Computing the Weight Distribution of a Set of Points Obtained by Scaling, Shifting, and Truncating a Lattice

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***Abstract*— A method is developed to compute the weight distribution of a set of points obtained from a lattice. The lattice is scaled (with possibly nonequal factors) along different dimensions, is shifted to an arbitrary point, and its lower dimensional subspaces are truncated within given shaping regions. Each branch in the lattice trellis diagram is labeled by the weight distribution of the corresponding coset incorporating the effects of scaling, shifting, and truncation. The weight distribution is obtained by multiplying the weight distribution of the serial branches and then adding the result over parallel paths.**

***Index Terms*—Weight distribution, trellis structure of lattices, squaring and cubic constructions, shaping, vector quantization.**

I. Introduction

Consider a discrete set of points S. A shaping region can be used to select a finite subset of S having a desirable property for the specific application in hand. For example: i) in geometrical source coding, a subset of the source space with the highest probability is selected [1]–[3], and ii) in constellation shaping, a subset of the channel space with the least energy is selected [4].

In dealing with problems of this type over a set S, it is usually useful to know how many points of S are located at a given distance from the origin. This is determined by the weight distribution of S which is defined as [5]

$$W_S(q) = \sum_{\boldsymbol{u}\in S} q^{d(\boldsymbol{u})} = \sum_x N(x) q^x \tag{1}$$

where $d(\boldsymbol{u})$ is the distance from the origin of point $\boldsymbol{u}$ and $N(x)$ is the number of points at a given distance x from the origin. We are concerned with distance measures having the additivity property. This means that for an n-tuple $\boldsymbol{u} = (u_0, u_1, \cdots, u_{n-1})$, we have

$$d(\boldsymbol{u}) = \sum_i d(u_i).$$

The given examples are based on the absolute distance and the square distance which for a given scalar u are equal to $|u|$ and u^2, respectively.

II. Basic Definitions

A real n-D (n-dimensional) lattice $\boldsymbol{\Lambda}_n$ is a discrete set of n-D vectors in $\boldsymbol{R}^n$ which form a group. A sublattice $\boldsymbol{\Lambda}_n^s$ of a lattice $\boldsymbol{\Lambda}_n$ is a subset of the elements of $\boldsymbol{\Lambda}_n$ that is itself a lattice. This is denoted by $\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s$. A lattice $\boldsymbol{\Lambda}_n$ is called an integer lattice if it is a sublattice of $\boldsymbol{Z}^n$ where $\boldsymbol{Z}$ denotes the set of the integers. A sublattice $\boldsymbol{\Lambda}_n^s$ induces a partition of $\boldsymbol{\Lambda}_n$ into equivalence classes modulo $\boldsymbol{\Lambda}_n^s$. The order of this partition is denoted as $|\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s|$. The lattice $\boldsymbol{\Lambda}_n$ is the union of $|\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s|$ cosets of $\boldsymbol{\Lambda}_n^s$.

In a novel approach, Forney in [6] expresses a lattice partition $\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s$ in terms of a tree with one initial node and $|\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s|$ final nodes. Then, a lattice is expressed in terms of a trellis diagram obtained by connecting two such trees at their final nodes. Each stage of such a trellis corresponds to a subspace of a given dimensionality. Different paths in the trellis correspond to different cosets of a given sublattice. An important class of lattices based on the squaring and cubic constructions is well suited to this type of representation.[1] The squaring construction based on the lattice partition $\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s$ is defined as the union U of all pairs (s_1, s_2) where s_1 and s_2 belong to the same coset of $\boldsymbol{\Lambda}_n^s$ in $\boldsymbol{\Lambda}_n$. This construction is denoted by $U = |\boldsymbol{\Lambda}_n/\boldsymbol{\Lambda}_n^s|^2$.

We consider integer lattices that are scaled with (possibly) nonequal factors along different dimensions. Such a scaling preserves the group property of the lattice. Scaling is achieved such that the spacing between lattice points along the ith dimension is equal to d_i. Such a lattice can be used for channel coding over a nonflat channel [7]. A shifted version of a lattice $\boldsymbol{\Lambda}_n$ is obtained by shifting $\boldsymbol{\Lambda}_n$ to a given n-D point $\boldsymbol{a}$. This shifting preserves the group property of the lattice if and only if $\boldsymbol{a} \in \boldsymbol{\Lambda}_n$.

