Asymptotics of Efficiency Loss in Competitive Market Mechanisms

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Abstract— **We consider the loss of efficiency in competitive market mechanisms used to allocate network resources. We model large heterogeneous populations of users assuming that each user has a random utility function. We show that if the utility functions are bounded, the competitive equilibrium will be nearly as efficient as the social optimum with high probability as the number of users increases. This is the case for inelastic capacity as well as elastic capacity, under some standard assumptions. This result extends to a network setup where sources and destinations are picked at random. If, however, the utility functions are not bounded, then the loss of efficiency does not converge to zero. Collaborating simulations are also presented.**

I. Introduction

A fundamental challenge in the operation of large-scale broadband communication networks is how to utilize the network resource (*e.g.*, bandwidth) efficiently. Sharing the same network bandwidth are heterogeneous users with a range of demand in terms of transmission rate. This gives rise to the need to capture each user's perceived utility from its allocation of the network resource. Furthermore, as a result of the large and sprawling nature of modern networks, it is difficult to manage the resource in a centralized fashion. An alternative approach is to allow the users to compete for the network resource, as a commodity, through a market mechanism.

We compare the efficiency of two paradigms: the first is a centralized resource allocation mechanism where one agent divides the network's resource such that the aggregate utility of all users is maximized, the second is a decentralized market mechanism where users bid selfishly and receive proportional amounts of resource [Kel97]. When efficiency is measured as the aggregate surplus of all the network's users, the centralized allocation mechanism achieves an optimal solution. Kelly [Kel97] shows that the market mechanism also admits a competitive equilibrium with an optimal solution, provided that the users do not anticipate the effect of their bids on the market price. However, there is no reason to assume that users will not anticipate the effect of their bids. Johari *et al.* show in [JT04] and [JMT05] that when users are price-anticipators, the market mechanism admits nevertheless a Nash equilibrium for which the efficiency loss is at most 25% for inelastic supply, and at most approximately 34% for elastic supply (where the users can pay for additional resource). Building upon these encouraging results, we show that, under some standard assumptions, the loss of efficiency actually tends to 0 when the number of potential users is large.

The subject of loss of efficiency between socially optimal and equilibrium outcomes—or price of anarchy, has been widely studied since the work of Koutsoupias and Papadimitriou [KP99]. Roughgarden and Tardos [RT02] show that the total latency experienced by selfish users is at most $4/3$ times the minimum possible total latency in the context of routing. Awerbuch *et al.* [AAE05] give bounds on the price of anarchy when network flows are unsplittable. Empirical study of selfish routing in realistic environments (*e.g.*, the Internet) already suggests that the loss of efficiency is often much less than the worst-case bound [QYZS03]. Friedman [Fri04] shows that the performance degradation due to selfish routing is generally small.

In this paper, we study the efficiency loss due to lack of cooperation among users, which can also be attributed to price-anticipating behavior in a market environment. Hence, we assume that the central agent has perfect knowledge of the true utilities and that every user acts optimally with the knowledge of the sum other users' bids. Loss of efficiency can also be the result of the central agent having a wrong perception of user utilities, some users manipulating their utility function, some users over- or under-bidding; these are not studied here. We will focus on asymptotic results for large populations of random users.

We show that, under some standard conditions on the utility functions and the distribution of the utility functions, the loss of efficiency tends to 0 almost surely as the number of users tends to infinity. Therefore, bigger—more populous—networks are also more efficient. This result holds whether the network's resource has inelastic or elastic supply. We first prove this result for a single link, then show how it encompasses general networks. However, under different assumptions, the reverse situation happens. We show that the loss of efficiency is bounded away from 0 with positive probability for a certain class of distributions over the random utility functions.

The paper is organized as follows. Section II presents the market-based resource allocation mechanism for a single link and the probabilistic setup. Within this setting, Section III

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shows that the efficiency loss tends to 0 as the number of users tends to infinity for bounded utility functions, along with simulation results for illustration. By slightly modifying the setup, Section IV gives situations where the loss of efficiency does not tend to 0. Section V shows that the efficiency loss still converges to 0 if the resource has a soft capacity—or is elastically supplied. Section VI shows how to generalize the previous results to networks. Section VII relates our work to other works in economics. Some conclusions are offered in Section VIII.

II. Background

In this section we provide the essential background on market mechanisms that will be used in the subsequent development. In Section II-A, we list our assumptions concerning the utility functions. In Section II-B, we recall the pricing mechanism, based on [Kel97], for a single link. In Section II-C, we specify the mathematical model for random utility functions.

A. Utility functions

Consider a set of users, labeled $1, \ldots, n$, that share an amount c of a given network resource. Henceforth, we will use the term *commodity* to refer to this resource, whether it is traffic, bandwidth, time-slots, etc. User i is characterized by the utility functions $u_i : [0, \infty) \to [0, \infty)$. We will make the following assumption concerning the utility functions.

Assumption 2.1: The utility functions u_i:

i. are concave and continuously differentiable,
ii. are strictly increasing, with $u_i(0) = 0$,
iii. have finite slope everywhere except possibly at 0, *i.e.*, $u_i'(\epsilon) < \infty, \forall \epsilon > 0$.

Observe that this assumption allows linear utility functions (*cf.* [Kel97] where a strict concavity assumption is made). Traffic that leads to such utility functions is often referred to as *elastic* traffic [She95].

In general, valid utility functions need not have a closed-form expression. We could dispense with the continuous derivative assumption to the detriment of ease of analysis. The strictly increasing condition ensures that the amount of resource c is entirely allocated. The condition $u_i(0) = 0$ is realistic as no commodity would typically imply no benefit. Note that we allow $u_i'(\epsilon) \to \infty$ as $\epsilon \downarrow 0$. For brevity, we will write $u_i'(0)$ for the *initial slope* of user i, *i.e.*, $\lim_{\epsilon \downarrow 0} u_i'(\epsilon)$.

B. Resource allocation mechanism for a single link

We now specify the assumptions of the market mechanism model, for more details, the reader is referred to [Kel97]. For the ease of exposition, the number of users and their utility functions is assumed fixed (and non-random) in this sub-section.

Let x_i be the amount of commodity allocated to user i. An allocation $x_1, \ldots, x_n$ achieves social optimality if it is a solution to the optimization problem

$$\begin{aligned} &\textbf{SYSTEM} \\ &\max_{x_1,\ldots,x_n} \quad \sum_{i=1}^{n} u_i(x_i) \\ &\text{subject to} \quad \sum_{i=1}^{n} x_i \le c, \\ &\qquad\qquad x_i \ge 0, \quad i = 1, \ldots, n. \end{aligned} \tag{1}$$

We call the objective function of the problem SYSTEM the *aggregate utility*. A central agent with knowledge of the utility functions can implement this allocation mechanism.

We will consider the bidding mechanism, introduced by Kelly [Kel97], where the users bid for the resource and receive proportional shares. A general version of the mechanism is described by Shubik [Shu73] and Shapley [Sha76]. Specifically,

1) every user i submits a bid b_i,
2) if $\sum_{i=1}^{n} b_i > 0$, then the price for the commodity is set to
$$\lambda = \frac{\sum_{i=1}^{n} b_i}{c},$$
3) each user receives an amount $x_i = b_i/\lambda$ of the commodity.

