

Block-based Eigensystem of the $1 \pm D$ and $1 - D^2$ Partial Response Channels

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Abstract—We find analytical expressions for the block-based input and output eigenvectors and eigenvalues of the systems with responses $1 \pm D$ and $1 - D^2$. The input eigenvectors form an orthonormal basis which is the optimum modulator for a channel with that transfer function. The output eigenvectors form an orthonormal basis with the same spectral nulls as the corresponding system. This basis can be used to produce line codes with spectral nulls. The eigenvectors are sinusoids.

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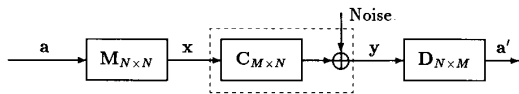


Fig. 1. Block diagram of the transmission system.

This reduces the computational complexity by allowing for fast transform algorithms to perform the modulation for a block of data.

Index Terms—Block-based eigensystem, partial response channels.

I. INTRODUCTION

We consider the problem of finding the input and the output eigenvectors and the eigenvalues of the block-based $1 \pm D$ and $1 - D^2$ systems. For an $M \times N$ -dimensional matrix C , the input eigenvectors \mathbf{m}_k , $k \in [0, N-1]$, are the eigenvectors of $C^t C$ with the eigenvalues ϕ_k^2 . The output eigenvectors $\hat{\mathbf{m}}_k$, $k \in [0, M-1]$, are the eigenvectors of CC^t . Assuming $M > N$, CC^t has N nonzero eigenvalues equal to the same ϕ_k^2 's and $M_0 = M - N$ eigenvalues equal to zero. We have

$$\begin{aligned} C\mathbf{m}_k &= \phi_k \hat{\mathbf{m}}_k, \\ C^t \hat{\mathbf{m}}_k &= \phi_k \mathbf{m}_k. \end{aligned} \quad (1)$$

Since $C^t C$ and CC^t are both symmetric, the input and the output eigenvectors form an orthonormal basis denoted by \mathbf{M} and $\hat{\mathbf{M}}$, respectively. In the following, we discuss two applications for the eigenvectors of the matrix C .

The first application involves signaling over a channel with the transfer matrix C . The block diagram of the transmission system under consideration is shown in Fig. 1. We use a discrete-time model and block-based processing. The block length is equal to N . Each block invokes $M = N + M_0$ channel uses, where M_0 is the memory length of the channel. M_0 zeros are transmitted between successive blocks; as a result, each block starts with zero initial conditions. Modulator matrix \mathbf{M} is the basis for the given constellation at the channel input. The additive noise is white Gaussian with zero mean and unit variance. The demodulator matrix \mathbf{D} is selected such that $\mathbf{DCM} = \mathbf{I}$, where \mathbf{I} is the $N \times N$ identity matrix. This results in a unity gain N -dimensional channel with additive Gaussian noise whose autocorrelation matrix is \mathbf{DD}^t . We assume that the decisions along different dimensions of the channel are made independently. In this case, the effective noise along the k th dimension, $k = 0, \dots, N-1$, is a Gaussian random process with power σ_k^2 , where σ_k^2 is the k th diagonal element of the matrix \mathbf{DD}^t .

It can be shown that the input eigenvectors of C are the optimum modulating basis at the channel input [1]. This basis minimizes the product of the noise powers along different dimensions. In this case, for a given total rate and given minimum distance-to-noise ratio at the demodulator output, the volume of the signal space and, consequently, the required energy at the channel input, are minimized.

The second application involves line coding. The output eigenvectors of a system C form an orthonormal basis with the same spectral nulls as the system. This basis can be used to produce line codes with spectral nulls. Fig. 2(a) shows the block diagram of such a line coder. Considering (1), multiplication (modulation) by $\hat{\mathbf{M}}$ can be achieved using the system shown in Fig. 2(b), where Φ^{-1} is a diagonal matrix with the diagonal elements $1/\phi_k$. As we will see later, for the systems under consideration, modulation by \mathbf{M} can be achieved using an even discrete sine transform algorithm. This reduces the computational complexity of a realization based on Fig. 2(b).

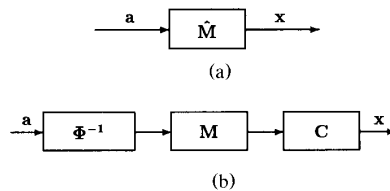


Fig. 2. (a) Block diagram of the line coder. (b) An equivalent form for (a).

The $1 - D$, $1 + D$, and $1 - D^2$ systems have special importance in partial response signaling [2]. The $1 - D$ system has a spectral null at zero frequency, $1 + D$ has a spectral null at the Nyquist frequency, and $1 - D^2$ has spectral nulls at both zero and the Nyquist frequencies. Let the energy per channel use be normalized to 1. The $1 \pm D$ systems have an $(N+1) \times N$ -dimensional transfer matrix with i th column, $i = 0, \dots, N-1$, equal to $[(0)^i, \sqrt{2}/2, \pm \sqrt{2}/2, (0)^{N-1-i}]^t$. The $1 - D^2$ system has an $(N+2) \times N$ -dimensional transfer matrix with i th column, $i = 0, \dots, N-1$, equal to $[(0)^i, \sqrt{2}/2, 0, -\sqrt{2}/2, (0)^{N-1-i}]^t$.

