Proc. IEEE SP Workshop Statistical Signal, Array Processing (Victoria, BC), pp. 300-303, Oct. 1992

A UNITARY TRANSFORMATION ALGORITHM FOR WIDEBAND ARRAY PROCESSING†

Shahrokh Valaee [1] and Peter Kabal [1,2]

[1] Dept. of Elect. Eng.
McGill University
Montreal, Quebec, H3A 2A7

[2] INRS-Télécommunications
Université du Québec
Verdun, Quebec, H3E 1H6

Abstract: A new method for broadband array processing is proposed. The method is based on unitary transformation of the signal subspaces. We apply a two-sided transformation on the cross correlation matrices of the array. It is shown that the Two-sided Correlation Transformation (TCT) generates unbiased estimates of the directions of arrival regardless of the bandwidth of the signals. The capability of the method for resolving two closely spaced sources is compared with that of the Coherent Signal-subspace Method (CSM). The resolution threshold for the new technique is smaller than the threshold for CSM.

1. Introduction

In this paper we address the processing of wideband signals. Wideband processing arises in many applications such as audio conferencing, spread spectrum transmission and passive sonar. A wideband signal has a large bandwidth relative to its center frequency. A common approach to wideband signal processing is to sample the spectrum of the signal at each sensor to form an array of narrowband signals. In the so-called incoherent signal-subspace processing method, the narrowband signals at each frequency are processed as a vector to get the estimates of the DOA's. Then the estimates at the frequency bins are combined to obtain the final result [1]. Correlated sources cannot be handled by this approach. Furthermore, the efficiency of such methods degrades for closely spaced sources and low signal-to-noise ratios (SNR).

The Coherent Signal-subspace Method (CSM) [2] is an alternative algorithm that improves the efficiency of the estimation by condensing the energy of narrowband signals in a pre-defined subspace. This process is called *focusing*. It has been shown, [2], that CSM also resolves correlated sources. A high resolution method such as MUSIC is then used to find the DOA's. Despite the fact that CSM is very effective in wideband signal detection and estimation, it suffers from the asymptotic bias of the peaks in the spatial spectrum. The bias increases with the bandwidth of the sources and the deviation of the focusing points from the true directions of arrival.

In this paper we introduce a new technique for broadband array processing. Our method is similar to CSM in the sense that a transformation of the signal subspaces is done through focusing matrices and then a high resolution spectral estimation algorithm, such as MUSIC, is applied to determine DOA. In the new method we use unitary matrices for transformation. In [3] it was shown that unitary transformations have good performance in terms of focusing loss and relative information index. In the new method, we also use unitary matrices for transformation. However, our method uses a two-sided transformation of the correlation matrix. The motivation for using the correlation matrices instead of the location vectors is based on the fact that most of the high resolution spectral estimation algorithms use an eigenstructure decomposition of the correlation matrix. We show that the new method has a lower resolution threshold SNR and a smaller peak bias. In the past, alternative methods have been introduced that reduce the peak bias [4, 5]. However, these methods are accompanied by a large increase on computational cost.

2. Problem Formulation

Consider an array of p sensors exposed to $q < p$ far-field wideband sources. The signals of the sources can be partially or fully correlated. The output of the sensors is shown by p-vector $\mathbf{x}(t)$ with the i-th component

$$x_i(t) = \sum_{l=1}^{q} s_l(t - \tau_i(\theta_l)) + n_i(t), \qquad 1 \le i \le p, \tag{1}$$

where s_l is the l-th source signal, θ_l is the angle of arrival for the l-th source and $\tau_i(\theta_l)$ is the propagation delay for the l-th source at the sensor i with respect to the reference point of the array. For a linear array with uniform spacing, $\tau_i(\theta_l) = (i-1)\frac{d}{c}\sin\theta_l$, where d is the spacing between two consecutive sensors, c is the propagation velocity and the reference point is at the first sensor. It is also assumed that the observation is corrupted by an additive noise which is represented in the model by $n_i(t)$.