In this setting, users can act as *price-takers* or *price-anticipators*. Price-taking users do not anticipate the effect of their bid on the price. Every price-taking user i maximizes his surplus as follows when $\sum_{i=1}^{n} b_i > 0$:

$$\begin{aligned} &\max_{b_i} \quad u_i\left(\frac{b_i}{\lambda}\right) - b_i \\ &\text{subject to} \quad b_i \ge 0. \end{aligned}$$

Every price-anticipating user i maximizes his surplus as follows when $\sum_{i=1}^{n} b_i > 0$:

$$\begin{aligned} &\max_{b_i} \quad u_i\left(\frac{b_i}{b_i + \sum_{j \ne i} b_j}\, c\right) - b_i \\ &\text{subject to} \quad b_i \ge 0. \end{aligned}$$

Kelly [Kel97] showed that there exists a competitive equilibrium when the users behave as price-takers[1]. Moreover, the resulting allocation is an optimal solution to SYSTEM. This effectively establishes the equivalence between two interpretations of social optimum: price-taking bidding equilibrium and centralized control.

Hajek and Gopalakrishnan [HG04] showed the existence of a unique Nash equilibrium under Assumption 2.1 when the users behave as price-anticipators. This Nash equilibrium is the solution to an optimization problem similar to SYSTEM, with modified utility functions.

Theorem 2.1 (Hajek and Gopalakrishnan, [HG04]): Assume that there are $n \ge 2$ users, each with a concave, strictly increasing, and continuously differentiable utility

[1] Kelly's result is under the assumption that the utility functions are *strictly* concave, which is slightly stronger than Assumption 2.1.

function. Then there exists a unique Nash equilibrium, and the resource allocation at the Nash equilibrium is the solution to the following optimization problem:

GAME

$$\begin{aligned} \max_{y_1,\dots,y_n} \quad & \sum_{i=1}^{n} \left[\left(1 - \frac{y_i}{c}\right) u_i(y_i) + \frac{1}{c} \int_0^{y_i} u_i(z)\,\mathrm{d}z \right] \\ \text{subject to} \quad & \sum_{i=1}^{n} y_i \le c, \\ & y_i \ge 0, \quad i = 1,\dots,n. \end{aligned} \tag{2}$$

A Nash equilibrium in this setting is a set of bids such that no user can increase his utility by unilaterally changing his bid. Let the allocation $y_1,\dots,y_n$ arise from bids at a Nash equilibrium between price-anticipating users. We adopt the following notion of *price of anarchy* [JT04], drawn from the analogy between Nash equilibrium and a state where each individual looks exclusively at his own interest:

$$\text{POA} \triangleq \frac{\sum_{i=1}^{n} u_i(y_i)}{\sum_{i=1}^{n} u_i(x_i)}.$$

It is also natural to define the more intuitive notion of *loss of efficiency*:

$$\text{LOE} \triangleq 1 - \text{POA}.$$

Intuitively speaking, this is a measure of loss benefit to the users collectively due to lack of cooperation.

Henceforth, when we write "socially optimal outcome," we mean a solution to SYSTEM, which is equivalent to a competitive equilibrium among price-taking users. When we say "Nash equilibrium outcome," we mean the solution to GAME, which is equivalent to the Nash equilibrium among price-anticipating users.

C. Modeling a population of random users

In this paper, we consider populations of random users. We will study the asymptotics of the loss of efficiency. By *asymptotic behavior* of efficiency loss, we mean its characteristics when the number of users having random utility functions tends to infinity. As we will see, many of the users may actually receive no commodity at the end of the allocation process, so by "user," we really mean "potential user."

We assume that the users enter into play one by one, sequentially. Their utility functions are drawn i.i.d. from the probability space $(\Omega, \mathcal{B}, \Pr)$, where Ω is a set of utility functions satisfying Assumption 2.1 and $\mathcal{B}$ is a σ-field over Ω. Let the constant ν be the supremum of $u'(0)$ among all functions $u \in \Omega$, *i.e.*, $\nu = \sup_{u\in\Omega}\{u'(0)\}$.

We will use U_i to denote the random utility functions and u_i for realizations of U_i which are elements of Ω. The following assumption is our basic assumption concerning $\Pr$, the probability function on Ω.

Assumption 2.2: The probability space satisfies one of the following conditions:

i. $\nu < \infty$,
ii. $\nu = \infty$ and $\Pr(U_i'(0) = \infty) > \delta$ for some $\delta > 0$, *i.e.*, there is a probability mass on the event $\{U_i'(0) = \infty\}$.

These conditions guarantee that there is always some probability mass on utility functions whose initial slope is near u. Note that given that Assumption 2.1 holds, the users cannot have a constant zero utility function.

Let us recursively generate the set of random utility functions for users $1,\dots,n$ as follows:

$$\begin{aligned} \mathbf{U}^{(1)} &\triangleq \{U_1\}, \\ \mathbf{U}^{(n)} &= \mathbf{U}^{(n-1)} \cup \{U_n\}, \quad n \ge 2. \end{aligned}$$

We will consider what happens when we let $n \to \infty$. Observe that, here and later, when we say that the number of users tends to infinity or write $n \to \infty$, we mean that additional users appear while formerly present users remain.

A note about notation is due. We will generally denote random variables with capital letters (*e.g.*, X_i, Y_i), and their realizations with small letters (*e.g.*, x_i, y_i). We will suppress the index $^{(n)}$ when obvious. The only exceptions to this rule are random variables denoted by Greek letters.

It would be convenient to consider the randomization mechanism where a scalar random variable multiplies a certain utility function. In that case,

$$U_i(x) = S_i\, u(x),$$

where S_i are assumed to be random scalars, and u is a deterministic utility function that satisfies Assumption 2.1. We further assume that sequence $S_1,\dots,S_n$ is independent, identically distributed, drawn according to some discrete or continuous probability distribution. We will call such a randomization mechanism a *scalar modulation* of the utility function u. We now present a scalar modulation example that satisfy Assumption 2.2, followed by an example that does not satisfy the assumption.

Example 2.1 (Modulation satisfying Assumption 2.2): Suppose that S_i has bounded support and that $u(x) = x$. It is easily verified that $U_i(x) = S_i\, x$ satisfies Assumption 2.2. Same holds for $U_i(x) = S_i \log(1 + x)$. In the case of $u(x) = \sqrt{x}$, we do not even have to assume that S_i is bounded. All that is needed is to assume that S_i is not identically zero. In that case, $U_i(x) = S_i \sqrt{x}$ satisfies Assumption 2.2 since the probability of having infinite slope at 0 is not zero and equals $\Pr(S_i \neq 0)$.

Example 2.2 (Modulation violating Assumption 2.2): The case of $U_i(x) = |S_i|\, x$ with S_i drawn from a Gaussian or exponential distribution violate Assumption 2.2 because

$$\Pr(U_i'(0) = \nu) = \Pr(S_i = \infty) = 0.$$

III. Asymptotic behavior of efficiency loss for a single link with hard capacity

In this section we establish the convergence of the loss of efficiency to zero under the model and conditions set forth in Section II. Our main theorem is stated under rather broad conditions and only requires that Assumptions 2.1 and 2.2 hold. We illustrate the result using simulation where we draw utility functions randomly from various probability distributions.

A. Loss of efficiency convergence theorem

In this section we prove our basic results concerning the convergence of the loss of efficiency. We start with the following definition.

Definition 3.1 (Active Users): The *active users* are the users that receive non-zero allocation. We will employ this term for the socially optimal outcome or the Nash equilibrium, depending on the context.

Theorem 3.1 (Convergence of loss of efficiency): Consider a resource allocation game with concave utility functions satisfying Assumption 2.1. The utility functions are drawn i.i.d. from a distribution satisfying Assumption 2.2. Then the loss of efficiency tends to 0 almost surely as the number of users tends to infinity.

Johari shows in [Joh04, Corollary 2.8] that if an infinite number of users each receive an infinitesimal fraction of the commodity, then the loss of efficiency approaches 0. Our proof bears some similarities.

