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A Family of Nyquist Filters Based on Generalized Raised-Cosine Spectra

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Abstract

Data transmission over bandlimited channels requires pulse shaping to eliminate or control Inter-Symbol Interference (ISI). A widely used filter for this purpose is the raised cosine filter which satisfies Nyquist's first criterion. We design a phase compensator so that the square-root raised-cosine filter also satisfies Nyquist's first criterion. Such a technique is particularly useful to accommodate two different structures for the receiver, one with a filter matched to the transmitting filter and one without any matched filter. In the case of a raised cosine spectrum with full excess bandwidth, we show that the phase compensator corresponds to a pure time delay.

We also extend the raised-cosine spectra to a more general family of Nyquist filters which their compensated square-root spectra satisfy Nyquist's criterion. In practice, it is important that the transmitting and receiving filters be well approximated with short impulse responses. From this point of view, the raised cosine spectrum is not necessarily the best choice for Nyquist filter design. A new family of Nyquist pulses are designed such that they have faster asymptotic decay compared to the raised-cosine filter impulse response.

1 Introduction

A conventional baseband Pulse Amplitude Modulation (PAM) signal can be represented as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_s) \tag{1}$$

where a_k's are the transmitted symbols and $g(\cdot)$ is a real valued "Nyquist pulse" which satisfies Nyquist's first criterion,

$$g(kT_s) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k = \pm 1, \pm 2, \cdots \end{cases} \tag{2}$$

$g(\cdot)$ represents the overall impulse response of the transmitting filter, receiving filter and the communication channel. Each transmitted symbol a_k can be recovered from the received signal $x(t)$, by taking samples of $x(t)$ at the time instances kT_s. In other words, choosing $g(\cdot)$ as a Nyquist pulse prevents Inter-Symbol-Interference (ISI) and allows sample-by-sample detection at the receiver. In the frequency domain, Nyquist's first criterion is written as:

$$\sum_{n=-\infty}^{\infty} G(f - \frac{n}{T_s}) = T_s \tag{3}$$

where $G(f)$ is known as a Nyquist filter. A particular Nyquist filter with wide practical applications is the *raised-cosine* filter

$$G_{rc}(f) = \begin{cases} T_s & |f| \le \frac{1-\alpha}{2T_s} \\ T_s \cos^2\left(\frac{\pi T_s}{2\alpha}(|f| - \frac{1-\alpha}{2T_s})\right) & \frac{1-\alpha}{2T_s} \le |f| \le \frac{1+\alpha}{2T_s} \\ 0 & |f| > \frac{1+\alpha}{2T_s} \end{cases} \tag{4}$$

where α is called the *roll-off factor* and takes values between zero and one. The parameter α also represents the fractional *excess bandwidth* occupied by the signal beyond the Nyquist frequency $1/2T_s$.

In practical applications, the overall magnitude response of the raised-cosine spectrum is split evenly between the transmitter and receiver. The phase response of the receiving filter compensates for the transmitting filter phase so that the overall filter has a linear phase:

$$\begin{aligned} G_{rc}(f) &= G_T(f) G_R(f) \\ |G_T(f)| = |G_R(f)| &= \sqrt{|G_{rc}(f)|} \\ \angle G_R(f) &= -2\pi f \tau_0 - \angle G_T(f) \end{aligned} \tag{5}$$

$G_T(f)$ and $G_R(f)$ are the transfer functions of the transmitting and receiving filters accordingly. The receiving filter in this case is matched to the transmitting filter to maximize the Signal-to-Noise ratio (SNR) at the sampling time instances at the receiver [1]. The transmitting and receiving filters are typically considered to be linear phase with a nominal delay of $\tau_0/2$ that is required to make the filters physically realizable. Here however, we consider more general phase response for these filters. The phase added to the square-root raised-cosine spectrum causes the transmitting filter to satisfy the Nyquist's condition. As a result, with or without a matched filter at the receiver, we can obtain ISI free transmission.

The ability to use a simple receiver filter (not a matched filter) is particularly useful to reduce the cost of the receiver [2]. There are also new voice-band modems designed for PCM channels where the channel at one end is terminated to the digital telephone network [3] [4] [5]. Since there is no access to the filters at the telephone network, the pulse shaping should be performed entirely at the other end (analog end user). The phase compensation technique, as it will be described here, provides a method to perform pulse shaping at one end of the channel and at the same time, stay compatible with the other structures where a matched filter is in place at the receiver.

