(z)$, $\sqrt{E_{sb}}$ can again be expressed in quadratic form $\boldsymbol{h}^T\boldsymbol{R}\boldsymbol{h}$ where $\boldsymbol{h} = [h(0)h(1) \cdots h(L)]^T$ and $\boldsymbol{R}$ is a positive definite symmetric matrix whose entries are given by

$$R(r, s) = \frac{1}{2\pi}\int_S e^{j\omega(r-s)}\, d\omega = \frac{1}{\pi}\int_{\omega_s}^{\pi} \cos(\omega(r-s))\, d\omega \tag{19}$$

for $0 \leq r, s \leq L$.

2) Intersymbol Interference Distortion: At output terminal l, the mean-square intersymbol interference distortion is given by $(1/N^2)\sum_{n\neq 0} t_{ll}^2(n)$ where $t_{ll}(n)$ is the inverse z transform of the input–output transfer function $T_{ll}(z)$. The mean-square intersymbol interference distortion depends on which output terminal is considered. However, given the discussion in Section IV-A, the transfer function is the same for many input–output terminal pairs when practical filters are used. Therefore, the mean-square intersymbol interference distortion will be the same at many output terminals.

Consider systems T2, T4, and T5. As mentioned in Section IV-A, $t_{ll}(n)$ is the same for every terminal l even with practical filters. Hence, it is sufficient to determine the mean-square intersymbol interference distortion at only one terminal. Moreover, $t_{ll}(n)$ is the inverse z-transform of $T(z)$ where $T(z^N)$ is defined in (10) for T2 and (14) for T4 and T5. Therefore, the mean-square intersymbol interference distortion is $1/4\sum_{n\neq 0} \nu^2(nN)$ for T2 and $1/4\sum_{n\neq 0}\nu^2(2nN)$ for T4 and T5 where $\nu(n) = h(n) * h(n)$ ($*$ is the convolution operator).

In systems T1 and T3, $t_{ll}(n)$ is generally different for each terminal l with practical filters. As mentioned in Section IV-A, these differences are due to the fact that the

[2]The delay values used for the combining and separation banks must add up to a multiple of N to preserve the crosstalk cancellation property (see also [15]).

prototypes are not band limited. We ignore the differences in $t_{ll}(n)$ and only consider the terminal at either a center frequency of 0 or π. At each of these terminals, $t_{ll}(n)$ is the inverse z transform of $T(z)$ where $T(z^N)$ is defined in (8) for T1 and in (12) for T3. Therefore, the mean-square intersymbol interference distortion at each of these terminals is $1/4 \sum_{n \neq 0} v^2(nN)$ for T1 and $1/4 \sum_{n \neq 0} w^2(nN)$ for T3 where $v(n) = h(n) * h(n)$ and $w(n) = h(n) * h(-n)$.

The factor representing the mean-square intersymbol interference distortion is denoted by E_{isi}. For systems T2, T4, and T5, E_{isi} is based on any terminal l. However, for T1 and T3, E_{isi} is based on the terminal at either a center frequency of 0 or π. From the preceding discussion, E_{isi} is given by

$$E_{\text{isi}} = \begin{cases} \sum\limits_{\substack{n=cN \\ n \neq 0}} [h(n) * h(n)]^2 & \text{for systems T1 and T2} \\ \sum\limits_{\substack{n=2cN \\ n \neq 0}} [h(n) * h(n)]^2 & \text{for systems T4 and T5} \\ \sum\limits_{\substack{n=cN \\ n \neq 0}} [h(n) * h(-n)]^2 & \text{for system T3.} \end{cases} \tag{20}$$

Note that E_{isi} is a function of $\boldsymbol{b}$ for T1, T2, T4, and T5 and is a function of $\boldsymbol{h}$ for T3.

3) Crosstalk Distortion: At output terminal l, the total crosstalk power due to the undesired input signals is $P_{\text{ctk}}(l)$. In developing a mathematical formula for $P_{\text{ctk}}(l)$, we assume that each of the input data signals is zero-mean, white, uncorrelated with other inputs and has the same signal power P_s. The crosstalk power at output terminal l contributed by a signal at input terminal k is given by the input signal power P_s multiplied by $1/N^2 \sum_n t_{kl}^2(n)$ where $t_{kl}(n)$ is the inverse z transform of the crosstalk function $T_{kl}(z)$. Also, the total crosstalk power at output terminal l is the sum of the crosstalk power contributed by each of the undesired signals and is given by

$$P_{\text{ctk}}(l) = \frac{P_s}{N^2} \sum_{\substack{k=0 \\ k \neq l}}^{N-1} \sum_n t_{kl}^2(n). \tag{21}$$

