Comparison of Distance Measurement Methods for Turbo Codes

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***Abstract*— Reliable distance measurement methods for turbo codes are a key element in the design of interleavers with high minimum distances, which are essential for lowering the flare effect at low error rates. The usefulness of such methods depends strongly on their computational complexity, especially for long interleavers with high minimum distances. This paper improves the reliability of the double-impulse iterative decoding method and compares it with Garello's true minimum distance method, the error-impulse method and the all-zero iterative decoding method. The comparison is based on the interleavers specified in the Digital Video Broadcasting with Return Channel via Satellite (DVB-RCS) standard, random interleavers and dithered relative prime (DRP) interleavers. A new interleaver for an MPEG packet of size 1504 information bits is designed for the DVB-RCS standard. The new interleaver provides an improvement of at least 0.4 dB at low error rates.**

I. Introduction

Efficient and accurate determination of the distance spectrum, or even the minimum distance $d_{\min}$, for long interleavers with high distance, is a challenging problem. Recently, four interesting distance measurement methods have been introduced: Garello's true distance spectrum method [1], Berrou's error-impulse method [2], Garello's all-zero iterative decoding method [3] and Crozier's double (and triple) impulse method(s) [4]. Minimum distances obtained with these methods are compared for double-binary turbo codes.

II. Distance measurement methods

The first significant approach to determine the minimum distance was introduced by Robertson [5]. This brute force approach is practical, even for long interleavers, as long as the minimum distance remains small. The complexity of this approach quickly becomes unacceptable if the minimum distance is due to high input-weights, which is often the case for well-designed interleavers. Similar approaches have been presented in [6] and [7]. Robertson's approach was improved in [8] by introducing a *back tracking* algorithm that efficiently computes the distances caused by low input-weights.

A. True distance measurement method

Garello *et al.* [1] introduced a novel method for single-binary turbo codes, which has been extended to tail-biting double-binary turbo codes in [9] and [10]. This method allows not only the accurate determination of $d_{\min}$, but it also allows the accurate determination of the first few terms of the distance spectrum for a specific interleaver. For high-distance interleavers, the complexity of this method increases rapidly as $d_{\min}$ increases.

B. Error-impulse method

Berrou *et al.* introduced in [2] a fast method based on the ability of a soft-in decoder to overcome error-impulse inputs. This method inserts a low-amplitude impulse into the all-zero codeword at a specific index to see if the decoder can correct it. This amplitude is increased in steps of 1 until the decoder fails. The lowest amplitude at which the decoder fails is recorded. Testing all data indices in the all-zero codeword in this manner gives a list of amplitudes. The lowest amplitude gives an estimate of $d_{\min}$. Since the iterative decoding complexity is linear in the codeword length, the complexity of this method is $O(\alpha\beta K^2)$, where α is the average number of iterations, β is the average number of tested amplitudes (typically proportional to $d_{\min}$) and K is the number of information bits. To achieve good convergence, the maximum α must be high. However, the average can be significantly reduced using early stopping techniques, especially for amplitudes lower than the estimated $d_{\min}$. It was shown in [2] that this method is guaranteed to find the true minimum distance if the decoder uses true maximum likelihood (ML) decoding. Unfortunately, turbo decoding is not guaranteed to perform a ML decoding, because of the iterative nature of the decoding process. Thus, the relationship between the distance obtained with this method and the true $d_{\min}$ remains uncertain. It has been observed that this method is usually pessimistic, but distances higher than $d_{\min}$ have also been found. Even so, the approach may prove to be very useful for finding good interleavers.

C. All-zero iterative decoding method

Garello *et al.* introduced in [3] a method similar to Berrou *et al.*'s method [2]. Instead of increasing the amplitude of the error-impulse in steps of 1 until the decoder fails to converge to the all-zero codeword [2], this method [3] intentionally sets the amplitude of the error-impulse to a high value so that the decoder can not converge to the all-zero codeword. Since the decoder fails, it estimates a non-zero input sequence. Encoding this input sequence leads to a non-zero codeword. The hamming weight of this codeword is an upper bound for

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the true $d_{\min}$. The complexity of this method is $O(\alpha K^2)$, but it only works well for interleavers with low minimum distances.