Proof: The proof proceeds in five main steps. In the first step, we express the optimality conditions for social optimum and Nash equilibrium. In the second step, we show that the price of the commodity at the Nash equilibrium outcome converges (to a random variable) as the number of users tends to infinity. In the third step, we show that the limit is almost surely the supremum of $\{u_i'(0)\}$. In the fourth step, we show that the quantity of commodity allocated to each active user tends to 0 asymptotically. Finally, we show that the loss of efficiency converges to 0 almost surely. We consider only the case where $\nu < \infty$. The case where there is a probability mass at $\nu = \infty$ follows a similar argument, we outline the proof at the end.

Step 1: Derivation of optimality conditions for social optimum and Nash equilibrium given a fixed number of users with fixed utility functions.

First, consider the social optimization SYSTEM (1) with a fixed n. The complementary slackness conditions are:

- if $x_i^{(n)} > 0$, then $\lambda^{(n)} x_i^{(n)} = u_i(x_i^{(n)})$,
- if $x_i^{(n)} = 0$, then $\lambda^{(n)} x_i^{(n)} \geq u_i(x_i^{(n)})$,
- if $\lambda^{(n)} > 0$, then $\sum_{i=1}^n x_i^{(n)} = c$,
- if $\lambda^{(n)} = 0$, then $\sum_{i=1}^n x_i^{(n)} \leq c$,

where $\lambda^{(n)}$ is the Lagrange multiplier of the capacity constraints (*cf.* [Kel97] and [JT04]). Observe that $\lambda^{(n)}$ must be strictly positive because of the first condition. By taking derivatives, we get the following optimality conditions.

- For all i such that $x_i^{(n)} > 0$, we have $u_i'(x_i^{(n)}) = \lambda^{(n)}$. In other words, all active users' utility functions have the same slope $\lambda^{(n)}$ at their respective allocation levels $x_i^{(n)}$. This condition is guaranteed by the concavity and continuous first derivative of $u_i(x_i^{(n)})$.
- For all i such that $x_i^{(n)} = 0$, we have $u_i'(0) \leq \lambda^{(n)}$.
- $\sum_{i=1}^n x_i^{(n)} = c$. This is true because of the strictly increasing utility assumption.

Note that a user i is active if and only if its initial slope is strictly greater than $\lambda^{(n)}$, *i.e.*, $u_i'(0) > \lambda^{(n)}$, because of the concave utility assumption. Therefore, $\lambda^{(n)}$ represents the shadow price of the commodity.

Similarly, let $y_i^{(n)}$ denote the solution to the Nash equilibrium optimization problem GAME (2). Consider the modified utility function from GAME:

$$\tilde{U}_i(y_i) = \left(1 - \frac{y_i}{c}\right) u_i(y_i) + \frac{1}{c}\int_0^{y_i} u_i(z)\,\mathrm{d}z.$$

It is easily verified that $\tilde{U}_i$ also satisfies all the criteria of Assumption 2.1. Therefore, we can derive optimality conditions similar to the above.

- For all i such that $y_i^{(n)} > 0$, we have
$$\left(1 - \frac{y_i^{(n)}}{c}\right) u_i'(y_i^{(n)}) = \mu^{(n)},$$
where $\mu^{(n)}$ is the analogue of $\lambda^{(n)}$. This follows from taking the derivative of $\tilde{U}_i(y_i)$.
- For all i such that $y_i^{(n)} = 0$, we have $u_i'(0) \leq \mu^{(n)}$.
- $\sum_{i=1}^n y_i^{(n)} = c$.

Notice the similarity between the two sets of optimality conditions for the socially optimal and equilibrium outcomes. We will show that indeed, the optimality conditions are the same as $n \to \infty$.

In the following steps, we allow the utility functions to be random and let n tend to infinity. Consequently, $\lambda^{(n)}$ becomes a function of $\mathbf{V}^{(n)}$, but we will still use $\lambda^{(n)}$ as a short-hand notation for $\lambda\left(\mathbf{V}^{(n)}\right)$. The same remark holds for the optimal value of the decision variables $Y_i^{(n)}$ with realizations denoted $y_i^{(n)}$.

Step 2: Show that the limit $\lim_{n\to\infty} \mu^{(n)}$ exists.

Observe that the sequence $\mu^{(n)}$ is monotone increasing because one user is added at each increment of n. Moreover, the first optimality condition implies that the sequence $\mu^{(n)}$ is bounded from above by ν. Suppose on the contrary that the cost decreases when we add user $n+1$, that is $\mu^{(n+1)} < \mu^{(n)}$. In this case, the allocated commodity for each user increases by the concavity and continuous first-derivative assumptions. This is impossible because one of the optimality conditions requires that c be entirely allocated for every n. Therefore, the random variable $\hat{\mu} \triangleq \lim_{n\to\infty} \mu^{(n)}$ exists.

Step 3: Show that $\mu^{(n)} \to \nu$ almost surely by contradiction.

We show that if $\mu^{(n)}$ does not converge to ν, the number of users allocated a non-negligible amount of resource tends to infinity. Fix $\epsilon > 0$ and let $K^{(n)}$ denote the number of users with $u_i'(0) > \nu - \frac{\epsilon}{2}$, then

$$\Pr\left(U_i'(0) > \nu - \frac{\epsilon}{2}\right) > 0,$$

by Assumption 2.2. Moreover, by the strong law of large numbers,

$$\frac{K^{(n)}}{n} \xrightarrow{\text{a.s.}} \Pr\left(U_i'(0) > \nu - \frac{\epsilon}{2}\right). \tag{3}$$

Suppose, on the contrary, that there exist $\epsilon > 0$ and $\delta > 0$ such that $\Pr(\hat{\mu} < \nu - \epsilon) > \delta$. Consider the event $\{\hat{\mu} < \nu - \epsilon\}$.

By definition, there must be at least $K^{(n)}$ active users. For these $K^{(n)}$ active users, the optimality conditions require that

$$\left(1 - \frac{y_i^{(n)}}{c}\right) u_i'(y_i^{(n)}) = \mu^{(n)}.$$

In turn, this requires that $Y_i^{(n)}$ be bounded away from 0 with some probability $\zeta > 0$. Otherwise, if $Y_i^{(n)} \to 0$ almost surely, contradiction ensues because the left-hand side tends almost surely to $u_i'(0) > \nu - \frac{\epsilon}{2}$, whereas the right-hand side tends to $\hat{\mu} < \nu - \epsilon$. In summary, we have shown that there are $K^{(n)}$ active users for whom $Y_i^{(n)}$ is bounded away from 0 with probability $\zeta > 0$. Since $K^{(n)} \to \infty$ as $n \to \infty$ by Equation (3), this contradicts the fact that we have a finite amount of resource c. Therefore, $\hat{\mu} = \nu$ almost surely.

Step 4: Show that $Y_i^{(n)} \to 0$ for every active user i.

Consider again the optimality condition for active users:

$$\left(1 - \frac{Y_i^{(n)}}{c}\right) U_i'(Y_i^{(n)}) = \mu^{(n)}.$$

In the limit as $n \to \infty$, the right-hand side converges to ν almost surely. The left-hand side is monotone decreasing in $Y_i^{(n)}$ and upper-bounded by ν, so $Y_i^{(n)}$ must converge to 0 almost surely as $n \to \infty$.

Step 5: Show that the efficiency loss converges almost surely to 0.

Finally, we see that the optimality conditions for the socially optimal and Nash equilibrium outcomes coincide when $Y_i^{(n)} \to 0$ almost surely because

$$\left(1 - \frac{Y_i^{(n)}}{c}\right) U_i'(Y_i^{(n)}) \to U_i'(Y_i^{(n)}).$$

Since the optimality conditions coincide in the limit, the solutions of the two problems also coincide. It follows that the loss of efficiency tends to 0 almost surely.