In Section 2, a general relationship between phase and amplitude response of a Nyquist filter is presented. The special case of the square-root raised-cosine spectrum is investigated. We also quantify the SNR degradation due to replacing the matched filter by a lowpass filter.

In Section 3, a generalized raised cosine spectrum is introduced. The square root of the generalized spectrum can be phase compensated to become a Nyquist filter. Furthermore, compared to the raised cosine filter, the generalized raised cosine filter can have a smoother spectrum and consequently a shorter approximated impulse response.

2 Nyquist filter design using phase compensation

The transmitting filter $G_T(f)$ can be expressed in terms of its magnitude and phase responses,

$$G_T(f) = |G_T(f)| \cdot \exp\big(j\theta(f)\big). \tag{6}$$

For a filter with real impulse response, $\theta(\cdot)$ is an odd real function of frequency. In our discussion we assume $G_T(f)$ is bandlimited to $|f| < 1/T_S$. As stated by Gibby and Smith in [6], for a bandlimited filter Eq. (3) can be simplified to the following conditions:

$$\begin{aligned} |G_T(f)|\cos\theta(f) + |G_T(f-1/T_s)|\cos\theta(f-1/T_s) &= K \quad (7)\\ |G_T(f)|\sin\theta(f) + |G_T(f-1/T_s)|\sin\theta(f-1/T_s) &= 0 \quad (8)\end{aligned}$$

where K is a real constant. As shown in [6], $\theta(f)$ can be expressed in terms of the magnitude response of $G_T(f)$:

$$\theta(f) = \arccos\left(\frac{K^2+|G_T(f)|^2-|G_T(f-1/T_s)|^2}{2K|G_T(f)|}\right) \tag{9}$$

Since the argument of the arccos$(\cdot)$ should be limited to one in absolute value, there may not be a real solution for $\theta(f)$. However, for the square-root raised-cosine filter, the appropriate phase response exists and has a simple closed form. To find the phase function $\theta(f)$, we substitute the square root of the raised-cosine spectrum into Eq. (9). Let us assume that the impulse response of the filter is properly normalized such that the constant value K in Eq. (9) is equal to $\sqrt{T_s}$. Note also that throughout our discussion here, we assume the time delay τ_0, introduced in Eq. (5), is zero. A time delay can always be added to an appropriately truncated filter response to make it causal and physically realizable.

The resulting phase response $\theta(f)$ is a piecewise linear phase function:

$$\theta(f) = \begin{cases} \dfrac{\pi T_s}{2\alpha}\left(-f-\dfrac{1-\alpha}{2T_s}\right) & -\dfrac{1+\alpha}{2T_s} \le f \le -\dfrac{1-\alpha}{2T_s} \\ 0 & |f| \le \dfrac{1-\alpha}{2T_s} \\ \dfrac{\pi T_s}{2\alpha}\left(-f+\dfrac{1-\alpha}{2T_s}\right) & \dfrac{1-\alpha}{2T_s} \le f \le \dfrac{1+\alpha}{2T_s} \end{cases} \tag{10}$$

The phase response is not necessarily linear for out-of-range frequencies, $|f| > \frac{1+\alpha}{2T_s}$. We can verify that the square-root raised-cosine spectrum, along with the above phase response, satisfies Nyquist's condition.

In the time domain, the impulse response of the filter is calculated as:

$$\mathcal{F}^{-1}\{\sqrt{T_s}|G_T(f)|\exp\big(j\theta(f)\big)\} = \alpha\sin(\frac{\pi t}{T_s})\cdot\text{sinc}(\frac{\alpha t}{T_s}-\frac{1}{2}) + \text{sinc}(\frac{t}{T_s})\cdot\cos(\frac{\pi t\alpha}{T_s}) \tag{11}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. It is worth noting that Eq. (11) contains two terms, both with regular zero crossings at integer multiples of T_s. The first term however, takes on a non-zero value at $t=0$.