D. Double-impulse iterative decoding method

Crozier *et al.* in [4] improved upon the all-zero iterative decoding method [3] by introducing second (and third) impulse(s). Forcing the information bit to be "1" at a single index helps the decoder to converge to two events, one in the first encoder (ENC1) and the other in the second encoder (ENC2). Since high distances are often a result of more than two events, forcing two information bits to be "1" helps the decoder to converge to more events. The reliability of this double-impulse method is improved by a full range search for the second impulse instead of a limited range search as in [4]. The full range search for the second impulse allows separate events in ENC1 and ENC2 to be directly influenced by an impulse. This method, as well as the other iterative methods, works best for interleavers with high spread. It has been found that this method works very well with max-log *a posteriori* probability (APP) decoding, where the knowledge of signal-to-noise ratio (SNR) is not required. However, if log-APP decoding is applied then a value for the noise variance, σ^2, must be selected. Let d^* be an upper bound on $d_{\min}$ and $w(\hat{\boldsymbol{c}})$ be the hamming weight of the estimated codeword $\hat{\boldsymbol{c}}$. The vector $\boldsymbol{y}$ represents the input to the decoder, assuming that the all-zero codeword is modulated using antipodal signaling. A short description of this method follows, where i and j are the indexes for the two information bits forced to be "1s" and $A_{\min}$ is an estimate of the minimum distance multiplicity:

- || choose σ^2, if log-APP decoding is used;
- || set $E_i = E_j \simeq 2d^*$. set $d_{\min} = d^*$. set $A_{\min} = 1$;
- || for $i = 0$ to $(K-1)$ do
  - || for $j = i$ to $(K-1)$ do
    - set $\boldsymbol{y} = (1,\ldots,1,-E_i,1,\ldots,1,-E_j,1,\ldots,1)$, where $(-E_i)$ and $(-E_j)$ are in positions i and j;
    - iterative decoding of $\boldsymbol{y} \Rightarrow \hat{\boldsymbol{x}}$;
    - encoding of $\hat{\boldsymbol{x}} \Rightarrow \hat{\boldsymbol{c}}$;
    - determine $w(\hat{\boldsymbol{c}})$;
    - if $w(\hat{\boldsymbol{c}}) < d_{\min}$, set $d_{\min} = w(\hat{\boldsymbol{c}})$ and $A_{\min} = 1$;
    - else if $w(\hat{\boldsymbol{c}}) = d_{\min}$, update $A_{\min}$;

    end for

  end for

With this method the decoder must decode $\frac{K\cdot(K+1)}{2}$ packets, which results in complexity of $O(\alpha K^3)$. However, the complexity is reduced when testing structured interleavers such as dithered relative prime (DRP) interleavers [11] for tail-biting turbo codes because the distance properties repeat every M-indexes (typical values for M are 4, 8, 16 and 32). Since only M-indexes in the first loop need to be tested, the complexity is $O(\alpha M K^2)$. It has been observed for this method and the all-zero iterative decoding method that a high value for α does not necessarily increase the reliability. Thus, it is usually sufficient to set α to a moderate value (e.g., 16 or 32).

III. Distance Results

The all-zero iterative decoding method and double-impulse iterative decoding method are referred to in this section as the single-impulse method (SIM) and the double-impulse method (DIM), respectively. Using the DVB-RCS turbo encoder, the true $d_{\min}$ values, obtained using the method in [10], are compared with the distances obtained with the error-impulse method (EIM), the SIM and the DIM. The EIM uses max-log-APP decoding with early stopping (ES), where W is the number of consecutive sets of hard decisions that must agree before stopping [12] [13]. Genie ES can also be used with the EIM. With genie ES the decoder stops when at any half-iteration the all-zero codeword is produced. The maximum number of full iterations was set to 256 for EIM and 16 for SIM and DIM. $\overline{d}_{\text{method}}$ and σ_{method} are the average distance and the standard deviation for a specific method, respectively, and $\overline{K} = \frac{K}{2}$ is the number of two-bit information symbols.

A. DVB-RCS standard interleavers

The distances shown in Table I are for the 12 DVB-RCS standard interleavers. The code rate is 1/3 and early stopping with different W values were investigated. The use of ES reduces the computational complexity of the EIM. The EIM is used with both normal ES and genie ES.

Table I shows that the distances estimated using the EIM with normal ES are pessimistic. The EIM with normal ES and $W = 16$ gives the same distances as with $W = 8$, thus in the following sections only $W = 8$ was used. The EIM with genie ES gives pessimistic results for packet sizes less than or equal to 228 symbols and optimistic results for packet sizes greater than or equal to 424 symbols. The SIM tends to be very optimistic. In fact, it always provides an upper bound on the true $d_{\min}$. The DIM was able to find the true $d_{\min}$ for all 12 DVB-RCS standard interleavers.

Table II shows the distances with the MPEG packet size for the seven standard code rates (R_c). Each distance estimated with EIM using normal ES is pessimistic. The EIM with genie ES gives optimistic results for the lower code rates and pessimistic results for the higher code rates. Note that the SIM continues to give very optimistic results with puncturing. The DIM was able to find the true $d_{\min}$ for all but one of the seven code rates. For the rate 3/4 case, the estimated minimum distance was only $d_{\min} + 1$.