In the frequency domain, after arrangement in vector form, (1) is represented by

$$\mathbf{x}(\omega) = \mathbf{A}(\omega,\theta)\mathbf{s}(\omega) + \mathbf{n}(\omega), \tag{2}$$

where $\mathbf{x}(\omega)$, $\mathbf{s}(\omega)$ and $\mathbf{n}(\omega)$ are the Fourier transforms of the observation, signal and noise vectors, respectively. The $p \times q$ matrix of steering vectors is given by $\mathbf{A}(\omega,\theta) = [\mathbf{a}(\omega,\theta_1)\ldots\mathbf{a}(\omega,\theta_q)]$. It is assumed that $\mathbf{A}(\omega,\theta)$ is of full rank. In other words the steering vectors $\mathbf{a}(\omega,\theta_i)$ are independent for every ω.

For the signal and noise we consider the following statistical structure. The signal samples are generated independently by a complex Gaussian distribution with an unknown covariance matrix $\mathbf{S}(\omega)$. The noise samples are an i.i.d. sequence of Gaussian random vectors with unknown covariance matrix $\sigma^2\mathbf{I}$ and are independent of the signal samples. From (2) and the assumptions on the signal and noise samples, the covariance matrix of the observation vector at the frequency ω is represented by

$$\mathbf{R}(\omega) = \mathbf{A}(\omega,\theta)\mathbf{S}(\omega)\mathbf{A}^H(\omega,\theta) + \sigma^2\mathbf{I}, \tag{3}$$

† This work was partly supported by a grant from Natural Sciences and Engineering Research Council of Canada (NSERC).

1992/7/7

where the superscript H represents the Hermitian transpose. An estimate of the covariance matrix $\mathbf{R}(\omega)$ at a given frequency, say ω_j, is done by time averaging,

$$\hat{\mathbf{R}}(\omega_j) = \frac{1}{N}\sum_{k=1}^{N}\mathbf{x}_k(\omega_j)\mathbf{x}_k^H(\omega_j), \tag{4}$$

where $\mathbf{x}_k(\omega_j)$ is the Fourier transform of the observation vector at the frequency ω_j and N is the number of observations.

The Fourier transforms operate on the observed data. In practice, a sufficiently long duration of sensor outputs is observed. Then the sampled data are divided into N batches. Each batch contains J samples. In each batch, an FFT algorithm is used to transform the data onto the frequency domain. Thus, N sets of transformed data are available where each set contains J frequency samples of the spectrum of the observation vector. We represent these samples by $\mathbf{x}_j, j = 1, \ldots, J$. In the sequel we suppress the frequency variable. Then $\mathbf{R}_j$ represents $\mathbf{R}(\omega_j)$, $\mathbf{S}_j$ represents $\mathbf{S}(\omega_j)$, $\mathbf{A}_j$ represents $\mathbf{A}(\omega_j, \theta)$ and so on.

In CSM, the observation vectors at different frequency bins are transformed into a given subspace (focusing). In particular, a new observation vector is formed from by

$$\mathbf{y}_j = \mathbf{T}_j\mathbf{x}_j, \qquad j = 1, \ldots, J, \tag{5}$$

where $\mathbf{T}_j$ satisfies

$$\mathbf{T}_j\mathbf{A}_j = \mathbf{A}_0, \qquad j = 1, \ldots, J, \tag{6}$$

where $\mathbf{A}_0$ is the focusing location matrix.

In a modified version of CSM [3], the transformation matrices are the solutions of the minimization problem

$$\min_{\mathbf{T}_j} \|\mathbf{A}_0 - \mathbf{T}_j\mathbf{A}_j\| \qquad j = 1, \ldots, J \tag{7}$$
$$\text{s.t.} \qquad \mathbf{T}_j^H\mathbf{T}_j = \mathbf{I},$$

where $\|.\|$ is the Frobenius matrix norm. Minimization of (7) gives

$$\mathbf{T}_j = \mathbf{V}_j\mathbf{W}_j^H, \tag{8}$$

where $\mathbf{V}_j$ and $\mathbf{W}_j$ are the left and right singular vectors of $\mathbf{A}_0\mathbf{A}_j^H$. The transformed observation vectors are used to construct the sample correlation matrices. An average of these aligned correlation matrices gives a universal sample correlation matrix which can be used for detection and estimation.