2.1 Special Case: Full excess bandwidth

In the case of full excess bandwidth, i.e. $\alpha = 1$, $\theta(f)$ simplifies to a linear function over the whole frequency range of the filter:

$$\theta(f) = \frac{-\pi T_s}{2}\, f \quad \text{for} \quad |f| \le \frac{1}{T_s} \tag{12}$$

In the time domain, adding the phase response $\theta(f)$ to the filter delays the impulse response by a quarter of the sampling period T_s.

$$g_{tr}(t) = \frac{\sqrt{T_s}\sin(\dfrac{2\pi t}{T_s})}{4\pi t(\dfrac{1}{2}-\dfrac{t}{T_s})} \tag{13}$$

Since $\theta(f)$ causes only a pure time delay in the filter impulse response, the phase compensation does not change the pulse shape. We can further conclude that in the case of full excess bandwidth, with or without a matched filter at the receiver, one can achieve ISI free transmission using the square-root full raised-cosine filter by merely redefining the sampling points at the receiver.

We use the eye pattern diagrams to compare the conventional raised-cosine filters with the compensated square-root raised-cosine filters. First consider a raised-cosine filter with $\alpha = 1$; Figure 1 shows the eye pattern of the pulse shape modulated by a binary data. At integer multiples of the sampling periods, each transmitted symbol can be recovered without any ISI. Figure 2 shows the eye pattern of a square-root full raised-cosine spectrum. Note that the center of the eye in Fig. 2 coincides with peak of the impulse response. For zero ISI, the sampling points should be shifted by one quarter of sampling time. However, for the special case of binary data, the center of the eye has the widest vertical opening. In fact, we can show that the lower boundaries of the eye pattern stay constant, equal to 1 and -1 for half of the sampling period around the center. The boundaries are shown in Fig. 2. Compared to the eye pattern of the conventional full raised-cosine filter, the eye pattern of the square-root filter has a larger vertical eye opening. Note however, that the above results are based on the binary PAM signaling and do not generalize to a multi-level PAM signaling.

2.2 SNR degradation

It is well known that the use of matched filter at the receiver of an additive white Gaussian noise (AWGN) channel maximizes the signal-to-noise ratio at the sampling instances [1].

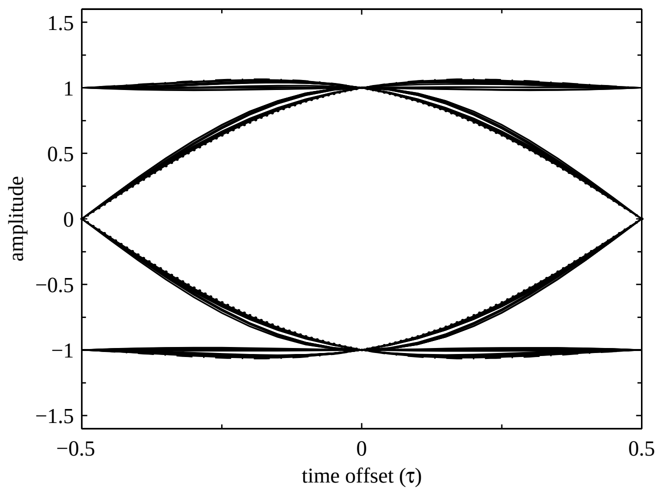

Fig. 1 conventional raised-cosine filter

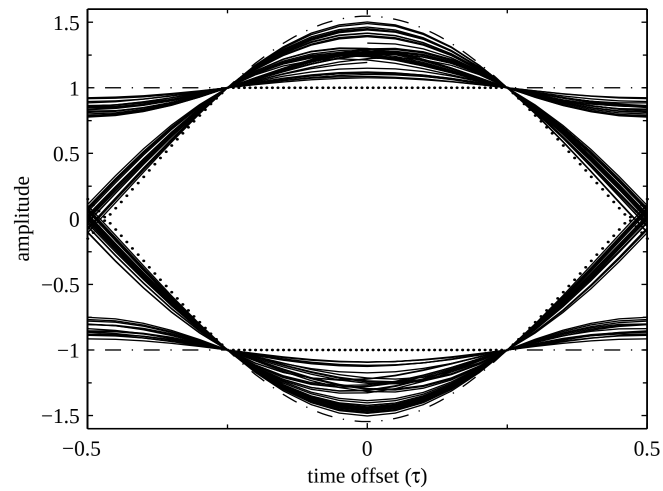

Fig. 2 square-root raised-cosine filter

Here, we quantify the SNR degradation due to the use of a non-matched filter at the receiver.