For the case $\nu = \infty$, we can verify that the sequence $\mu^{(n)}$ is monotone increasing, and that, similarly to Step 3, $\mu^{(n)} \to \infty$ almost surely as $n \to \infty$. Therefore, as in Step 4, $Y_i^{(n)} \to 0$ almost surely. It follows that the optimality conditions coincide and the loss of efficiency converges to 0 almost surely. ∎

B. Simulations with random utility functions

To validate the results, we simulate the loss of efficiency for different numbers of users. For scalar modulated linear utility function, Figure 1 shows that the mean loss of efficiency, as well its empirical variance, decrease as the number of users increases. Likewise, Figures 2 and 3 show the simulated the loss of efficiency for scalar modulated non-linear utility functions. Since there is no closed-form solution for these utility functions, we resort to generic optimization algorithms, which cause prohibitively long simulation times for large numbers of variables. For this reason, we restrict the number of users to be small.

We have shown in this section that, with a certain class of random utility functions, a decentralized market mechanism can achieve the same asymptotic efficiency as a central authority enforcing a socially optimal resource allocation. The economic interpretation is that price-anticipating behavior has little effect on efficiency in large networks where resource is allocated using the market mechanism of Section II-B.

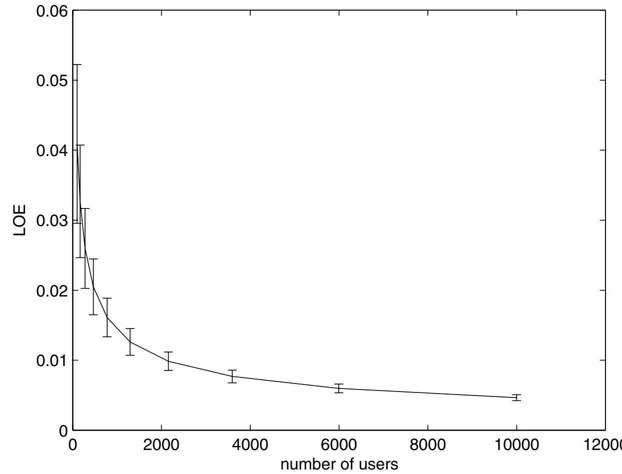

Fig. 1. LOE versus n for $S_i\, x$ utility functions and S_i sampled uniformly in $[0, 1]$. The error bars have width equal to one standard deviation.

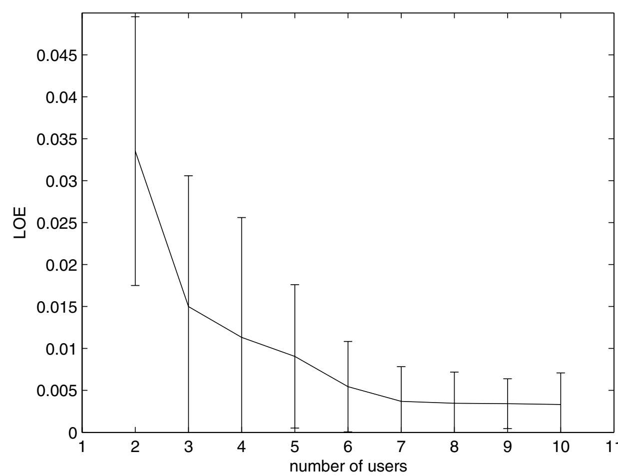

Fig. 2. LOE versus n for $S_i\,\sqrt{x}$ utility functions and S_i sampled uniformly in $[0, 1]$.

IV. Situations where the loss of efficiency does not tend to zero

We show that the loss of efficiency does not tend almost surely to 0 in the case of linear utility functions $U_i(x) = S_i\, x$, with slopes S_i drawn i.i.d. from some class of distributions with support unbounded from above. The case of linear utility functions is interesting because it lies at the "boundary" of the class of concave functions of Assumption 2.1. For this task, we first relate the loss of efficiency to a ratio of order statistics, then apply results about the asymptotic distribution of order

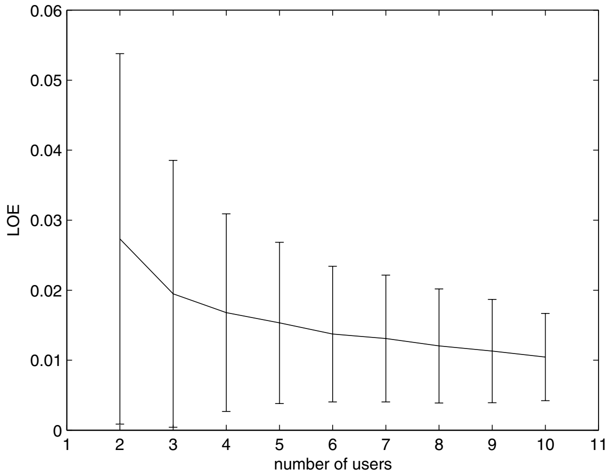

Fig. 3. LOE versus n for $S_i\, x/(1+x)$ utility functions and S_i sampled uniformly in $[0,1]$.

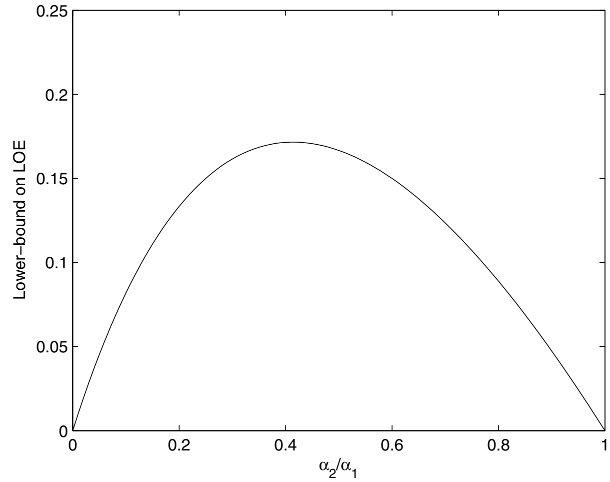

Fig. 4. Lower-bound on LOE as a function of $\alpha_2^{(n)}/\alpha_1^{(n)}$.

statistics. The theory of order statistics is treated in [DN03] and [LLR83] among other places.

Definition 4.1: Let $\mathbf{S}^{(n)} \triangleq \{S_1, \ldots, S_n\}$. The order statistics $\alpha_1(\mathbf{S}^{(n)})$ and $\alpha_2(\mathbf{S}^{(n)})$ are the largest and second-largest elements of $\mathbf{S}^{(n)}$. For brevity sake, we will shorten $\alpha_i(\mathbf{S}^{(n)})$ to $\alpha_i^{(n)}$.

It can be verified that when every S_i has unbounded support, the sequences of highest slopes, $\alpha_1(\mathbf{S}^{(n)})$, $\alpha_2(\mathbf{S}^{(n)})$, and the sequence of shadow prices $\lambda(\mathbf{S}^{(n)})$ do not converge as n tends to infinity. This prompts us to study and describe the asymptotic behavior of the efficiency loss in terms of the ratio of the two highest slopes: $\alpha_1^{(n)}/\alpha_2^{(n)}$.