3. Two-sided Correlation Transformation method

In this section we introduce a new method for the processing of wideband signals using an array of sensors. The method is based on transformation of the signal-subspaces into a subspace called the *focusing subspace*. Like the CSM algorithm, the transformation of the subspaces is done using focusing matrices. The focusing matrices are unitary and are chosen to minimize the distance between the focusing subspace and the signal-subspaces at the frequency bins. However, unlike CSM, in the new method, the transformation of the subspaces is done through two-sided transformations applied to the correlation matrices.

3.1 The TCT criterion

Our method is based on transformation of the source-only correlation matrix of the array defined by

$$\mathbf{P}_j = \mathbf{A}_j\mathbf{S}_j\mathbf{A}_j^H, \qquad j = 0, 1, \ldots, J, \tag{9}$$

where $\mathbf{P}_0$ corresponds to the focusing frequency and the rest of $\mathbf{P}_j$'s are constructed from the data at the frequency bins. In particular we minimize

$$\min_{\mathbf{U}_j} \|\mathbf{P}_0 - \mathbf{U}_j\mathbf{P}_j\mathbf{U}_j^H\| \tag{10}$$
$$\text{s.t.} \qquad \mathbf{U}_j^H\mathbf{U}_j = \mathbf{I}, \qquad j = 1, \ldots, J.$$

The solution of (10) is obtained as

$$\mathbf{U}_j = \mathbf{X}_0\mathbf{X}_j^H \tag{11}$$

where $\mathbf{X}_0$ and $\mathbf{X}_j$ are the matrices containing the eigenvectors of $\mathbf{P}_0$ and $\mathbf{P}_j$, respectively.

In computing $\mathbf{U}_j$ the matrices $\mathbf{A}_j$ and $\mathbf{S}_j$ are assumed to be known. In practice a low resolution beamformer is applied to estimate the number and the directions of arrival of the sources. Closely separated and correlated sources may not be resolved in this step. Using the results of this pre-processing, an estimate of the location matrices is obtained. To find the source correlation matrix, first the noise power at the j-th frequency bin is estimated using

$$\hat{\eta}_j^2 = \frac{1}{p-\hat{q}}\sum_{i=\hat{q}+1}^{p}\lambda_i(\hat{\mathbf{R}}_j) \tag{12}$$

where $\hat{q}$ is the number of sources in the pre-estimation. The source correlation matrix at the frequency bin j is found from

$$\hat{\mathbf{S}}_j = (\mathbf{A}_j^H\mathbf{A}_j)^{-1}\mathbf{A}_j^H[\hat{\mathbf{R}}_j - \hat{\eta}_j^2\mathbf{I}]\mathbf{A}_j(\mathbf{A}_j^H\mathbf{A}_j)^{-1}. \tag{13}$$

Using the estimates of $\mathbf{A}_j$ and $\mathbf{S}_j$, the correlation matrices $\mathbf{P}_j$, $j = 1, \ldots, J$, are formed.

3.2 Bias of the estimation

It has been shown that the CSM algorithm generates an estimate of the directions of arrival that is asymptotically biased [3]. The bias increases with the bandwidth of the processing and the deviation of the estimated angles from the true DOA. To see this recall

$$\mathbf{R}_{\mathrm{CSM}} = \sum_{j=1}^{J}\mathbf{T}_j\mathbf{R}_j\mathbf{T}_j^H. \tag{14}$$

To study the mechanism that generates the asymptotic bias, a noiseless environment is considered. In such a case the correlation matrix is

$$\begin{aligned}\mathbf{R}_{\mathrm{CSM}} &= \sum_{j=1}^{J}\mathbf{T}_j\mathbf{P}_j\mathbf{T}_j^H \\ &= \sum_{j=1}^{J}\mathbf{V}_j\mathbf{W}_j^H\mathbf{A}_j\mathbf{S}_j\mathbf{A}_j^H\mathbf{W}_j\mathbf{V}_j^H\end{aligned} \tag{15}$$

where we have used equation (8). As can be seen, the focussed correlation matrix is a function of the pre-estimate of the DOA's and the bandwidth of sources. In general $\mathbf{R}_{\mathrm{CSM}}$ has nonzero eigenvalues in the noise subspace. In other words the received power is distributed in a p-dimensional space. In TCT, the transformed subspaces for different frequencies are aligned and hence the eigenvalues at the noise subspace are zero. Recall