To include the crosstalk power for every terminal l, we formulate an overall crosstalk factor E_{ctk} given by

$$\begin{aligned} E_{\text{ctk}} &= \frac{1}{P_s} \sum_{l=0}^{N-1} P_{\text{ctk}}(l) \\ &= \frac{1}{N^2} \sum_{l=0}^{N-1} \sum_{\substack{k=0 \\ k \neq l}}^{N-1} \sum_n t_{kl}^2(n) \\ &= \sum_{l=0}^{N-1} \sum_{\substack{k=0 \\ k \neq l}}^{N-1} \sum_{n=cN} [a_k(n) * b_l(n)]^2. \end{aligned} \tag{22}$$

Recall that $a_k(n)$ and $b_l(n)$ are the impulse responses of the kth combining filter and the lth separation filter, respectively. Note that E_{ctk} is a function of $\boldsymbol{b}$ for T1, T2, T4, and T5 and a function of $\boldsymbol{h}$ for T3.

For computational purposes, the number of terms invoved in the expression for E_{ctk} can be decreased by exploiting the symmetry of the crosstalk power and the fact that there may be some crosstalk functions that are exactly zero. The total crosstalk power for output terminal l operating at a center frequency ω_l is the same as that for a terminal operating at $\pi - \omega_l$ (except for $\omega_l = \pi/2$ in some systems). Hence, only the output terminals operating at frequencies in the range $[0, \pi/2]$ need be considered. After taking advantage of the symmetry described above, we can further exclude the terms in E_{ctk} corresponding to the crosstalk functions which are exactly zero.

4) Overall Error Function: The overall error function to be minimized is the weighted sum of the individual factors relating to the stopband, mean-square intersymbol interference distortion and total crosstalk power. At this point, note that the zero solution ($\boldsymbol{b} = 0$ or $\boldsymbol{h} = 0$) is the global minimum. To avoid reaching this solution, we append a term $(\boldsymbol{b}^T\boldsymbol{b} - 1)^2$ or $(\boldsymbol{h}^T\boldsymbol{h} - 1)^2$ to the overall error function. Hence, the overall error function $E(\boldsymbol{b})$ (applies to T1, T2, T4, and T5) and $E(\boldsymbol{h})$ (applies to T3) are

$$\begin{aligned} E(\boldsymbol{b}) &= \gamma_1 E_{sb} + \gamma_2 E_{isi} + \gamma_3 E_{ctk} + \gamma_4(\boldsymbol{b}^T\boldsymbol{b} - 1)^2 \\ E(\boldsymbol{h}) &= \gamma_1 E_{sb} + \gamma_2 E_{isi} + \gamma_3 E_{ctk} + \gamma_4(\boldsymbol{h}^T\boldsymbol{h} - 1)^2 \end{aligned} \tag{23}$$

where the γ_i represent nonnegative weighting factors. With $\gamma_3 = 0$ (no crosstalk factor), the same $E(\boldsymbol{b})$ and hence, the same filter results for systems T1 and T2 and for T4 and T5.

5) Optimization Procedure: We use a quasi-Newton approach [17] to get a local minimum of E. It is an iterative method specified by the two equations

$$\begin{aligned} \boldsymbol{H}_k \boldsymbol{s}_k &= -\nabla \boldsymbol{E}(\boldsymbol{d}_k) \\ \boldsymbol{d}_{k-1} &= \boldsymbol{d}_k + \lambda_k \boldsymbol{s}_k \end{aligned} \tag{24}$$

where k is the iteration index, $\boldsymbol{H}_k$ is the Hessian matrix, $\boldsymbol{s}_k$ is the direction of descent, $\nabla \boldsymbol{E}$ is the gradient of E, and λ_k is a scaling factor which specifies the extent to which movement along the direction of descent occurs to get an update. Note that $\boldsymbol{d}$ is the vector of variables to be optimized and is updated in each iteration. Then, $\boldsymbol{d} = \boldsymbol{b}$ for T1, T2, T4, and T5 and $\boldsymbol{d} = \boldsymbol{h}$ for T3. We express the gradient $\nabla \boldsymbol{E}$ in closed form and evaluate it at $\boldsymbol{d}_k$ in each iteration. Although the Hessian matrix can be expressed in closed form, we use the Broyden–Fletcher–Goldfarb–Shanno (BFGS) update [17] in each iteration. In the actual implementation, we use a routine from the IMSL library [18] to perform the minimization. An initial condition is supplied as an input. Also, subroutines to calculate the error function and its gradient are supplied by the user.