B. Random interleavers

Tables I and II indicate that the distances estimated with the EIM and genie ES can be misleading. They are a mix of optimistic and pessimistic results. Thus, in this section the investigation of EIM was limited to normal ES with $W = 8$. The four methods were tested with 1000 random interleavers for the code rate 1/3.

Table III indicates that the longer the interleaver is, the better the distance estimated with EIM. One possible explanation is that increasing the size of a random interleaver does not imply a significant increase in $d_{\min}$, but it improves the

TABLE I

DISTANCES SHOWN HERE ARE FOR THE CODE RATE 1/3. DVB-RCS STANDARD INTERLEAVERS WERE USED WITH NORMAL ES, WHERE W IS THE NUMBER OF CONSECUTIVE SETS OF HARD DECISIONS THAT MUST AGREE BEFORE STOPPING. THE SIGN – INDICATES THAT THERE IS NO DIFFERENCE BETWEEN THE CORRESPONDING RESULT AND THE RESULT ESTIMATED WITH NORMAL ES AND W=4.

	true $d_{\min}$	d_{EIM} with normal ES			d_{EIM} with genie ES	d_{SIM}	d_{DIM}
$\overline{K}$		$W=4$	$W=8$	$W=16$			
48	21	17	–	–	18	21	21
64	25	16	–	–	17	31	25
212	31	25	24	24	30	188	31
220	31	23	22	22	28	257	31
228	30	25	–	–	29	84	30
424	30	24	–	–	32	364	30
432	31	25	–	–	33	497	31
440	28	25	24	24	32	509	28
752	33	28	–	–	38	204	33
848	36	28	27	27	37	634	36
856	33	28	–	–	38	332	33
864	36	27	–	–	35	332	36

TABLE II

DISTANCES FOR MPEG SIZE ($\overline{K}$=752 SYMBOLS) USING DVB-RCS STANDARD INTERLEAVER AND STANDARD PUNCTURING. NORMAL ES WITH VARIOUS W WERE USED. THE SIGN – INDICATES THAT THERE IS NO DIFFERENCE BETWEEN THE CORRESPONDING RESULT AND THE RESULT WITH NORMAL ES AND W=4.

	true $d_{\min}$	d_{EIM} with normal ES			d_{EIM} with genie ES	d_{SIM}	d_{DIM}
R_c		$W=4$	$W=8$	$W=16$			
1/3	33	28	–	–	38	204	33
2/5	27	23	–	–	30	397	27
1/2	19	17	–	–	20	219	19
2/3	12	11	–	–	14	394	12
3/4	9	7	–	–	8	32	10
4/5	9	7	6	6	9	110	9
6/7	6	4	–	–	5	12	6

TABLE III

DISTANCES SHOWN HERE ARE FOR THE CODE RATE 1/3. THE m RANDOM INTERLEAVERS WERE TESTED USING NORMAL ES WITH $W=8$. THE SIGNS – AND ● INDICATE THAT THERE IS NO DIFFERENCE BETWEEN THE ESTIMATED $\overline{d}_{\text{method}}$, σ_{method} AND THE TRUE ONES.

		true $d_{\min}$		EIM($W=8$)		SIM		EIM(W=8) vs. true $d_{\min}$		SIM vs. true $d_{\min}$	
$\overline{K}$	m	$\overline{d}_{\min}$	$\sigma_{\min}$	$\overline{d}_{\mathrm{EIM}}$	σ_{EIM}	$\overline{d}_{\mathrm{SIM}}$	σ_{SIM}	agr.	disagr.	agr.	disagr.
48	1000	10.545	1.445	10.057	1.122	10.547	1.447	627	373	998	2
212	1000	10.743	1.509	10.604	1.435	–	●	888	112	1000	0
752	1000	10.854	1.543	10.808	1.507	–	●	961	39	1000	0
1024	1000	10.874	1.554	10.854	1.546	–	●	981	19	1000	0
1504	1000	10.781	1.541	10.768	1.536	–	●	988	12	1000	0

TABLE IV

DISTANCES SHOWN HERE ARE FOR THE CODE RATE 1/3. THE m DRP INTERLEAVERS WERE TESTED WITH NORMAL ES AND $W=8$.