III. Computation of the Weight Distribution

The basic idea behind the computational method proposed in this correspondence is as follows: Consider two sets of points of dimensionalities n_1, n_2 and their Cartesian product which is a set of dimensionality $n = n_1 + n_2$. Let $N_n(x)$ denote the cardinality of the set of the n-D points at a given distance x from the origin. Using the additivity property of the distance measure, we obtain

$$N_n(x) = \sum [N_{n_1}(x_1) \times N_{n_2}(x_2)] \tag{2}$$

where the summation is computed over all the pairs (x_1, x_2) satisfying $x = x_1 + x_2$. Considering this property, and also the additivity property of the distance, we conclude that the weight distribution of the Cartesian product of two sets is equal to the product of their weight distributions.

We make use of the lattice trellis diagram in our computation. Each branch in the diagram is labeled by the weight distribution of the corresponding coset incorporating the effects of scaling, shifting, and truncation. The key point is that if two branches are connected through a given state, then their weight distributions multiply. Note that a cascade of branches counts for the Cartesian product of the subsets corresponding to those branches. We also note that the parallel paths in the trellis count for the union of the their respective subsets and the corresponding weight distributions add together. Based on these observations, we conclude that the final weight distribution is obtained by multiplying the weight distribution of the serial branches and then adding the result over parallel paths. This results in a partial weight distribution for the intermediate states in the trellis which act as a multiplicative factor in the weight distribution of the paths coming out of that state. In the following, this idea is explained by the use of two examples based on lattices obtained from the squaring construction.

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[1] The Barnes–Wall lattices $D_4, \boldsymbol{E}_8, \cdots$ are based on the squaring construction and the Leech lattice $\boldsymbol{\Lambda}_{24}$ is based on the the cubic construction.

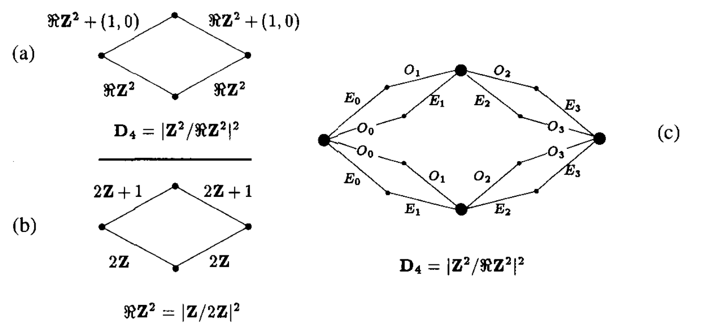


Fig. 1. The squaring construction for the D_4 lattice. E, O stands for even, odd and the subscript denotes the index of the space dimension.

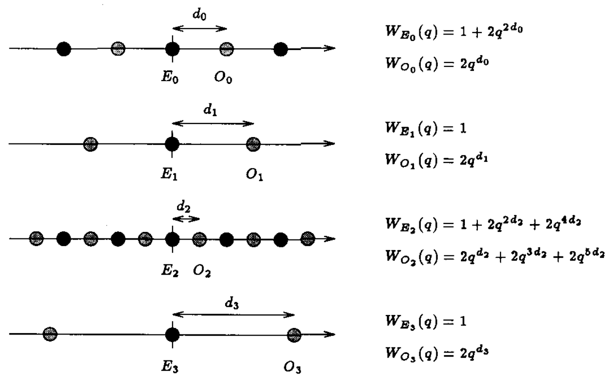
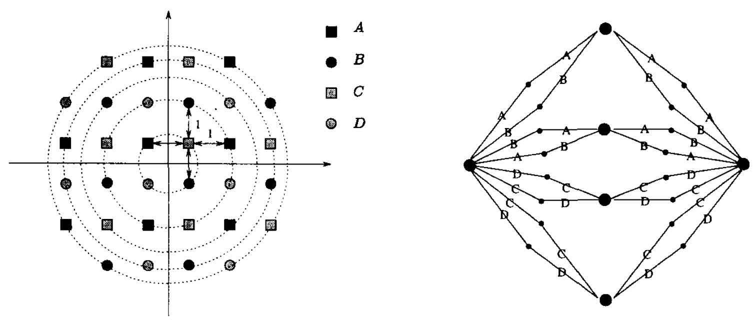


Fig. 2. 1-D truncated subsets for the example based on the D_4 lattice.



$$W_A(q) = W_B(q) = W_C(q) = W_D(q) = q^{0.5} + 2q^{2.5} + q^{4.5} + 2q^{6.5} + 2q^{8.5}$$

Fig. 3. The squaring construction for the E_8 lattice. Subsets A, B, C, D are obtained from the partition $Z^2/2Z^2$, shifted to the point $(1/2, 1/2)$, and truncated within a circle. The corresponding weight distributions are with respect to the square distance.