VI. Design Examples

When performing an unconstrained minimization of the error function, we use the optimization procedure described above. Note that the initial conditions affect the final local minimum. For systems T1, T2, T4, and T5, the initial condition we use corresponds to an equiripple

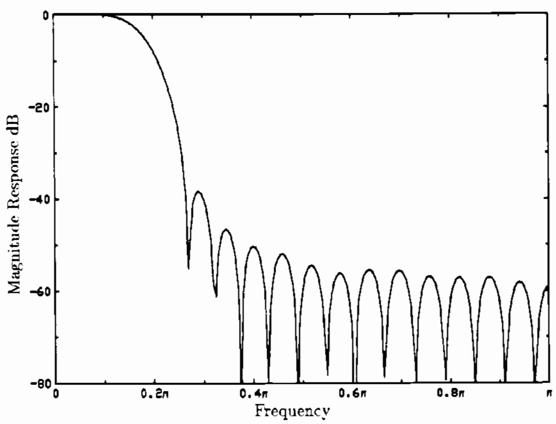


Fig. 3. Magnitude response of the low-pass filter for system T1. The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

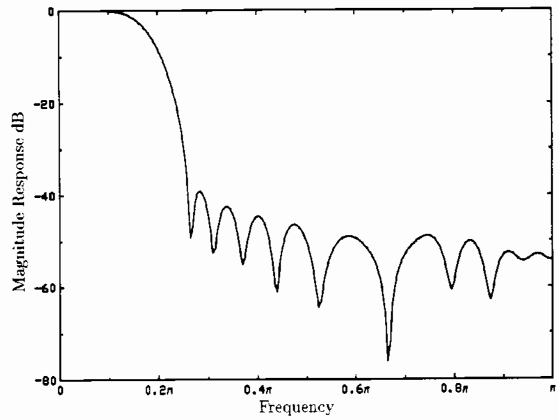
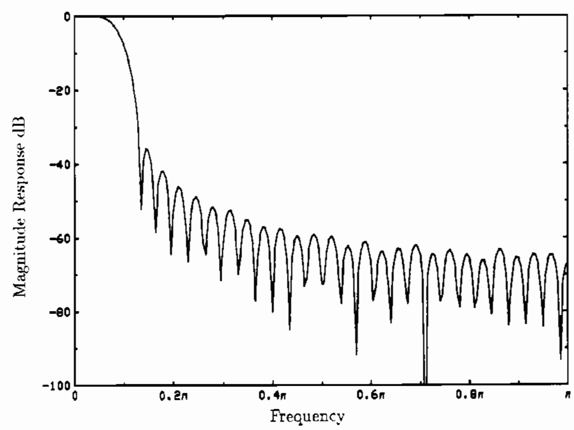


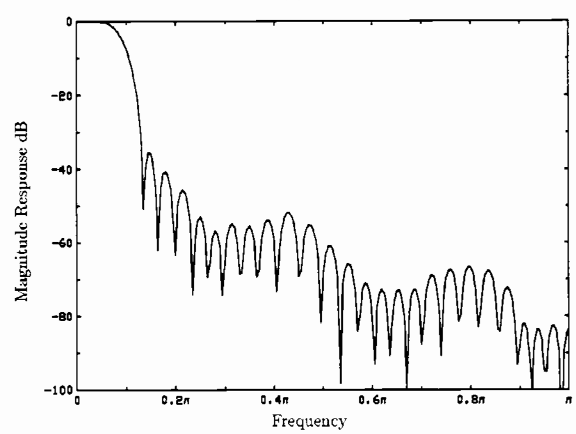
Fig. 4. Magnitude response of the low-pass filter for system T3. The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

linear phase filter (with unity gain at zero frequency) having a frequency response that is a minimax approximation of the square root of a raised cosine spectrum. For transmultiplexer T3, the initial condition we use corresponds to an equiripple minimum phase filter (with unity gain at zero frequency) that is designed by the approach in [14]. Examples of magnitude response plots are shown in Fig. 3 (system T1), Fig. 4 (system T3), and Figs. 5(a) and (b) (system T4) for the case $N = 6$ and $\beta = 0.52$. Fig. 3 shows the magnitude response of a 33 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Fig. 4 shows the magnitude response of a 30 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Fig. 5 shows the magnitude responses of a 59 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Note that the magnitude response in the passband is flat to within 0.013 dB (Fig. 3), 0.003 dB (Fig. 4), and 0.014 dB (Figs. 5(a) and (b)).

The fact that some crosstalk terms which form the crosstalk function $T_{kl}(z^N)$ are exactly zero is reflected in the frequency response of the low-pass prototype. Consider Fig. 5 which shows the magnitude responses of the optimized filters for system T4 with and without a crosstalk weight γ_3. The stopband response is significantly different for the two filters. When a positive crosstalk weight is applied, the stopband response is shaped so as to suppress the nonzero crosstalk terms. An analysis of system T4 revealed that none of the crosstalk functions $T_{kl}(z^N)$ is exactly zero. However, some of the terms in the crosstalk function $T_{kl}(z^N)$ are zero. Among the crosstalk functions in T4 for the case $N = 6$, the terms involving sidebands whose center frequencies are separated by $\pi/3$, $2\pi/3$, and π are never zero. The other terms involving sidebands whose center frequencies are separated by $\pi/6$, $\pi/2$, and $5\pi/6$ are consistently zero. This manifests itself in that the magnitude response in the stopband around the frequencies of $\pi/3$, $2\pi/3$, and π exhibit a higher attenuation than neighbouring regions. It is the higher attenuation in these regions that suppress the nonzero crosstalk terms. Similarly, transmultiplexer T3 has nonzero crosstalk terms involving sidebands separated by $2\pi/3$ when $N = 6$. When the crosstalk weight $\gamma_3 = 1$, the stopband response of the resulting filter is better than for a design with $\gamma_3 = 0$ about the frequency $2\pi/3$.