Lemma 4.1 (Bound on LOE as a function of $\alpha_2^{(n)}/\alpha_1^{(n)}$): Suppose that the utility functions are of the form $u_i(x) = t_i\, x$, $t_i > 0$ and for $n \geq 2$. The efficiency loss is bounded from below as follows:

$$\text{LOE}^{(n)} \geq \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \frac{\left(1 - \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}{\left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}, \quad \forall n \geq 2.$$

Proof: The result is algebraic in nature, and will be proved for any n and any realization of the random quantities involved. We let $s_1, \ldots, s_n$ denote realizations of $S_1, \ldots, S_n$. At the outset, observe that for linear utility functions, the aggregate utility at the Nash equilibrium is highest when there are fewest active users. Hence, the efficiency loss is smallest when there are fewest (*i.e.*, two) active users, which occurs when

$$\alpha_1^{(n)} \geq \alpha_2^{(n)} > \mu^{(n)} > s_j,\ \forall s_j \in \{s_1, \ldots, s_n\} \setminus \{\alpha_1^{(n)}, \alpha_2^{(n)}\}.$$

In this case, the optimality conditions (as in Step 1 of Theorem 3.1) yield the set of equations:

$$\left(1 - x_1^{(n)}\right) \alpha_1^{(n)} = \mu^{(n)},$$
$$\left(1 - x_2^{(n)}\right) \alpha_2^{(n)} = \mu^{(n)},$$
$$x_1^{(n)} + x_2^{(n)} = c.$$

From simple calculations, we obtain

$$x_1^{(n)} = \left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^{-1} c,$$
$$x_2^{(n)} = \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^{-1} c.$$

From the observation at the outset, it follows that

$$\text{POA}^{(n)} \leq \frac{1 + \left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)^2}{1 + \left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}, \quad \forall n \geq 2,$$

$$\iff \text{LOE}^{(n)} \geq \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} \frac{\left(1 - \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}{\left(1 + \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}}\right)}, \quad \forall n \geq 2. \qquad (4)$$

■

The lower-bound on the loss of efficiency from Equation (4) is plotted in Figure 4.

Corollary 4.2 (Necessary condition for zero LOE): A necessary condition for the loss of efficiency to tend almost surely to 0 is:

$$\liminf_{n\to\infty} \Pr\left(\frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} = 0 \text{ or } 1\right) = 1.$$

Our next aim is to derive asymptotic characteristics of the ratio $\alpha_2^{(n)}/\alpha_1^{(n)}$ using the theory of order statistics. The largest and second-largest slopes, $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$, are said to be the first and second maximal *order statistics* of the set $\mathbf{S}^{(n)}$. Given the deterministic sequences $\{l_n \mid l_n > 0\}$ and $\{m_n\}$, we define the *linearly normalized order statistics* [LLR83]:

$$\rho_1^{(n)} \triangleq l_n\,(\alpha_1^{(n)} - m_n),$$
$$\rho_2^{(n)} \triangleq l_n\,(\alpha_2^{(n)} - m_n).$$

These linear transformations are indispensable in order to avoid degenerate distributions as $n \to \infty$. The following theorem, due to Gnedenko, describes the asymptotic distribution of the suitably normalized maximal order statistic.

Theorem 4.3 (Extremal types, [LLR83]): Let S_i be independent random variables drawn from a distribution $F(z)$ and $\alpha_1^{(n)} = \max(S_1, \ldots, S_n)$. If there exist deterministic sequences $\{l_n \mid l_n > 0\}$ and $\{m_n\}$ such that the normalized maximal order statistic $\rho_1^{(n)}$ converges in distribution, *i.e.*,

$$\Pr\left(\rho_1^{(n)} \le z\right) \xrightarrow[n\to\infty]{} G(z),$$

for some non-degenerate G, then G belongs to one of the following three families of extreme value distributions,[2] for some $\theta > 0$:

$$\text{(Gumbel)} \quad G_1(z) = e^{-e^{-z}}, \quad z \in \mathbb{R},$$
$$\text{(Fréchet)} \quad G_2(z;\theta) = \begin{cases} e^{-z^{-\theta}}, & z > 0, \\ 0, & z \le 0, \end{cases}$$
$$\text{(Weibull)} \quad G_3(z;\theta) = \begin{cases} e^{-(-z)^{\theta}}, & z \le 0, \\ 1, & z > 0. \end{cases}$$

Furthermore, we say that $F(z)$ belongs to the domain of attraction of the corresponding extreme value distribution.

The next theorem gives the necessary and sufficient conditions for a cumulative probability distribution to belong to the domain of attraction of each of the three extreme value distributions.

Theorem 4.4 (Domains of attraction, [LLR83]): Let $F(z)$ be the distribution function of the sequence of i.i.d. random variables $\{S_i\}$. Let us denote the upper-bound of the support by $z_F \triangleq \sup\{z \mid F(z) < 1\}$. Let us write $F \in \mathcal{D}(G_i)$ when F belongs to the domain of attraction of G_i, $i = 1, 2, 3$. Then,

- $F \in \mathcal{D}(G_1)$ if and only if there exists a strictly positive function $g(t)$ such that
$$\lim_{t\uparrow z_F} \frac{1 - F(t + g(t)z)}{1 - F(t)} = e^{-z}, \text{ for all } z \in \mathbb{R}. \quad (5)$$
- $F \in \mathcal{D}(G_2)$ if and only if $z_F = \infty$ and there exists $\theta > 0$ such that
$$\lim_{t\to\infty} \frac{1 - F(tz)}{1 - F(t)} = z^{-\theta}, \text{ for all } z > 0. \quad (6)$$
- $F \in \mathcal{D}(G_3)$ if and only if $z_F < \infty$ and there exists $\theta > 0$ such that
$$\lim_{t\downarrow 0} \frac{1 - F(z_F - tz)}{1 - F(z_F - t)} = z^{\theta}, \text{ for all } z > 0. \quad (7)$$

[2]Also known as max-stable distributions.

This theorem tells us that the slopes S_i drawn from a distribution $F \in \mathcal{D}(G_3)$ must be bounded from above. This case is covered by Theorem 3.1.

Example 4.1 (Pareto distribution): Consider the Pareto distribution function $F(z) = 1 - \kappa\, z^{-\theta}$, with parameters $\theta, \kappa > 0$, and with support $[\kappa^{1/\theta}, \infty)$. If S_i has a Pareto distribution, then it is easily verified that the distribution of utility functions $U_i(x) = S_i\, x$ does not satisfy Assumption 2.2: we have $\nu = \infty$, but $\Pr(U_i'(0) = \infty) = 0$. If we set $l_n = (\kappa\, n)^{-1/\theta}$ and $m_n = 0$, we obtain [LLR83]:

$$\Pr\left((\kappa\, n)^{-1/\theta}\, \alpha_1^{(n)} \le z\right) \xrightarrow[n\to\infty]{} G_2(z;\theta).$$

In other words, $F \in \mathcal{D}(G_2)$.

