Alternative Proofs for "On Unique Localization of Constrained-Signal Sources"

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Abstract— **In this correspondence, we provide simple proofs for the theorems in "On Unique Localization of Constrained-Signal Sources" by M. Wax. The approach is based on the topological dimension of a set. All the possible observation matrices form the legitimate set. The observation matrices that can have nonunique solutions form the ambiguity set. The components of the legitimate and ambiguity set are random matrices. We find the conditions under which the dimensionality of the ambiguity set is smaller than the dimensionality of the legitimate set. In such a case, the probability of the ambiguity set is zero and with probability one, a unique solution can be found for the localization problem.**

I. Introduction

In [1], Wax showed that when the signals are constrained to certain loci in the complex plane, the number of uniquely resolvable signals is larger than the number of sensors. He considered two classes of constraints, and for each class, he found the conditions under which almost surely a unique solution for the localization problem can be found. Here, we give alternative and simple proofs to the theorems in [1]. The same approach is also used to find a sufficient condition for unique localization of unconstrained signals [2].

Assume that an array of p sensors receives the wavefield of q sources in N consecutive snapshots. The sources can be partially or fully correlated. In a noise-free environment, the $p \times N$ observation matrix $\mathbf{X}$ can be shown as

$$\mathbf{X} = \mathbf{A}(\theta)\mathbf{S} \tag{1}$$

where $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_q)]$ is the $p \times q$ location matrix, and $\mathbf{S}$ is the $q \times N$ source signal matrix. Uniqueness can be stated as "Given the observation matrix $\mathbf{X}$, a unique vector θ can be found that satisfies (1) for any source signal matrix $\mathbf{S}$." The following constraints have been imposed on the array location vectors:

- A1) The array manifold $\{\mathbf{a}(\theta) : \theta \in \Omega\}$, where Ω is the field of view, is known.
- A2) Any p distinct steering vectors from the array manifold are linearly independent.

The source signal matrix can be expressed as $\mathbf{S} = [\mathbf{S}_1, \mathbf{S}_2]$, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are $q \times \eta$ and $q \times (N - \eta)$ matrices with η being the rank of $\mathbf{S}$. Similar to [1], we can show that the uniqueness problem needs only to be considered for the observation matrix $\mathbf{X}_1 = \mathbf{A}(\theta)\mathbf{S}_1$. In the sequel, we assume that the dimensionality of $\mathbf{X}$ and $\mathbf{S}$ are equal to $p \times \eta$ and $q \times \eta$, respectively. In this paper, two kinds of constraints are considered for the signals. These constraints will be discussed separately in the following sections. In Section III, we provide a simpler proof for the sufficient condition for unique localization of unconstrained signals [2].

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II. Real Constraints

Assume that the signals are constrained to the loci [1]

$$f_k(s_k(t_i), \lambda_k^{(\mu)}) = 0, \quad k = 1, \ldots, q \tag{2}$$

where

- $s_k(t_i)$ signal of the kth source at time instant t_i
- f_k smooth map from the complex plane to the real line
- $\lambda_k^{(\mu)}$ vector of parameters with μ real unknowns.

Theorem 1: Let θ be fixed, and let $\mathbf{S}$ be a $q \times \eta$ random matrix drawn from the set of all rank-η matrices having elements constrained by (2) and jointly distributed according to some absolutely continuous distribution. A general array satisfying A1) and A2) can then almost always, with the exception of a set of signals of measure zero, uniquely localize q sources, provided that

$$q < \frac{\eta}{\eta + \mu + 1} 2p. \tag{3}$$

Proof: The location matrix $\mathbf{A}(\theta)$ is uniquely defined by the parameter vector θ. Thus, the number of free parameters to describe $\mathbf{A}(\theta)$ is q. The source signal matrix $\mathbf{S}$, is a $q \times \eta$-dimensional complex matrix. In general, a $q \times \eta$ complex matrix is described by $2q\eta$ real parameters. Since the signals are constrained by (2), the number of free parameters is reduced by the number of constraints $q\eta$. However, each constraint adds μ new unknowns. Thus, the total number of free parameters of $\mathbf{S}$ is equal to $(2q\eta - q\eta + q\mu)$.

The $p \times \eta$ matrices $\mathbf{X}$, which satisfy (1) and (2), form the "legitimate set," which is denoted by $\mathcal{G}$. The legitimate set is a subset of the subspace of $p \times \eta$-dimensional complex matrices. Since θ and $\mathbf{S}$ can be chosen independently in (1), the dimensionality of the legitimate set is equal to $(q + q\eta + q\mu)$.

A nonunique solution appears for the localization problem if

$$\mathbf{X} = \mathbf{A}(\theta)\mathbf{S} = \mathbf{A}(\theta')\mathbf{S}' \tag{4}$$

for different θ and θ'. Let us define

$$\mathbf{C} = \mathbf{A}(\theta)\mathbf{S} - \mathbf{A}(\theta')\mathbf{S}'. \tag{5}$$

Note that $\mathbf{C}$ is a $p \times \eta$ complex matrix and can be uniquely described by $2(q + q\eta + q\mu)$ real parameters.