(a)



(b)

Fig. 5. Magnitude response of the low-pass filter for system T4. (a) The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$. (b) The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

Additional experiments were conducted by changing only the parameter γ_4 (the weighting factor for the term that avoids a zero solution) and observing the performance in terms of intersymbol interference and crosstalk distortions. The value $\gamma_4 = 0.01$ was chosen to arrive at a good solution in a reasonable number of iterations. Reducing γ_4 significantly below this value gives a local minimum with a poorer performance (in terms of intersymbol interference and crosstalk distortions). Increasing γ_4 beyond 0.01 merely increases the number of iterations.

As an alternative to the quasi-Newton procedure, the steepest descent algorithm was also attempted with the same initial conditions. At the beginning, there was a rapid decrease in the error. Then, there was a very slow decrease in the error but no signs of convergence even after many iterations.

VII. Transmultiplexer Performance

The performance of the transmultiplexers is evaluated and compared for minimax filters and for filters designed by the method in this paper. The transmultiplexers have six bands ($N = 6$) and use filters having an excess bandwidth of 52% ($\beta = 0.52$). For systems T1–T3, the aim is to achieve a minimum stopband attenuation of about 40 dB. A stopband attenuation of about 35 dB is used for systems T4 and T5 since an excessively long prototype would be required for a 40-dB attenuation when using the minimax method.

For systems T1, T2, T4, and T5, a minimax linear phase $H(z)$ is designed by the McClellan–Parks algorithm [13] such that its frequency response approximates the square root of a raised cosine spectrum. The factorable minimax method in [14] is used for T3. The resulting prototypes $H(z)$ and $H(z^{-1})$ are not linear phase. For T1 and T2, the prototype has 77 taps. For T3, a 30 tap filter results. For T4 and T5, a 99 tap prototype is used. Equiripple designs are obtained by a weighting function equal to unity.

We also design nonequiripple responses for the transmultiplexers. For systems T1, T2, T4, and T5, the weighting function $W(\omega)$ is unity in the passband and the transition band. In the stopband, an increasing weight is used:

$$W(\omega) = \frac{200}{2\pi}(\omega - \omega_s) + 1 \tag{25}$$

for $\omega_s \le \omega \le \pi$. In the case of T3, the factorable minimax method in [14] is based exclusively on stopband control and hence, allows for weighting only in the stopband. We use $W(\omega)$ as above for $\omega_s \le \omega \le \pi$. These filters, with a stopband attenuation increasing towards π, should achieve a higher crosstalk suppression. In all cases, the minimum stopband attenuation (at the stopband edge) is essentially the same for the equiripple and nonequiripple filters. However, the attenuation at the high frequencies for the nonequiripple designs is 58 dB (77 tap prototype for T1 and T2), 52 dB (30 tap filter for T3) and 54 dB (99 tap filter for T4 and T5).

Using the new method involving an unconstrained minimization of the error function E, we design a 33 tap filter for systems T1 and T2, a 30 tap filter for system T3 and a 59 tap filter for transmultiplexers T4 and T5. For systems T1, T2, T4, and T5, the initial condition for the optimization corresponds to an equiripple linear phase filter (with unity gain at zero frequency) having a frequency response that is a minimax approximation of the square root of a raised cosine spectrum. For system T3, the initial condition corresponds to an equiripple minimum phase filter (with unity gain at zero frequency) design by the factorable minimax method. The weighting factors used are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$ and $(100, 1, 1, 0.01)$. The design examples in the previous section correspond to those used here in the performance study. The minimum stopband attenuations (at the stopband edge) are approximately equal whether crosstalk is taken into account or not ($\gamma_3 = 1$ or $\gamma_3 = 0$).