For a fixed parameter $\theta > 0$, which is suppressed to lighten notation, we can derive the following limiting distributions characteristic of the Fréchet extreme value distribution [LLR83]:

$$F_{\rho_1}(z_1) = e^{-z_1^{-\theta}}, \quad z_1 > 0,$$

and the following joint, marginal, and conditional probability density functions [LLR83]:

$$\begin{aligned} f_{\rho_1,\rho_2}(z_1, z_2) &= \theta^2 (z_1\, z_2)^{-1-\theta} e^{-z_2^{-\theta}}, \quad 0 < z_2 < z_1, \\ f_{\rho_1}(z_1) &= \theta\, z_1^{-1-\theta}\, e^{-z_1^{-\theta}}, \quad z_1 > 0, \\ f_{\rho_2|\rho_1}(z_2, z_1) &= \theta z_2^{-1-\theta} e^{-z_2^{-\theta}} e^{z_1^{-\theta}}, \quad 0 < z_2 < z_1. \end{aligned} \quad (8)$$

It also follows that

$$F_{\rho_2|\rho_1}(z_2, z_1) = e^{-z_2^{-\theta}} e^{z_1^{-\theta}}. \quad (9)$$

By conditioning and using the above relations, we can bound from below the probability that $\alpha_2^{(n)}/\alpha_1^{(n)}$ is between ϵ and $1 - \epsilon$ in the limit $n \to \infty$, for $\epsilon \in (0, 1/2)$. Since $\rho_i^{(n)} = (\kappa\, n)^{-1/\theta}\, \alpha_i^{(n)}$, for $i = 1, 2$, we have

$$\begin{aligned} &\liminf_{n\to\infty} \Pr\left(\epsilon < \frac{\alpha_2^{(n)}}{\alpha_1^{(n)}} < 1 - \epsilon\right) \\ &= \liminf_{n\to\infty} \Pr\left(\epsilon < \frac{\rho_2^{(n)}}{\rho_1^{(n)}} < 1 - \epsilon\right) \\ &= \liminf_{n\to\infty} \int_0^\infty \Pr\left(\epsilon z_1 < \rho_2^{(n)} \le (1-\epsilon) z_1 \mid \rho_1^{(n)} = z_1\right) \\ &\qquad \mathrm{d}\big(\Pr(\rho_1^{(n)} = z_1)\big) \\ &\text{(by Fatou's Lemma)} \\ &\ge \int_0^\infty \left[F_{\rho_2|\rho_1}((1-\epsilon) z_1, z_1) - F_{\rho_2|\rho_1}(\epsilon z_1, z_1)\right] f_{\rho_1}(z_1)\, \mathrm{d}z_1 \\ &\text{(by Equations (9) and (8))} \\ &\ge \int_0^\infty \left[e^{-(1-\epsilon)^{-\theta} z_1^{-\theta}} - e^{-\epsilon^{-\theta} z_1^{-\theta}}\right] e^{z_1^{-\theta}} \theta z_1^{-1-\theta} e^{-z_1^{-\theta}} \mathrm{d}z_1. \end{aligned}$$

Since the above integrand is positive for $0 < \epsilon < 1/2$, we conclude that in the limit as $n \to \infty$, the ratio $\alpha_2^{(n)}/\alpha_1^{(n)}$ is bounded away from both 0 and 1 with some positive probability. In this case, by Corollary 4.2, the expected efficiency loss is bounded away from 0.

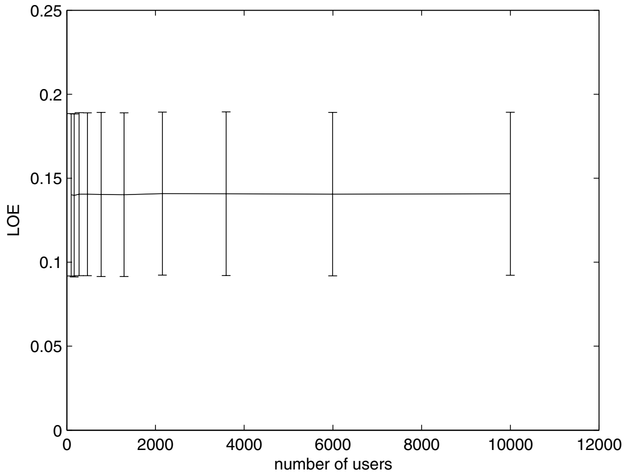

Fig. 5. LOE versus n for $S_i\,x$ utility functions and S_i sampled with Pareto distribution ($\theta = 2$).

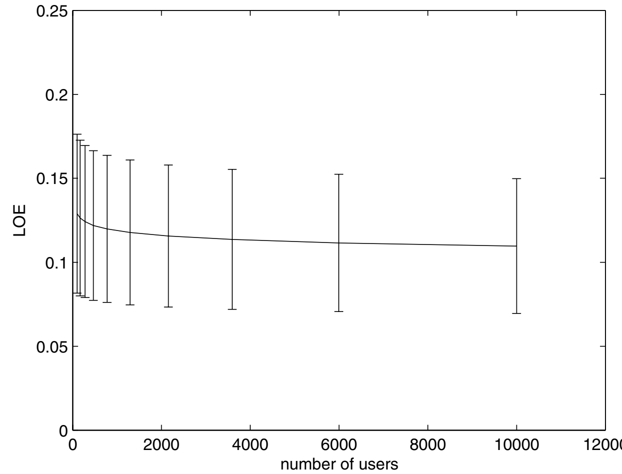

Fig. 7. LOE versus n for $S_i\,x$ utility functions, S_i sampled with exponential distribution.

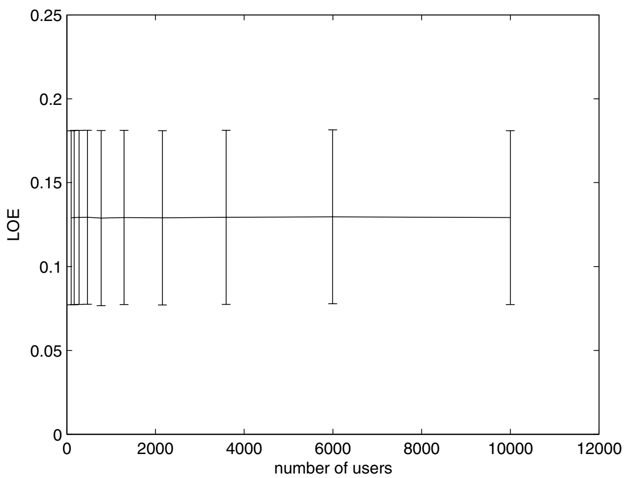

Fig. 6. LOE versus n for $S_i\,x$ utility functions and S_i sampled with Cauchy distribution ($\theta = 1$).

To obtain numerical estimates for the asymptotic lower-bound on the loss of efficiency, we rely on simulations results such as Figure 5. We note that the standard deviation of the loss of efficiency does not decrease to 0. This implies that even if the number of users is large, significant variation in the loss of efficiency is expected.

Example 4.2 (Cauchy distribution): The preceding results for Pareto distribution readily extend to the case of one-sided Cauchy distribution, where $F(z) = \frac{2}{\pi}\arctan(z)$, $z > 0$, with $l_n = \tan\left(\frac{\pi}{n}\right)$ and $m_n = 0$ [LLR83]. In this case, $F \in \mathcal{D}(G_2)$ with the parameter $\theta = 1$. Similarly to the Pareto distribution, the loss of efficiency does not converge to 0, nor its empirical standard deviation (see Figure 6).

In fact, for every Fréchet-type random variable, we can find a linear transformation with $m_n = 0$ [DN03]. Therefore, the same result as for Pareto distribution example holds, which we summarize in the following corollary.

Corollary 4.5 (Non-zero asymptotic loss of efficiency): For scalar-modulated linear utility functions, $U_i(x) = S_i\,x$, with S_i drawn i.i.d. from some distribution $F \in \mathcal{D}(G_2)$, the loss of efficiency does not tend to 0 almost surely as n increases to infinity.

The proof follows the same lines as Example 4.1.

Let us turn our attention to the other two extreme value distributions. By definition, Weibull-type random variables have finite support (Equation (7)); hence, Theorem 3.1 holds. For Gumbel-type random variables such as exponential and Gaussian random variables, difficulties arise because the normalizing sequences m_n increase with n [DN03]. It is not clear if we can prove a similar result. Furthermore, notice that the Gumbel-type distributions have a light tail (Equation (5)) whereas Fréchet-type distributions have a heavy tail (Equation (6)). Nonetheless, simulations suggest that even for Gumbel-type random variables, a non-zero loss of efficiency can be expected (see Figure 7).