						EIM(W=8) vs. true $d_{\min}$		SIM vs. true $d_{\min}$		DIM vs. true $d_{\min}$	
$\overline{K}$	m	$\overline{d}_{\min}$	$\overline{d}_{\mathrm{EIM}}$	$\overline{d}_{\mathrm{SIM}}$	$\overline{d}_{\mathrm{DIM}}$	agr.	disagr.	agr.	disagr.	agr.	disagr.
48	20	22	15.2	51.9	22	0	20	0	20	20	0
212	20	31	23.2	154.9	31	0	20	0	20	20	0
752	20	36	26.4	280.5	36	0	20	0	20	20	0

reliability of the extrinsic information, which in turn helps convergence.

The SIM produced the true $d_{\min}$ in all cases, except for two short interleavers of size 48 symbols. It is interesting to note that the two events (one in ENC1 and one in ENC2) leading to $d_{\min}$ for both interleavers were short events. However, the estimated minimum distance tends to be a tight upper bound on $d_{\min}$ for random interleavers.

Since the SIM is a subset of the DIM, it is enough to test these two interleavers of size 48 symbols with the DIM. The DIM found the true $d_{\min}$ for these two interleavers. Thus, the DIM found the true $d_{\min}$ in all cases.

C. Dithered relative prime interleavers

The four methods were tested with 20 DRP interleavers [11] for the code rate 1/3. All 20 interleavers are unique in the tail-biting sense, meaning that any shift or offset or combination of both to any interleaver does not result in any other interleaver. The EIM used normal ES with $W = 8$.

Table IV shows that EIM and SIM were not able to find the true $d_{\min}$ for any of the tested DRP interleavers. The results show that EIM and SIM are very pessimistic and optimistic, respectively. The DIM was able to find the true $d_{\min}$ for all tested interleavers.

D. Search for DRP interleavers using DIM

The DIM was used to find DRP interleavers of size 752 symbols (MPEG size) for the DVB-RCS standard. The dithered window size (M) [11] was set to 16 symbols. The newly designed DRP interleaver has a $d_{\min}$ of 40, a codeword multiplicity of 1222 and information bit multiplicity of 8084.

Fig. 1 shows simulated error rate results for rate 1/3 turbo codes using 8 iterations. QPSK modulation and an AWGN channel were used. The constituent decoders used enhanced max-log-APP decoding [12] with an extrinsic scale factor of 0.75. Fig. 1 shows an improvement with the DRP interleaver greater than 0.4 dB at FERs below 10^{-6} compared to the DVB-RCS standard interleaver.

IV. Conclusion

Four distance measurement methods were compared based on the DVB-RCS standard interleavers, random interleavers and dithered relative prime (DRP) interleavers. The error-impulse method (EIM) is usually pessimistic, whereas the single-impulse method (SIM) is usually optimistic and always gives a true upper bound on $d_{\min}$. Both methods can be useful for estimating low minimum distances, typically associated with random interleavers. For random interleavers with low distances, the reliability of both methods increases with the size of the interleaver. The double-impulse method (DIM) was able to find the true minimum distance in almost all cases and is a very powerful tool for finding structured interleavers with high minimum distances, such as high-distance DRP interleavers. A new MPEG-sized interleaver was designed for the DVB-RCS standard. The new DRP interleaver performed at least 0.4 dB better than the standard interleaver for frame error rates below 10^{-6}.

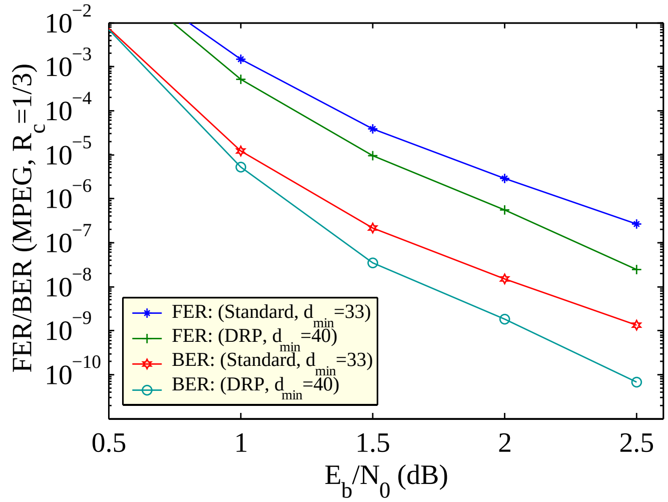

Fig. 1. FER and BER for MPEG packets ($\overline{K}$=752 symbols) of code rate 1/3 (QPSK/AWGN). The number of iterations is 8. Enhanced max-log-APP decoding with a fixed scale factor of 0.75 for the extrinsic was used. The number of packet errors at 2.5 dB were 50 and 15 for the standard and the new DRP interleavers, respectively.

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