In measuring the performance of the transmultiplexers, we consider the normalized peak distortion D_P and the normalized root mean-square (rms) distortion D_{RMS} for the intersymbol interference. For the lth terminal, $D_P(l)$ is

$$D_P(l) = \frac{\sum\limits_{\substack{n \\ n \ne 0}} |t_{ll}(n)|}{|t_{ll}(0)|} \tag{26}$$

and $D_{\mathrm{RMS}}(l)$ is

$$D_{\mathrm{RMS}}(l) = \sqrt{\frac{\sum\limits_{\substack{n \\ n \ne 0}} t_{ll}^2(n)}{t_{ll}^2(0)}}. \tag{27}$$

Note that the factor E_{isi} in the error function only considers the mean-square distortion. The quantity $D_P(l)$ as well as $D_{\mathrm{RMS}}(l)$ will be the same for all terminals in T2, T4, and T5. There will be some variation among the terminals in T1 and T3.

The normalized crosstalk power at terminal l, $D_{\mathrm{CRP}}(l)$ is the performance measure for the crosstalk. It is expressed as

$$D_{\mathrm{CRP}}(l) = \frac{P_{\mathrm{ctk}}(l)}{\frac{P_s}{N^2}\sum\limits_n t_{ll}^2(n)} = \frac{\sum\limits_{\substack{k=0 \\ k \ne l}}^{N-1} \sum\limits_n t_{kl}^2(n)}{\sum\limits_n t_{ll}^2(n)}. \tag{28}$$

The output signal at terminal l contains two components, one desired term resulting from the corresponding input and an undesired factor due to crosstalk. At terminal l, the power of the desired component is the input signal power P_s multiplied by $(1/N^2)\,\Sigma_n\, t_{ll}^2(n)$. Dividing the total crosstalk power by the power of the desired component establishes the normalized crosstalk power $D_{\mathrm{CRP}}(l)$ which can be thought of as a crosstalk to signal ratio.

Tables II–IV show the values of $D_P(l)$, $D_{\mathrm{RMS}}(l)$ and

TABLE II
PEAK DISTORTION (IN DECIBELS) FOR TRANSMULTIPLEXERS T1 TO T5. ENTRIES ALONG A ROW REFER TO OUTPUT TERMINALS l = 0, 1, AND 2, RESPECTIVELY. THE OPTIMIZED DESIGNS ARE DONE WITH $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$

Transmultiplexer	$D_p(l)$ in Decibels Minimax Design Constant $W(\omega)$			$D_p(l)$ in Decibels Minimax Design Increasing $W(\omega)$			$D_p(l)$ in Decibels Optimized Design $\gamma_3 = 0$			$D_p(l)$ in Decibels Optimized Design $\gamma_3 = 1$		
T1	−30	−29	−31	−29	−29	−30	−56	−55	−54	−56	−56	−56
T2	−30	−30	−30	−29	−29	−29	−56	−56	−56	−56	−56	−56
T3	−∞	−39	−39	−∞	−48	−48	−99	−49	−49	−92	−78	−82
T4	−26	−26	−26	−23	−23	−23	−56	−56	−56	−57	−57	−57
T5	−26	−26	−26	−23	−23	−23	−56	−56	−56	−56	−56	−56

TABLE III
RMS DISTORTION (IN DECIBELS) FOR TRANSMULTIPLEXERS T1 TO T5. ENTRIES ALONG A ROW REFER TO OUTPUT TERMINALS l = 0, 1, AND 2, RESPECTIVELY. THE OPTIMIZED DESIGNS ARE DONE WITH $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$

Transmultiplexer	$D_{RMS}(l)$ in Decibels Minimax Design Constant $W(\omega)$			$D_{RMS}(l)$ in Decibels Minimax Design Increasing $W(\omega)$			$D_{RMS}(l)$ in Decibels Optimized Design $\gamma_3 = 0$			$D_{RMS}(l)$ in Decibels Optimized Design $\gamma_3 = 1$		
T1	−36	−34	−37	−34	−34	−34	−60	−60	−60	−60	−60	−60
T2	−36	−36	−36	−34	−34	−34	−60	−60	−60	−60	−60	−60
T3	−∞	−45	−45	−∞	−54	−54	−105	−57	−57	−96	−83	−88
T4	−31	−31	−31	−31	−31	−31	−62	−62	−62	−63	−63	−63
T5	−31	−31	−31	−31	−31	−31	−62	−62	−62	−62	−62	−62

TABLE IV
NORMALIZED CROSSTALK POWER (IN DECIBELS) FOR TRANSMULTIPLEXERS T1 AND T5. ENTRIES ALONG A ROW REFER TO OUTPUT TERMINALS l = 0, 1, AND 2, RESPECTIVELY. THE OPTIMIZED DESIGNS ARE DONE WITH $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$

Transmultiplexer	$D_{CRP}(l)$ in Decibels Minimax Design Constant $W(\omega)$			$D_{CRP}(l)$ in Decibels Minimax Design Increasing $W(\omega)$			$D_{CR}(l)$ in Decibels Optimized Design $\gamma_3 = 0$			$D_{CRP}(l)$ in Decibels Optimized Design $\gamma_3 = 1$		
T1	−47	−47	−∞	−65	−65	−∞	−70	−70	−∞	−87	−87	−∞
T2	−47	−47	−47	−65	−65	−65	−70	−70	−70	−87	−87	−87
T3	−39	−40	−41	−47	−49	−48	−46	−48	−45	−74	−77	−73
T4	−25	−25	−25	−40	−40	−40	−54	−54	−54	−65	−65	−65
T5	−26	−26	−26	−43	−44	−41	−49	−50	−52	−60	−60	−61

$D_{CRP}(l)$ (in decibels) for the transmultiplexers when $N = 6$. Only the values for the first three output terminals are provided since symmetry gives the same results for the other three terminals.[3] We proceed to analyze the results and compare the two design methods.