A. Example for the D_4 Lattice, Absolute Distance

The lattice D_4 is obtained by applying the squaring construction to the lattice partition $Z^2/\Re Z^2$ where $\Re$ denotes the rotational operator [6]. The corresponding trellis diagram is shown in Fig. 1(a) where $\Re Z^2$ and $\Re Z^2 + (1, 0)$ denote the cosets of $\Re Z^2$ in Z^2. The lattice $\Re Z^2$, which is the set of 2-tuples with both components either even or odd, is obtained by applying the squaring construction to the lattice partition $Z/2Z$. Fig. 1(b) shows the corresponding trellis diagram where $2Z$ (set of even numbers) and $2Z + 1$ (set of odd numbers) denote the cosets of $2Z$ in Z. The final trellis diagram of D_4 is shown in Fig. 1(c).

Assume that the projection of the final 4-D set into its 1-D subspaces are the truncated subsets shown in Fig. 2. The functions W_{O_i}, W_{E_i}, $i = 0, 1, 2, 3$, denote the weight distribution of the corresponding odd, even subsets with respect to the absolute distance. Using the trellis diagram of D_4 given in Fig. 1(c), the final weight distribution is computed as

$$\begin{aligned} W(q) = {} & W_{E_0} W_{O_1} W_{O_2} W_{E_3} + W_{E_0} W_{O_1} W_{E_2} W_{O_3} \\ & + W_{O_0} W_{E_1} W_{O_2} W_{E_3} + W_{O_0} W_{E_1} W_{E_2} W_{O_3} \\ & + W_{E_0} W_{E_1} W_{E_2} W_{E_3} + W_{E_0} W_{E_1} W_{O_2} W_{O_3} \\ & + W_{O_0} W_{O_1} W_{E_2} W_{E_3} + W_{O_0} W_{O_1} W_{O_2} W_{O_3}. \end{aligned} \tag{3}$$

This weight distribution has application in the vector quantization of an independent Laplacian source with nonequal values of power along different dimensions [8].

B. Example for the E_8 Lattice, Square Distance

The lattice E_8 is obtained by applying the squaring construction to the lattice partition $D_4/\Re D_4$ [6]. This is shown in Fig. 3. The 2-D subsets are selected from the half-integer grid truncated within a circle. This truncation is in accordance with the structure of an optimally shaped constellation [9]. Using the trellis diagram, the weight distribution with respect to the square distance is computed as

$$W(q) = 16\,[q^{0.5} + 2q^{2.5} + q^{4.5} + 2q^{6.5} + 2q^{8.5}]^4. \qquad (4)$$

Without truncation, weight distribution of the subsets A, B, C, D would be equal to

$$q^{0.5} + 2q^{2.5} + q^{4.5} + 2q^{6.5} + 2q^{8.5} + \cdots = \frac{1}{4}[\theta_2(q)]^2 \qquad (5)$$

where $\theta_2(q)$ is one of the Jacobi theta functions [5]. This results in the following weight distribution:

$$W_{E_8+(1/2)^8}(q) = \frac{1}{16}[\theta_2(q)]^8. \qquad (6)$$

IV. Summary

A method to compute the weight distribution of a set of points obtained by scaling, shifting, and truncation of a lattice is presented. Computation is based on using the lattice trellis diagram. The proposed method is quite general and can be used in conjunction with different types of distance measures having additivity property and any lattice which is constructed using a trellis diagram.

References

[1] D. J. Sakrison, "A geometric treatment of the source encoding of a Gaussian random variable," *IEEE Trans. Inform. Theory*, vol. IT-14, pp. 481–486, May 1968.
[2] T. R. Fischer, "Geometric source coding and vector quantization," *IEEE Trans. Inform. Theory*, vol. 35, pp. 137–145, Jan 1989.
[3] D. G. Jeong and J. D. Gibson, "Uniform and piecewise uniform lattice vector quantization for memoryless gaussian and Laplacian sources," *IEEE Trans. Inform. Theory*, vol. 39, pp. 786–804, May 1993.
[4] G. D. Forney, Jr. and L. F. Wei, "Multidimensional constellations—Part I: Introduction, figures of merit, and generalized cross constellations," *IEEE J. Selected Areas Commun.*, vol. 7, pp. 877–892, Aug. 1989.
[5] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988.
[6] G. D. Forney, "Coset codes—Part II: Binary lattices and related codes," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1152–1187, Sept. 1988.
[7] S. Kasturia, J. T. Aslanis, and J. M. Cioffi, "Vector coding for partial response channels," *IEEE Trans. Inform. Theory*, vol. 36, pp. 741–762, July 1989.
[8] D. G. Jeong and J. D. Gibson, "Image coding with uniform and piecewise-uniform vector quantizers," submitted to *IEEE Trans. Image Processing*.
[9] A. K. Khandani and P. Kabal, "Shaping multidimensional signal spaces—Part I: Optimum shaping, shell mapping," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1799–1808, Nov. 1993.