Assume that the transmitter uses a modified square-root raised-cosine filter. The received signal is passed through a filter $H(f)$ and is sampled at integer multiples of T_s. Let us also assume the channel adds only white Gaussian noise to the transmitted signal.

$$\sigma_n^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df, \tag{14}$$

where $N_0/2$ is the power spectral density to the noise. The signal power at the sampling instance can be written as:

$$\sigma_s^2 = \left|\int_{-\infty}^{\infty} G_T(f)H(f)df\right|^2 \tag{15}$$

where $G_T(f)$ is the transmitting filter. We consider two different receiving filters:

1. A filter matched to the transmitting filter: $H(f) = G_T^*(f)$.
2. A brick-wall filter with $|f| \leq \dfrac{1+\alpha}{2T_s}$ pass-band region.

Using the fact that the transmitting filter is normalized, the SNR after sampling is

$$\text{SNR (matched)} = \frac{2}{N_0} \tag{16}$$

$$\text{SNR (brick-wall)} = \frac{2}{(1+\alpha)N_0} \tag{17}$$

Equations (16) and (17) show that there will be a $10\log(1+\alpha)$ dB loss if the matched filter is replaced by an ideal brick-wall filter. For instance, for $\alpha = 0.5$ the loss is 1.76 dB.

3 Generalized Raised Cosine Nyquist Filters

In this section we introduce a general family of Nyquist filters. Similar to the raised cosine filter, the new filters can satisfy Nyquist's criterion with and without matched filtering at the receiver. The raised cosine filter is a special case of these filters.

Let us define a real continuous odd function $V(x)$ which satisfies the following conditions:

$$V(x) = -V(-x) \qquad \forall x \tag{18}$$

$$V(x) = 1 \qquad 1 \leq x \tag{19}$$

Consider the following filter, defined for all values of f:

$$H_G(f) = T_s \ \cos^2\Big(\frac{\pi}{4}V\big(\frac{2T_s}{\alpha}(|f| - \frac{1}{2T_S})\big) + \frac{\pi}{4}\Big) \tag{20}$$

We call $H_G(f)$ a generalized raised-cosine filter. To make $H_G(f)$ a monotonic spectrum, we assume that $V(x)$ is a monotonic function for $-1 < x < 1$. The odd symmetry property of $V(x)$ guarantees that $H_G(f)$ satisfies Nyquist's first criterion:

$$H_G(f) + H_G(f - \frac{1}{T_s}) = T_s \tag{21}$$

The raised cosine filter as defined in Eq. (4) is a special case of $H_G(f)$ where

$$V(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ +1 & x > +1 \end{cases} \tag{22}$$

By adding an appropriate phase, the square root of $H_G(f)$ can be converted into a Nyquist filter. Following the same steps described in Section 2, we define the square-root generalized raised-cosine filter as:

$$H_{\mathrm{sqrt}}(f) = \sqrt{|H_G(f)|}\exp[j\phi(f)] \tag{23}$$

where $\phi(f)$ is defined as:

$$\phi(f) = \begin{cases} \frac{\pi}{4}V\Big(\frac{2T_s}{\alpha}(-f - \frac{1}{2T_S})\Big) + \frac{\pi}{4} & f \leq 0 \\ -\frac{\pi}{4}V\Big(\frac{2T_s}{\alpha}(f - \frac{1}{2T_S})\Big) - \frac{\pi}{4} & f > 0 \end{cases} \tag{24}$$

Since $\sqrt{|H_G(f)|}$ is an even function and $\phi(f)$ is an odd function of f, the impulse response of the filter is real. The phase compensated square-root filter satisfies the Nyquist's conditions (7) and (8). The family of Nyquist filters described in [2] can also be represented using the generalized raised-cosine filters.