A. Intersymbol Interference Suppression

In Section IV-A, we identified two potential sources of intersymbol interference. These are 1) the limitation of the design procedure in giving filters such that the Nyquist criterion is not exactly satisfied and 2) the fact that the prototypes are not band limited. These causes of intersymbol interference are reflected in Tables II and III. In the forthcoming analysis, we refer to these sources of intersymbol interference as source (1) and source (2). Also, our observations are confined to the first three terminals of the transmultiplexers. However, these observations will hold for the corresponding last three terminals due to symmetry.

[3]Note that for system T5 with $\gamma_3 = 1$, the optimization algorithm did not converge. A fixup involved using only the crosstalk terms having sidebands separated by no more than $\pi/2$.

First, consider the minimax designs. Source (1) is the only potential cause of intersymbol interference in systems T2, T4, and T5 and at terminal 0 of T1 and T3. There is no intersymbol interference at terminal 0 of T3 since the factorable minimax method assures a Nyquist characteristic. For the other cases, a minimax design that approximates the square root of a Nyquist characteristic leads to intersymbol interference. Regarding terminals 1 and 2 of transmultiplexer T1, both sources (1) and (2) contribute to intersymbol interference. However, the small variation in the values of D_p and D_{RMS} for T1 shows that source (2) is not severe. At terminals 1 and 2 of T3, only source (2) contributes to intersymbol interference. The low normalized peak and rms distortions for terminals 1 and 2 of T3 again show that source (2) is not severe. In fact, T3 outperforms the other systems indicating that source (1) is the dominant cause of intersymbol interference. Applying an increasing frequency weight in the stopband does not affect the normalized peak and rms distortions significantly except for terminals 1 and 2 of system T3. An enhanced stopband response (due to increasing frequency weight) diminishes the effect of source

(2) and leads to lower normalized peak and rms distortions at terminals 1 and 2 of system T3.

Now, consider the optimized design for systems T1, T2, T4, and T5. Source (1) leads to intersymbol interference in all the systems. Source (2) only affects terminals 1 and 2 of the system T1. However, source (1) is the dominant cause of intersymbol interference. This is exemplified by the fact that there is very little variation in the values of D_P and D_{RMS} for T1. The normalized peak and rms distortions are not significantly different for the cases $\gamma_3 = 0$ and $\gamma_3 = 1$.

In the case of an optimized design for T3, the intersymbol interference at terminal 0 is only due to source (1). However, both sources (1) and (2) affect terminals 1 and 2. In contrast to systems T1, T2, T4, and T5, source (2) is the major cause of intersymbol interference. This is revealed by the large difference in the normalized peak and rms distortions for terminals 1 and 2 compared with terminal 0. The initial condition used in the optimization corresponds to a filter $H(z)$ that assures exact zero crossings in the impulse response of $H(z)H(z^{-1})$. The use of this initial condition results in an optimized filter $H(z)$ that sacrifices the zero crossing property of $H(z)H(z^{-1})$. However, the resulting intersymbol interference distortion is very low at terminal 0. A crosstalk weight ($\gamma_3 = 1$) leads to more distortion at terminal 0 and less distortion at terminals 1 and 2 compared to the case $\gamma_3 = 0$. For terminals 1 and 2 of T3, the band-limitedness property is used to cancel terms in the input-output transfer function involving sidebands whose center frequencies are separated by $2\pi/3$. Source (2) contributes to intersymbol interference at these terminals. The enhanced stopband attenuation about $2\pi/3$ that results from the use of a positive crosstalk weight diminishes the effect of source (2). This results in a lower intersymbol interference distortion at terminals 1 and 2.

B. Crosstalk Suppression

The QAM systems (T1–T3) generally achieve a much lower normalized crosstalk power than the VSB transmultiplexers (T4 and T5) primarily because QAM systems exhibit many more crosstalk functions that are exactly zero. An exception arises for the optmized design with $\gamma_3 = 0$. In this case, T4 and T5 achieves a lower normalized crosstalk power than T3. However, this occurs by using a filter in T4 and T5 that has more taps and a better overall stopband response than the filter used in T3. Also, we notice that the crosstalk power is exactly zero for terminal 2 of T1. Among the QAM systems, T1 and T2 outperform T3 but at the expense of more filter coefficients (the disparity in the number of coefficients is much more for the minimax designs). For a minimax design, an increasing frequency weight diminishes the crosstalk power as anticipated. For the optimized design, a positive crosstalk weight ($\gamma_3 = 1$) results in a substantially lower crosstalk power than for a zero crosstalk weight.