$$\begin{aligned}\mathbf{R}_{\mathrm{TCT}} &= \sum_{j=1}^{J}\mathbf{U}_j\mathbf{P}_j\mathbf{U}_j^H \\ &= \sum_{j=1}^{J}\mathbf{X}_0\mathbf{X}_j^H\mathbf{P}_j\mathbf{X}_j\mathbf{X}_0^H\end{aligned} \tag{16}$$

where we have used (11). Suppose the diagonal matrix of the eigenvalues of $\mathbf{P}_j, j = 1, \ldots, J$, are shown by $\mathbf{\Gamma}_j, j = 1, \ldots, J$, respectively. Then for any pre-estimate of the DOA's the inner product in (16) can be simplified to give

$$\begin{aligned}\mathbf{R}_{\mathrm{TCT}} &= \sum_{j=1}^{J}\mathbf{X}_0\mathbf{\Gamma}_j\mathbf{X}_0^H \\ &= \mathbf{X}_0(\sum_{j=1}^{J}\mathbf{\Gamma}_j)\mathbf{X}_0^H \\ &= \mathbf{X}_0\mathbf{\Gamma}_0\mathbf{X}_0^H\end{aligned} \tag{17}$$

Since the dimension of Γ_j is equal to q, the focussed correlation matrix $\mathbf{R}_{TCT}$ will have eigenvalues in a q-dimensional subspace.

Computer simulation indicates that the CSM algorithm extends the signal power into the noise subspace. As an example, we considered a configuration with 16 sensors exposed to 4 signals in a noiseless environment. We applied both the CSM and TCT algorithms to obtain the focusing matrices. The eigenvalues of the corresponding matrices are tabulated in Table 1. It is seen that $\mathbf{R}_{CSM}$ has nonzero eigenvalues in the noise subspace. This nonuniform extension of the signal into the noise subspace acts as a non-white noise which creates biased estimates. It is important to note that the trace of $\mathbf{R}_{CSM}$ is equal to the trace of $\mathbf{R}_{TCT}$. In other words, the sum of the eigenvalues in Table 1 is identical for each matrix. This suggests that in TCT the energy of the signals is not lost. The TCT method condenses the total received power in a q-dimensional subspace to improve the performance.

To further discuss the asymptotic bias we consider the special case of perfect focusing. In perfect focusing the transformed correlation matrices, $\mathbf{U}_j\mathbf{A}_j$, are superimposed on $\mathbf{A}_0$. In such a case the focusing correlation matrix is an average of the correlation matrices at the frequency bins. In other words the following equality is satisfied,

$$\mathbf{A}_0\mathbf{S}_0\mathbf{A}_0^H = \frac{1}{J}\sum_{j=1}^{J}\mathbf{U}_j\mathbf{A}_j\mathbf{S}_j\mathbf{A}_j^H\mathbf{U}_j^H. \tag{18}$$

In practice perfect focusing is not possible. The transformed matrices are clustered around $\mathbf{A}_0$. However, as far as the equality (18) is satisfied for the true DOA, unbiased estimation is possible. It is important to notice that (18) is a general condition for unbiased estimation regardless of the transformation matrices. This is also in agreement with the work of Swingler and Krolik, [6]. Consider a single source scenario. Swingler and Krolik used the diagonal unitary transformation matrices,

$$\mathbf{T}_j = \mathrm{diag}[1, e^{-j(\omega_0-\omega_j)\tau_0}, \ldots, e^{-j(p-1)(\omega_0-\omega_j)\tau_0}], \tag{19}$$

where τ_0 is the pre-estimated propagation delay. Using this matrix, the equation (18) can be shown as

$$\mathbf{s}_0\mathbf{a}_0\mathbf{a}_0^H = \frac{1}{J}\sum_{j=1}^{J}\mathbf{s}_j\mathbf{b}_j\mathbf{b}_j^H, \tag{20}$$

where the transformed column vector $\mathbf{b}_j$ is given by

$$\mathbf{b}_j = [1 \; e^{-j\omega_0\tau_0+j\omega_j(\tau_0-\tau_1)} \; \ldots \; e^{-j(p-1)\omega_0\tau_0+j(p-1)\omega_j(\tau_0-\tau_1)}]^T \tag{21}$$

where τ_1 is the true DOA and the superscript T stands for transpose. The direction of arrival is estimated by equating