The impulse response of the filter $H_{\mathrm{sqrt}}(f)$ does not have a closed form in general. However, the inverse Fourier transform of $H_{\mathrm{sqrt}}(f)$ can be simplified as follows:

$$\begin{aligned}\mathcal{F}^{-1}\{\sqrt{T_s}H_{\mathrm{sqrt}}(f)\} &= \mathrm{sinc}(\frac{t}{T_s}) \cdot \cos(\frac{\pi t\alpha}{T_s}) \\ &+ \frac{\alpha}{2}\int_{-1}^{1} \sin(\frac{\pi}{2}V(x) - \frac{\pi t\alpha}{T_s}x + \frac{\pi t}{T_s})\, dx\end{aligned} \tag{25}$$

The first term has regular zero crossings at integer multiples of T_s except at $t = 0$. The second term has regular zero-crossings for all integer multiples of T_S. Note that $\sin(\frac{\pi}{2}V(x) - \frac{\pi t\alpha}{T_s}x)$ is an odd function of x , therefore its finite integral between -1 and 1 is zero. In the case of conventional raised-cosine filter, $I(t)$ has a closed form as shown in Eq.(11). In more general cases, $I(t)$ can be evaluated numerically. Decomposing the impulse response into two terms as shown in Eq. (25) simplifies the numerical evaluation.

3.1 A family of Nyquist filters with smoother spectra

As shown in Eq. (5), pulse shaping can be split between the transmitter and the receiver so that the overall response of the transmitting and receiving filters satisfies the Nyquist condition.

$$g_{rc}(t) = g_{tr}(t) * g_{re}(t) \tag{26}$$

Truncating a Nyquist pulse does not affect its Nyquist zero-crossing property. However, the convolution of the truncated responses will in general no longer satisfy Nyquist's first criterion. In that case, pulses with faster decay can reduce the ISI caused by truncation.

It is a well known result that if the amplitude response of a filter along with its first $(K-1)$ derivatives are all continuous but its Kth derivative is discontinuous, the filter impulse response asymptotically decays as $1/t^{K+1}$ [7]. Based on this result, the impulse response of a raised cosine filter decays asymptotically as $1/t^3$. Using the generalized raised cosine spectrum, we can design filters with higher rates of decay of their impulse responses. To achieve this goal, we design $V(x)$ such that it has higher degrees of continuity. We consider a family of polynomials $P_n(x)$ to design $V(x)$ with $(n-1)$ continuous derivatives. Let us assume

$$V(x) = \begin{cases} -1 & x \leq -1 \\ P_n(x) & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{27}$$

To satisfy condition (19), $P_n(x)$ should be an odd function of x which results in the following expression:

$$P_n(x) = \sum_{k=1}^{n} a_k \cdot x^{2k-1} \tag{28}$$

Coefficients a_k's are determined such that $V(x)$ and its first $(n-1)$ derivatives are continuous at $x = \pm 1$. The first derivative of $P_n(x)$, is a polynomial of degree $(2n-2)$ with $(n-1)$ repeated roots at $x = 1$ and by symmetry $(n-1)$ repeated roots at $x = -1$. Since the first derivative of $P_n(x)$ has no roots in $\{\, x \mid -1 < x < 1 \,\}$ interval, $P_n(x)$ is monotonic in this interval.

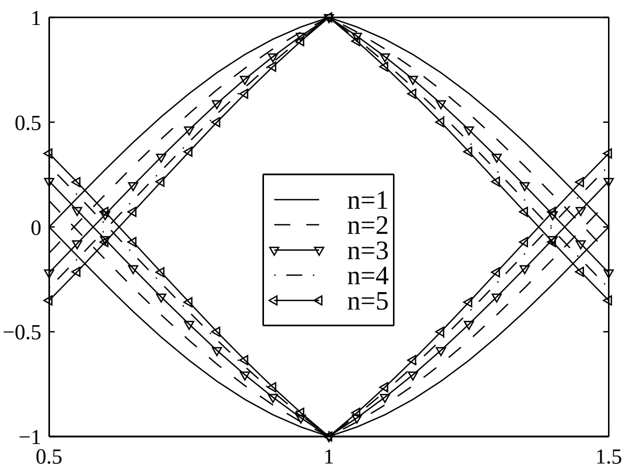

Fig. 3 Eye diagram for several generalized raised-cosine filters with $\alpha = 1$

The actual polynomials $P_n(x)$ are given in Table 1. Using a Taylor series, we can show that the generalized raised-cosine spectrum corresponding to $P_n(x)$ has $(2n-1)$ continuous derivative and its impulse response decays asymptotically as $1/t^{2n+1}$.