The "ambiguity set" is defined by the $p \times \eta$ complex matrices that satisfy

$$\mathcal{D} = \{\mathbf{A}(\theta)\mathbf{S} : \mathbf{A}(\theta)\mathbf{S} - \mathbf{A}(\theta')\mathbf{S}' = 0, \text{ for } \theta, \theta' \in \Omega, \text{ and } \mathbf{S}, \mathbf{S}' \text{ satisfying (2)}\}. \tag{6}$$

The constraints of the ambiguity set can be shown as

$$\mathbf{C} = \mathbf{0}. \tag{7}$$

In such a case, the number of free parameters will be reduced by the number of constraints in (7). Since $\mathbf{C}$ is a complex $p \times \eta$ matrix, the number of free parameters is reduced by $2p\eta$. Thus, the number of free parameters to describe (4) is equal to $2(q + q\eta + q\mu) - 2p\eta$. For uniqueness, this number, which is the dimensionality of the ambiguity set, should be strictly smaller than the dimensionality of the legitimate set. Hence

$$2(q + q\eta + q\mu) - 2p\eta < q + q\eta + q\mu \tag{8}$$

which is the same result as

$$q < \frac{\eta}{\eta + \mu + 1} 2p. \tag{9}$$

In such a case, the ambiguity set is a proper subset of the legitimate set. Since the ambiguity set is a set of random matrices, its conditional probability given the legitimate set is zero. Hence, if the number of signals satisfies (9) everywhere except in a set with probability zero, a unique solution for the localization problem exists.

III. Complex Constraints

In this case, the signals are constrained to the loci [1]

$$g_k(s_k(t_i), \lambda_k^{(\mu)}) = 0, \quad k = 1, \ldots, q \tag{10}$$

where g_k is a smooth map from the complex plane to itself, and $\lambda_k^{(\mu)}$ is a vector of parameters with μ real unknowns.

Theorem 2: Let θ be fixed, and let $\mathbf{S}$ be a $q \times \eta$ random matrix drawn from the set of all rank-η matrices having elements constrained by (10) and jointly distributed according to some absolutely continuous distribution. A general array satisfying A1) and A2) can then almost always, with the exception of a set of signals of measure zero, uniquely localizes q sources, provided that

$$q > \frac{\eta}{\mu + 1} 2p. \tag{11}$$

Proof: With a similar argument, the number of free parameters of $\mathbf{X}$ is $q + q\mu$, which is the dimensionality of the legitimate set. Here, the constraint functions are complex, and therefore, the number of free parameters of the ambiguity set is $2(q + q\mu) - 2p\eta$. This number should be smaller than the total number of free parameters of the legitimate set

$$2(q + q\mu) - 2p\eta < q + q\mu. \tag{12}$$

Thus, the number of uniquely localizable sources is bounded by

$$q < \frac{\eta}{\mu + 1} 2p. \tag{13}$$

If the number of signals satisfies (13), a unique solution for the localization problem exists with probability one.

IV. Unconstrained Signals

The method of the preceding sections can also be used to find the maximum number of uniquely localizable unconstrained signals. This problem has been discussed in [2]. There, it has been proved that with probability one a unique solution for the localization problem can be found if

$$q < \frac{2\eta}{2\eta + 1} p. \tag{14}$$

In the present section, we give a simpler proof for this condition.

Similar to previous sections, we define the legitimate set by the set of all observation matrices that satisfy (1). Here, no constraints are imposed on the signal matrix $\mathbf{S}$. The dimensionality of the legitimate set is equal to the free real parameters that are used to describe $\mathbf{X}$. The signal matrix $\mathbf{S}$ is a $q \times \eta$ dimensional and can be uniquely described by $2q\eta$ real parameters. Since θ and $\mathbf{S}$ can be chosen independently in (1), the dimensionality of the legitimate set is equal to $q + 2q\eta$.

The ambiguity set is defined by

$$\mathcal{D} = \{\mathbf{A}(\theta)\mathbf{S} : \mathbf{A}(\theta)\mathbf{S} - \mathbf{A}(\theta')\mathbf{S}' = 0, \quad \text{for } \theta, \theta' \in \Omega\}. \tag{15}$$

With a similar argument, we can show that the dimensionality of the ambiguity set is equal to $2(q + 2q\eta) - 2p\eta$. With probability one, a unique solution for the localization problem can be found if

$$\dim \mathcal{D} < \dim \mathcal{G}. \tag{16}$$

Using the dimensionality of $\mathcal{D}$ and $\mathcal{G}$, this condition can be shown as

$$q < \frac{2\eta}{2\eta + 1} p. \tag{17}$$

Thus, if q satisfies (17), almost surely, a unique solution exists for the localization problem.

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