$$\begin{aligned} \mathbf{s}_0\, e^{-j\omega_0\hat{\tau}_1} &= \sum_{j=1}^{J}\mathbf{s}_j e^{-j\omega_0\tau_0+j\omega_j(\tau_0-\tau_1)} \\ &= e^{-j\omega_0\tau_0}\sum_{j=1}^{J}\mathbf{s}_j e^{j\omega_j(\tau_0-\tau_1)}, \end{aligned} \tag{22}$$

where $\hat{\tau}_1$ is the estimate of τ_1. Applying the technique of Swingler and Krolik to our case, the equation (22) can be simplified to get

$$\begin{aligned} \mathbf{s}_0\, e^{-j\omega_0\hat{\tau}_1} &= e^{-j\omega_0\tau_0}\sum_{j=1}^{J}\mathbf{s}_j[1 + j\omega_j(\tau_0-\tau_1)] \\ &= \mathbf{s}_0\, e^{-j\omega_0\tau_0}\,[1 + j(\tau_0-\tau_1)\frac{1}{\mathbf{s}_0}\sum_{j=1}^{J}\mathbf{s}_j\omega_j] \\ &= \mathbf{s}_0\, e^{-j\omega_0\tau_0}\,[1 + j(\tau_0-\tau_1)\bar{\omega}] \\ &= \mathbf{s}_0\, e^{-j\omega_0\tau_0}\, e^{j(\tau_0-\tau_1)\bar{\omega}} \end{aligned}$$

eigenvalues	$\mathbf{R}_{CSM}$	$\mathbf{R}_{TCT}$
λ_1	900.2749	918.7666
λ_2	806.7057	808.0986
λ_3	241.6814	244.1410
λ_4	155.3949	140.9939
λ_5	4.3967	0.0000
λ_6	1.3504	0.0000
λ_7	0.4041	0.0000
λ_8	0.3457	0.0000
λ_9	0.3113	0.0000
λ_{10}	0.2721	0.0000
λ_{11}	0.2201	0.0000
λ_{12}	0.1929	0.0000
λ_{13}	0.1539	0.0000
λ_{14}	0.1243	0.0000
λ_{15}	0.0949	0.0000
λ_{16}	0.0767	0.0000

Table 1 The eigenvalues of the correlation matrices $\mathbf{R}_{CSM}$ and $\mathbf{R}_{TCT}$ for a configuration of 4 sources arriving at 16 sensors in a noiseless environment.

where $\bar{\omega}$ is the centroid frequency. Equating the exponents gives

$$\omega_0\hat{\tau}_1 = \omega_0\tau_0 - (\tau_0 - \tau_1)\bar{\omega},$$

which is the same result as Swingler and Krolik, [6].

The criterion (18) is a general condition for unbiased estimation. We have shown in [7] that the TCT algorithm forms a very good approximation of (18). Thus the bias is minimized in the TCT method.

4. Simulation results and performance comparison

In this section we investigate a configuration with two equipower uncorrelated sources impinging from the angles 11 and 13 degrees off broadside. The signal-to-noise ratio is 10 dB. A linear array of 8 sensors is used. The spacing between adjacent sensors is equal to half the wavelength at the center frequency. Sources are sampled with 33 frequency bins in the frequency domain. We imported the actual correlation matrix to the CSM and TCT algorithms and used the high resolution MUSIC algorithm for DOA estimation. A preliminary beamformer output gives a peak at 12 degrees. Two extra estimated angles are added at 9 and 15 degrees (same method as [3]). The results of the estimation for 40 and 100 percent bandwidth and for different focusing frequencies are given in Table 2. The bias columns in this table are the Euclidean norm of the bias vectors. The TCT algorithm does not have bias regardless of the bandwidth.

For this example, we examine the resolution capability of the two algorithms. We increase the number of sensors to 16 and consider a 40 percent relative bandwidth. It is assumed that only 20 batches of data are available. Again at each batch a 64-point DTF is applied to obtain 33 frequency samples in the frequency domain. The resolution criterion is defined as the difference between the average of the spatial spectrum at the peak points in the MUSIC algorithm and the spatial spectrum at the valley. It is measured in a dB scale for different SNR's. The results are given in Fig. 1. As it is seen the performance of TCT is about 6 dB better than CSM. The spatial spectrum of the two methods are overlapped in Fig. 2 for further comparison.