C. Comparison of Minimax and Optimized Designs

The new optimized design approach is highly beneficial for systems T1, T2, T4, and T5. A much lower intersymbol interference and crosstalk distortion is achieved (even with a crosstalk weight of zero) with many fewer filter taps as compared to a minimax design. In addition, the optimized design allows for the flexibility of taking crosstalk into account by setting $\gamma_3 > 0$.

For the peformance study of system T3, the number of filter coefficients for the minimax and optimized designs are the same. Moreover, the minimax filters serve as initial conditions for the optimized design. The main advantage of the optimized design over the minimax design primarily lies in using a positive crosstalk weight to substantially diminish the crosstalk power. The optimized filters designed with a positive crosstalk weight lead to a lower crosstalk distortion (at all terminals) and a lower intersymbol interference distortion (at terminals 1 and 2) as compared to minimax filters. Without a crosstalk weight, there is no clear advantage of the optimized design. In fact, the factorable minimax approach with an increasing stopband weight and the optimized design with $\gamma_3 = 0$ lead to a similar performance. Finally, in contrast to the minimax approach, an optimized design will not give an $H(z)$ such that $H(z)H(z^{-1})$ is a Nyquist filter with exact zero crossings thereby resulting in residual intersymbol interference at terminal 0.

VIII. Filter Design for Subband Systems

Given the design methods for the transmultiplexers, we now attempt to find out whether these methods also carry over to the complementary subband systems. The complementary subband systems have an input-output relationship $\hat{X}(z) = (1/N)T(z^N)X(z)$ if the prototypes are band limited where $T(z^N)$ is defined in (8), (10), (12), and (14). In addition perfect reconstruction is accomplished by satisfying the Nyquist criterion. With practical prototypes, there is residual aliasing in that the input-output relationship becomes $\hat{X}(z) = (1/N)T(z^N)X(z)$ + terms due to aliasing. In a practical design, the stopband edge frequency is restricted as in the case of transmultiplexers. Given the input-output relationship, the minimax design approaches carry over to the subband systems. In formulating a suitable error function, the factors E_{sb}, and E_{isi} and the factor that avoids a zero solution ($(\boldsymbol{b}^T\boldsymbol{b} - 1)^2$ or $(\boldsymbol{h}^T\boldsymbol{h} - 1)^2$) are the same as for the transmultiplexers. The remaining question is about how to take aliasing into account. In general, the output of a subband system is a combination of a filtered input and filtered frequency shifted versions of the input. Even for a zero-mean white input, the filtered input is correlated with the filtered frequency shifted versions of the input. This makes it difficult to express the total power at the output due to aliasing in relation to the power of the desired component due to the input especially for an arbitrary N. However, filters can be designed by minimizing the error function having the factors E_{sb}, E_{isi} and the factor that avoids a zero solution. The filters that were previously designed with $\gamma_3 = 0$ can be used in the complementary subband systems.

The subband systems S1 to S5 were tested using the optimized low-pass prototypes. The subband systems have six bands $N = 6$. The filters have a roll-off factor $\beta = 0.52$ and are designed with $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$. For S1 and S2, the prototype has 33 taps. For S3, a 30 tap filter results. For S4 and S5, a 59 tap prototype is used. An input with a flat frequency spectrum is applied to each of the subband systems. The frequency spectrum of the output signal is flat to within about 0.03 dB (systems S1 and S2), 0.15 dB (system S3), 0.05 dB (system S4) and 0.08 dB (system S5).

A subband system with two bands which accomplishes a natural cancellation of aliasing is the focus of [5], [19]. The error functions are weighted linear combinations of two components. The first component is the stopband energy which in [5] is expressed as an integral and which in [19] is approximated as a sum over a dense grid. The second component is the mean-square distortion at the output. The actual expressions in [5] and [19] differ in that a time domain approach is used in the former and a frequency domain approach is used in the latter. The error function for our subband systems consisting of a weighted linear combination of the terms E_{sb}, E_{isi} and the term that avoids a zero solution is based on a time domain approach as in [5].