Apart from the rate of decay of the filter impulse response, there are other issues to be considered. Since timing recovery at the receiver is not always perfect, we encounter timing phase jitter. The eye pattern diagram of a pulse shape can be used to assess the immunity of the pulse to timing phase jitter. Using polynomials given in Table 1, we design five generalized raised-cosine filters with $\alpha = 1$ and normalized sampling period. Figure 3 shows the lower boundaries of the binary eye diagrams associated with the impulse responses of these filters. As we increase the number of continuous derivatives of the spectrum, the width of the eye diagram decreases (see Table 1).

To study the effect of truncating the impulse response of the filter, we consider the following conditions:

- We assume that the compensated square-root generalized raised-cosine filters are used at the transmitter and the receiver.
- To calculate samples of the impulse response of the transmitting and receiving filters, we evaluate Eq. (25) numerically.
- Each impulse response is calculated for $N \cdot M$ points where M is the up-sampling ratio and N is the number lobes in each truncated impulse response. Therefore, the truncated impulse response has a length of $M \cdot N \cdot T_s$ sec.
- The overall impulse response is determined by convolving the impulse response of the transmitting and receiving filters.
- The maximum ISI (in dB scale) that is introduced by truncating the individual responses is calculated using random binary input data. We assume perfect timing recovery at the receiver.

Table 1 Comparison of several generalized raised cosine filter with $\alpha = 1$

n	$P_n(x)$	$h_G(t)$ asymptotic decay	Eye-width	Truncation effects $N = 8$ ISI(dB)	$N = 20$ ISI(dB)
1	x	t^{-3}	1.000	−50.8	− 64.5
2	$2^{-1}(-x^3 + 3x)$	t^{-5}	0.911	−54.2	− 76.7
3	$2^{-3}(3x^5 - 10x^3 + 15x)$	t^{-7}	0.843	−73.9	−112.4
4	$2^{-4}(-5x^7 + 21x^5 - 35x^3 + 35)$	t^{-9}	0.791	−59.0	−107.1
5	$2^{-7}(35x^9 - 180x^7 + 378x^5 - 420x^3 + 315x)$	t^{-11}	0.750	−49.4	−130.0

Table 1 shows the results for different generalized raised-cosine filters with $\alpha = 1$. As shown in Table 1, the conventional raised-cosine filter (the first row of the table) is not necessarily the best choice. Increasing the degree of continuity of $P_n(x)$ can reduce the ISI. However, we should bear in mind that increasing the rate of asymptotic decay of an impulse response does not necessarily result in a lower ISI. A pulse with a high rate of decay may have a larger ISI due to the first few lobes of the response. Based on results given in Table 1, a generalized raised-cosine filter based on $P_3(x)$ has a low ISI due to truncation and also has a reasonable eye-width.

Table 2 compares filters designed based on $P_n(x)$ with $\alpha = 0.5$. These filters have a smaller eye-width compared to those with $\alpha = 1$. For the same truncation length, filters with $\alpha = 0.5$ cause more ISI compared to the filters with $\alpha = 1$.

Table 2 Comparison of several generalized raised cosine filter with $\alpha = 0.5$

n	$h_G(t)$ asymptotic decay	Eye-width	Truncation effects $N = 8$ ISI(dB)	$N = 20$ ISI(dB)
1	t^{-3}	0.784	−33.6	−48.9
2	t^{-5}	0.674	−40.2	−61.7
3	t^{-7}	0.608	−26.5	−84.0
4	t^{-9}	0.565	−21.8	−79.6
5	t^{-11}	0.535	−19.1	−68.4

4 Concluding Remarks

We have introduced a phase compensation technique for the square-root raised-cosine filter to design a Nyquist filter. In the special case of full raised-cosine spectrum, we showed that the square-root filter satisfies Nyquist's first criterion, provided appropriate time delay in the impulse response. Therefore, the square-root full raised-cosine transmitting filter can be used with or without a matched filter at the receiver. We have suggested a generalized class of the Nyquist filters which extend the class of raised-cosine spectra. The generalized raised-cosine filters can be easily phase compensated to satisfy Nyquist's first criterion. One particular application of the generalized raised-cosine filter is to design filters which can be well approximated with short impulse responses. A family of polynomials was introduced to achieve this goal.

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