II. EIGENSYSTEM OF THE $1 \pm D$ AND $1 - D^2$ SYSTEMS

The input eigenvectors of the $1 - D$ system are equal to

$$m_k(n) = \sqrt{\frac{2}{N+1}} \sin \frac{\pi(k+1)(n+1)}{N+1}, \quad k, n = 0, \dots, N-1. \quad (2)$$

The corresponding eigenvalues are given by

$$\phi_k^2 = 1 - \cos \frac{\pi(k+1)}{N+1}. \quad (3)$$

This can be verified by considering (2) as a periodic function with period $N+1$. This function is zero at $n = i(N+1) - 1, \forall i$. This means that the signal itself provides zero initial conditions for the N -dimensional blocks. Consequently, the response of the system in each block is equal to its steady state. Note that in steady state, a sinusoid is the input eigenfunction of any linear system.

To give a formal proof, we consider $C^t C$ as the transfer matrix of a linear time-invariant system with the transfer function $H(D) = 0.5(1 - D)(1 - D^{-1})$. This is the transform of $c(n) * c(-n)$, where $c(n) = \{1/\sqrt{2}, -1/\sqrt{2}\}$ is the impulse response of the $1 - D$ system (power is normalized to 1) and $*$ denotes the convolution. To have consistency with the block-based processing, we apply a causal input and truncate the output of positive time. In this case, if $m(n)$ is an input eigenvector and $M(D)$ is its transform, we have

$$H(D)[M(D) - m(0)] + m(0)(1 - 0.5D) = \phi^2 M(D). \quad (4)$$

Calculating (4) at time zero results in

$$\phi^2 = 1 - 0.5 \frac{m(1)}{m(0)}. \quad (5)$$

Substituting $H(D) = 0.5(1 - D)(1 - D^{-1})$ and (5) in (4) results in

$$M(D) = \frac{m(0)}{1 - \frac{m(1)}{m(0)}D + D^2}. \quad (6)$$

Eqs. (6) and (5) can be satisfied by the eigenvectors and eigenvalues given in (2) and (3). Using (2) in (1), the output eigenvec-

After this paper was accepted for publication, we became aware of the work of Honig *et al.* [4], which presents a result similar to (2).

tors of the $1 - D$ system are found as

$$\hat{m}_k(n) = \sqrt{\frac{2}{N+1}} \cos \frac{\pi(k+1)(2n+1)}{2(N+1)},$$

$$n = 0, \dots, N, \quad k = 0, \dots, N-1. \quad (7)$$

The input and output eigenvectors of the $1 + D$ system are obtained by multiplying (2) and (7) with $(-1)^n$. The eigenvalues of the $1 + D$ system are the same as the $1 - D$ system given in (3).

An N -dimensional $1 - D^2$ channel, N even, can be considered as two time-multiplexed $N/2$ -dimensional $1 - D$ channels. Consequently, the eigenvalues are in pair equal to

$$\phi_k^2 = 1 - \cos \frac{\pi(k+1)}{(N/2)+1}, \quad k = 0, \dots, (N/2) - 1. \quad (8)$$

The two eigenvectors corresponding to a pair of eigenvalues are of the general form $\alpha_1 m_k(2n) + \alpha_2 m_k(2n+1)$, where $\alpha_1^2 + \alpha_2^2 = 1$ and $m_k(n)$ is the eigenvector of the $1 - D$ channel given in (2).

The product of the nonzero eigenvalues of C is equal to

$$\prod_{k=0}^{N-1} \phi_k^2 = |C^t C|, \quad (9)$$

where $|C^t C|$ is the determinant of $C^t C$. This product is an important parameter of the systems based on C . For example, in the transmission system shown in Fig. (1), the volume of the Voronoi region around each constellation point at the channel input is proportional to $(\prod_k \phi_k)^{-1}$ and the required energy is proportional to $(\prod_k \phi_k)^{-2/N}$.

For the $1 \pm D$ channels, assuming $|C^t C| = 2^{-N} \times A_N$ and expanding the determinant, we obtain $A_N = A_{N-1} + 1$. Solving this recursive equation with the initial value $A_1 = 2$ results in $A_N = N + 1$. Consequently, for the $1 \pm D$ channels, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times (N + 1). \quad (10)$$

For the $1 - D^2$ channel, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times [(N/2) + 1]^2. \quad (11)$$

For all three channels, modulation with the input eigenvectors can be performed by using the even discrete sine transform. For modulation with the output eigenvectors, we can use the block diagram shown in Fig. (2). Using (7), modulation with the output eigenvectors can also be achieved using an $N + 1$ -point even discrete cosine transform. In this case, samples of the modulating vector are shifted by one sample and filled with zero. Ref. [3] shows how both the discrete sine transform and the discrete cosine transform can be efficiently calculated.

III. SUMMARY

The input and output eigenvectors and the eigenvalues of the $1 \pm D$ and $1 - D^2$ systems are calculated. The product of the nonzero eigenvalues are found in closed form. In all cases, the multiplication by the input or output eigenvectors can be achieved by using fast transform algorithms.

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