Figure 8 shows the relative frequency for pairs of values $(\text{LOE}, \alpha_1/\alpha_2)$ in a simulation where the user slopes S_i are drawn at random according to the exponential distribution $F(z) = 1 - e^{-z}$, $z > 0$. The distinctions between Gumbel-type (*e.g.*, exponential, Gaussian) and Fréchet-type (*e.g.*, Pareto, Cauchy) random variables is further emphasized by inspecting Figures 8, 9, and 10. They also highlight the empirical difference in the distribution of α_2/α_1 for various distributions of utility functions. For Fréchet-type distributions, there is a noticeably broader range of values taken by α_2/α_1. In Figures 8, 9 and 10, we can also clearly observe the outline of the lower-bound plotted in Figure 4.

V. Single link with elastic supply

In this section we consider the case of elastic supply or soft capacity. We introduce a cost function $\mathrm{cost}(c)$ that reflects the

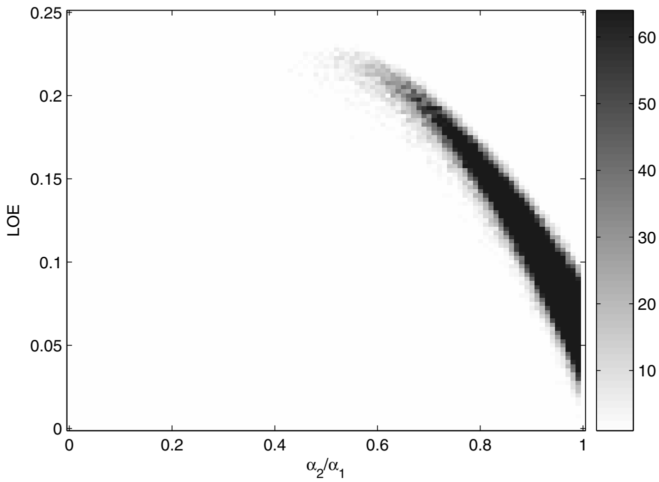

Fig. 8. Empirical joint relative frequency of LOE and α_2/α_1 out of 5×10^4 experiments for $S_i\, x$ utility functions, S_i sampled with exponential distribution, and $n = 10^3$ users.

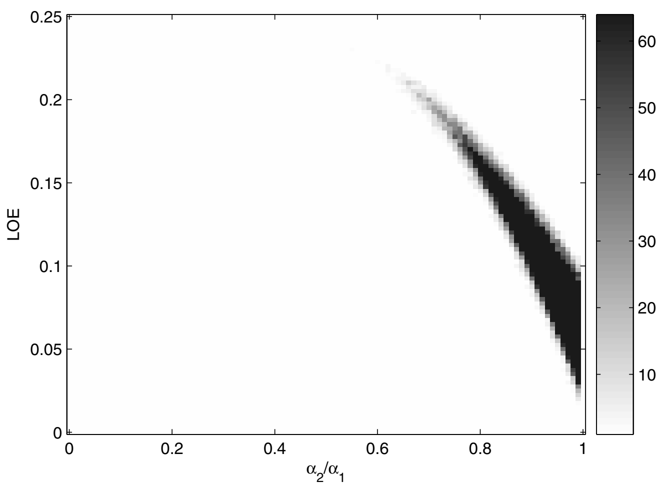

Fig. 9. Empirical joint relative frequency of LOE and α_2/α_1 out of 5×10^4 experiments for $S_i\, x$ utility functions, S_i sampled with Gaussian distribution, and $n = 10^3$ users.

cost incurred to all the users when c units of commodity are produced.

Assumption 5.1: There exists a price function $p(c)$ such that

$$\mathrm{cost}(c) = \int_0^c p(z)\,\mathrm{d}z.$$

Moreover, $p(c) : (0, \infty) \to (0, \infty)$ is convex and differentiable, and $p'(c) < \infty$ for every $c < \infty$.

A. Socially optimal outcome

In the soft capacity case, the socially optimal outcome, which maximizes the aggregate surplus, is the solution to the following problem [JMT05]:

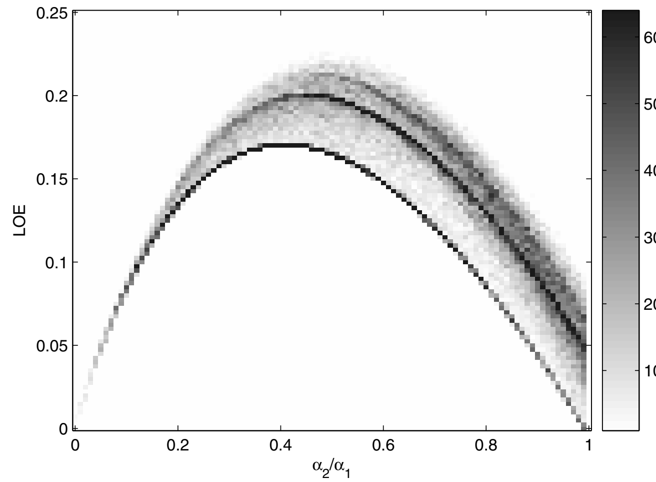

Fig. 10. Empirical joint relative frequency of LOE and α_2/α_1 out of 5×10^4 experiments for $S_i\, x$ utility functions, S_i sampled with Pareto distribution, and $n = 10^3$ users.

$$\begin{aligned} \max_{x_1,\ldots,x_n} \quad & \sum_{i=1}^{n} u_i(x_i) - \mathrm{cost}\left(\sum_{i=1}^{n} x_i\right) \\ \text{subject to} \quad & x_i \geq 0, \quad i = 1, \ldots, n, \end{aligned}$$

where the objective function is the *aggregate surplus*. The optimality conditions for this problem are:

- if $x_i^{(n)} > 0$, then $u_i'(x_i^{(n)}) = p(c^{(n)})$,
- if $x_i^{(n)} = 0$, then $u_i'(0) \leq p(c^{(n)})$,
- $\sum_{i=1}^{n} x_i^{(n)} = c^{(n)}$.

Kelly *et al.* [KMT98] present a bidding mechanism that, when combined with the assumption that users do not anticipate the effect of their bids on the price, maximizes the aggregate surplus. In this mechanism, every user submits a bid as before, but the pricing mechanism changes. When $\sum_{i=1}^{n} b_i > 0$, the selling price per unit of the commodity, λ, is set to satisfy the equation

$$\lambda = p\left(\frac{\sum_{i=1}^{n} b_i}{\lambda}\right) = p(c^{(n)}). \tag{10}$$

Every user receives a proportional share of the commodity as before: $x_i^{(n)} = b_i/\lambda$. For more details about the bidding, pricing and allocation mechanism, the reader is referred to [JMT05] and [KMT98].

When $\sum_{i=1}^{n} b_i > 0$, every price-taking user i maximizes his surplus as follows:

$$\begin{aligned} \max_{b_i} \quad & u_i\left(\frac{b_i}{\lambda}\right) - b_i \\ \text{subject to} \quad & b_i \geq 0. \end{aligned} \tag{11}$$

The variable λ is taken as given and assumed independent of b_i. Kelly *et al.* [KMT98] showed that when the users behave as price-takers, there exists a set of bids that is optimal for every individual user and that maximizes the aggregate surplus.