IX. Summary and Conclusions

This paper studies performance aspects of the transmultiplexers synthesized in [1]. First, the practical limitations concerning the input-output transfer function and the crosstalk functions limit the performance. The intersymbol interference is not exactly cancelled since practical filters are not band limited and the Nyquist criterion need not be exactly satisfied. Also, the crosstalk functions need not be exactly zero since practical filters are not band limited. The crosstalk terms in $T_{kl}(z^N)$ that involve spectral overlap with band-limited prototypes continue to be cancelled with practical filters. Moreover, it is shown that many crosstalk functions are zero independently of the low-pass prototype. The QAM systems exhibit many more zero crosstalk functions than the VSB transmultiplexers.

A new design procedure for an FIR low-pass prototype is formulated such that the practical degradations due to intersymbol interference and crosstalk are taken into account. The design procedure involves the optimization of an error function that is perfomed by a quasi-Newton technique. The function proposed is based on 1) achieving a low stopband energy, 2) suppressing the mean-square intersymbol interference, and 3) diminishing the crosstalk power. With an initial condition corresponding to a low-pass filter with an approximate or exact square root Nyquist frequency response, the resulting optimized filter leads to low intersymbol interference and crosstalk distortions.

The performance of the five transmultiplexers was compared for both minimax filtes and the optimized filters resulting from the new design approach. The intersymbol interference distortion is generally the lowest for system T3. This is due to the fact that for T3, a minimax design leads to filters that exactly satisfy the Nyquist criterion and the optimized design uses minimax filters as the initial condition. The normalized crosstalk power was observed to be generally lower for the QAM systems as compared to the VSB systems.

In comparing the design methods, we observed that lower intersymbol interference and crosstalk distortions with fewer filter coefficients are achieved by the optimized design when compared to minimax filters in the case of systems T1, T2, T4, and T5. Therefore, the optimized design is preferred for T1, T2, T4, and T5. In the case of T3, the advantage of the optimized design lies in using a crosstalk weight. This leads to a much lower crosstalk power than the minimax design for the same number of filter coefficients. Also, the resulting intersymbol interference distortion is very low although the Nyquist criterion is not exactly satisfied by the optimized design. When no crosstalk weight is applied, the optimized and minimax design approaches lead to a similar performance. For T3, there is a tradeoff between achieving a very low crosstalk distortion (optimized design) and exactly satisfying the Nyquist criterion (minimax design).

The complementary nature of transmultiplexers and subband systems allow for the conversion of the transmultiplexers into new subband systems. The minimax designs for the transmultiplexers carry over to the subband complements. Moreover, the optimized designs without a crosstalk weight also carry over to the subband complements.

Appendix
Number of Exact Crosstalk Cancellations for a Specific Case

Consider a center frequency ω_c that is an even multiple of $2\pi/N$ (excluding 0 and π) in system T1 with N being a multiple of 4. For a signal sent at ω_c, exact crosstalk cancellation with other signals set at odd multiples of $2\pi/N$ is achieved. Since there are $N/4$ frequencies that are odd multiples of $2\pi/N$ and two signals are sent at each of these frequencies, a total of $N/2$ crosstalk functions are exactly zero. In T1, there are a total of $(N - 4)/4$ center frequencies that are even multiples of $2\pi/N$. The crosstalk between the signal sent at ω_c and one of the signals sent at other frequencies that are even multiples of $2\pi/N$ will be zero depending on the delay factors. Furthermore, the crosstalk between the two signals sent at ω_c will be exactly zero. Now, we have an additional $(N - 4)/4$ crosstalk functions that are exactly zero bringing the total to $(3N - 4)/4$. In addition, the crosstalk between one of the signals sent at ω_c and the singles sent at 0 and π will be exactly zero depending on the delay factors. Depending on the signal sent at ω_c, the overall number of exact crosstalk cancellations is either $(3N - 4)/4$ or $(3N + 4)/4$.

Acknowledgment

The authors would like to thank Prof. M. L. Blostein of McGill University for his suggestion about formulating a design procedure that can take crosstalk into account.

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Ravi P. Ramachandran (S'87–M'90) was born in Bangalore, India, on July 12, 1963. He received the B.Eng. degree with great distinction from Concordia University, Montreal, P.Q., Canada, in 1986 and the M.Eng and Ph.D. degrees from McGill University, Montreal, P.Q., Canada, in 1986 and 1990, respectively.

From January to June 1988, he was a Visiting Postgraduate Researcher at the University of California, Santa Barbara. He is presently with the Speech Research Department at AT&T Bell Laboratories, Murray Hill, NJ. His main research interests are in speech coding, data communications, and digital signal processing.

Peter Kabal (S'70–M'75) received the B.A.Sc., M.A.Sc., and Ph.D. degrees in electrical engineering from the University of Toronto.

He holds a position as an Associate Professor in the Department of Electrical Engineering at McGill University, Montreal, P.Q., Canada, and as a Visiting Professor at INRS-Telecommunications (a research institute affiliated with the Université du Québec), Verdun, P.Q. His current research interests focus on digital signal processing as applied to speech coding, adaptive filtering, and data transmission.