BW = 0.4	CSM			TCT		
f_0	11	13	bias	11	13	bias
0.8	11.89	12.13	1.24	11.00	13.00	0.00
0.9	11.25	12.75	0.35	11.00	13.00	0.00
1.0	11.01	12.99	0.01	11.00	13.00	0.00
1.1	10.88	13.12	0.17	11.00	13.00	0.00
1.2	10.78	13.22	0.31	11.00	13.00	0.00
BW = 1.0	**CSM**			**TCT**		
f_0	11	13	bias	11	13	bias
0.8	-	12.01	-	11.01	12.99	0.01
0.9	11.42	12.58	0.59	11.00	13.00	0.00
1.0	11.12	12.88	0.17	11.00	13.00	0.00
1.1	10.95	13.05	0.07	11.00	13.00	0.00
1.2	10.84	13.16	0.23	11.00	13.00	0.00

Table 2 The estimation results for the first example.

5. Conclusion

In this paper, we have introduced a new method for broadband array processing. Our method is based on two-sided unitary transformation of the correlation matrices. The algorithm is called the Two-sided Correlation Transformation (TCT). The motivation for this work was to reduce the error of the subspace fitting and to remove the bias of estimation that occurs with the Coherent Signal-subspace Method (CSM). The bias of estimation in CSM depends on the focusing points and the bandwidth of processing. We have also shown that using the TCT algorithm, an unbiased estimation of the directions of arrival is possible. The estimation is unbiased regardless of the processing bandwidth. Comparison of the resolution threshold for the two methods show that the TCT algorithm operates with smaller threshold signal to noise ratio.

REFERENCES:

[1] M. Wax, T. Shan and T. Kailath, "Spatio-temporal spectral analysis by eigenstructure methods," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 817-827, August 1984.

[2] H. Wang and M. Kaveh, "Coherent signal-subspace processing for the detection and estimation of angles of arrival of multiple wide-band sources," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 823-831, August 1985.

[3] H. Hung and M. Kaveh, "Focusing matrices for coherent signal-subspace processing," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-36, pp. 1272-1281, August 1988.

[4] K. M. Buckley and L. J. Griffiths, "Broad-band signal-subspace spatial-spectrum (BASS-ALE) estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-36, pp. 953-964, July 1988.

[5] J. Krolik and D. N. Swingler, "Multiple broad-band source location using steered covariance matrices," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, pp. 1481-1494, Oct. 1989.

[6] D. N. Swingler and J. Krolik, "Source location bias in the coherently focused high-resolution broad-band beamformer," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-37, pp. 143-145, Jan. 1989.

[7] S. Valaee and P. Kabal, "Wideband array processing using a two-sided correlation transformation," submitted to *IEEE Trans. Signal Processing*.

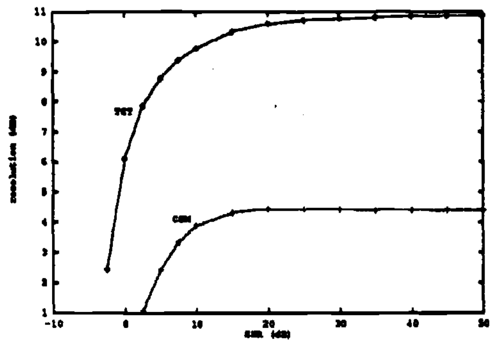

Fig. 1 The resolution comparison between the two algorithms, TCT and CSM.

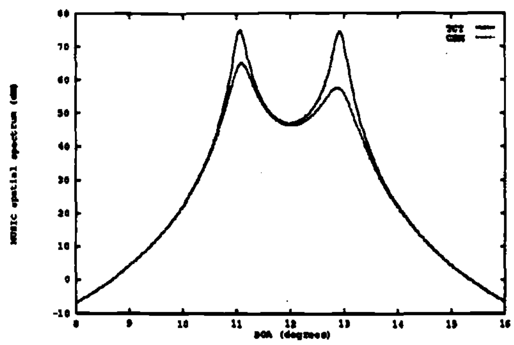

Fig. 2 The MUSIC spatial spectrum for the two methods, TCT and CSM.