B. Nash equilibrium outcome

In contrast to the price-taking situation, every user i can instead anticipate the price and maximize his surplus as follows (assuming that $\sum_{i=1}^n b_i > 0$):

$$\begin{aligned} \max_{b_i} \quad & u_i\left(\frac{b_i}{p(d)}\right) - b_i \\ \text{subject to} \quad & b_i + \sum_{j\neq i} b_j = d \cdot p(d), \\ & b_i \geq 0, \end{aligned} \tag{12}$$

where d is used to denote the amount of commodity. This amount may be different from the amount c allocated in the social optimum. Under Assumption 5.1, Johari *et al.* [JMT05] show the existence of a unique Nash equilibrium set of bids for the case of price-anticipating users. It is also shown that the necessary and sufficient conditions characterizing the Nash equilibrium allocation $(y_1^{(n)}, \ldots, y_n^{(n)})$ are:

- if $y_i^{(n)} > 0$, then
$$\left(1 - \frac{y_i^{(n)}\, p'(d^{(n)})}{p(d^{(n)}) + d^{(n)}\, p'(d^{(n)})}\right) u_i'(y_i^{(n)}) = p(d^{(n)}),$$
- if $y_i^{(n)} = 0$, then $u_i'(0) \leq p(d^{(n)})$,
- $\sum_{i=1}^n y_i^{(n)} = d^{(n)}$.

Moreover, the loss of efficiency, defined in terms of aggregate *surplus* instead of aggregate utility, is at most $6 - 4\sqrt{2}$ [JMT05].

C. Convergence of efficiency loss

Theorem 5.1 (Convergence of LOE with elastic supply): Under Assumptions 2.1, 2.2, and 5.1, the loss of efficiency converges to 0 almost surely as the number of users tends to infinity.

Proof: (Outline) Let $D^{(n)}$ be the random variable for the total quantity of commodity allocated at the Nash equilibrium outcome, and let $d^{(n)}$ denote its realization. Analogously to the proof of Theorem 3.1, we first show that $p(D^{(n)})$ converges almost surely to the supremum of $\{u_i'(0)\}$ as $n \to \infty$. As a second step, we show that the quantity of commodity allocated to each active user tends to 0 as $n \to \infty$. Then, we show that the necessary and sufficient conditions for the social optimal outcome and the Nash equilibrium outcome coincide as $n \to \infty$, from which follows the fact that the loss of efficiency converges to 0 almost surely. We first consider the case $\nu < \infty$ of Assumption 2.2. The proof for the case $\nu = \infty$ follows by a similar argument.

Step 1: Show that $\lim_{n\to\infty} p(D^{(n)})$ exists and equals ν, the supremum of $\{u_i'(0)\}$.

It can be shown that the sequence $D^{(n)}$ is monotone increasing, and that it is bounded when $\nu < \infty$. Hence, it has a finite limit $\hat{D}$. It follows, by continuity of p, that $p(D^{(n)})$ has a finite limit $p(\hat{D})$. For a fixed $\epsilon > 0$, let $K^{(n)}$ be the number of users with $u_i'(0) > \nu - \frac{\epsilon}{2}$, then there exists some $\gamma > 0$ so that $K^{(n)}/n \to \gamma$ as $n \to \infty$ by Assumption 2.2 and the law of large numbers.

Next, we prove that $p(D^{(n)}) \to \nu$ almost surely. Observe that since the limit random variable $p(\hat{D})$ exists and since $p(D^{(n)})$ is monotone, almost sure convergence here is equivalent to convergence in probability. We will show the latter by contradiction. Suppose on the contrary that there exist an $\epsilon > 0$ and a $\delta > 0$ such that $\Pr(p(\hat{D}) < \nu - \epsilon) > \delta$. In the event that $p(\hat{D}) < \nu - \epsilon$, there are at least $K^{(n)}$ active users. We show that each of these $K^{(n)}$ users receives a non-negligible amount of resource. Consider the optimality condition for active users:

$$1 - \frac{Y_i^{(n)}\, p'(D^{(n)})}{p(D^{(n)}) + D^{(n)}\, p'(D^{(n)})} = \frac{p(D^{(n)})}{U_i'(Y_i^{(n)})}, \tag{13}$$

where we have divided both sides by $U_i'(Y_i^{(n)}) > 0$. On the left-hand side, the expression in the denominator is bounded away from 0 with probability 1 as $n \to \infty$ since $\Pr(\hat{D} = 0) = 0$ (by a straight forward application of the Borel-Cantelli Lemma). Therefore, the left-hand side converges to 1 if $Y_i^{(n)} \to 0$. However, if $Y_i^{(n)} \to 0$, the right-hand side tends to

$$\frac{p(\hat{D})}{U_i'(0)} \leq \frac{\nu - \epsilon}{\nu - \frac{\epsilon}{2}},$$

which is bounded away from 1. This implies that $Y_i^{(n)}$ on the left-hand side is bounded away from 0 with some positive probability, so that $D^{(n)} \to \infty$ almost surely because $K^{(n)} \to \infty$. This contradicts the assumption that $p(\hat{D}) < \nu - \epsilon$. Hence, $p(D^{(n)}) \to \nu$ almost surely.

Step 2: Show that the optimality conditions for the social optimal and Nash equilibrium outcomes coincide.

From the optimality condition (13), we also conclude that $Y_i^{(n)} \to 0$ almost surely. This implies that the optimality conditions for the socially optimal outcome coincide with those of the Nash equilibrium outcome. Finally, it follows that efficiency loss tends to 0 almost surely as $n \to \infty$.

For the case where $\nu = \infty$, we proceed as follows. First, consider the event that $D^{(n)} < \infty$ for all n. Then, we can show that the number of users receiving non-negligible quantities of commodity tends to infinity, which is a contradiction. Therefore, we have $D^{(n)} \to \infty$ almost surely. This implies, through the optimality conditions, that $Y_i^{(n)} \to 0$. Finally, the convergence of the efficiency loss follows from comparing the two sets of optimality conditions. ■

Figure 11 illustrates the convergence of the efficiency loss with elastic supply.

VI. Networks

Theorems 3.1 can be generalized to networks with arbitrary topology. For this section, we will only consider linear utility functions and define the loss of efficiency in terms of aggregate utility. We give the outline of a proof, restricted to scalar-modulated linear utility functions of the form: $U_i(x) = S_i\, x$.

Our network will be modeled as a simple undirected graph $G = (N, E)$. The set of nodes N comprises of the possible sources and destinations. Each edge e_k of E is a link, parallel

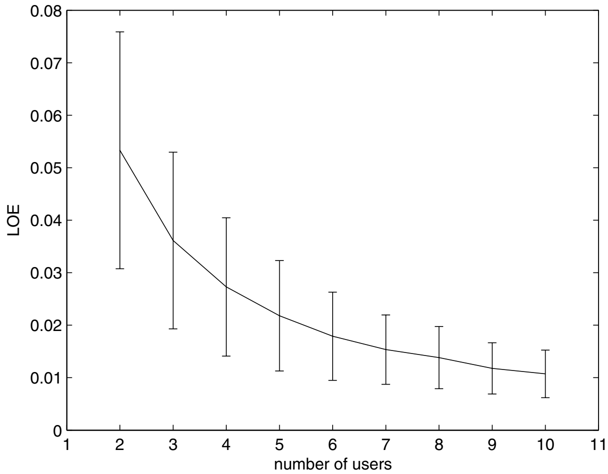

Fig. 11. LOE versus n for $S_i \log(1+x)$ utility functions, S_i sampled with uniform distribution, and elastic supply with $p(c) = c$.

edges and self-loops are not allowed. Let c_k denote the capacity of link e_k.

We assume that each user i picks a source and destination pair $\{\sigma_i, \tau_i\}$ in N according to some non-degenerate distribution, *i.e.*, such that for every pair $v_1, v_2 \in N$, we have $\Pr(\{\sigma_i = v_1\} \cap \{\tau_i = v_2\}) > 0$. Furthermore, the random source and destination are picked independently of the random utility function U_i. Next, we assume that each user i routes his traffic on a single-path, π_i, *i.e.*, flows are unsplittable. To determine π_i, user i picks one of the shortest paths from σ_i to τ_i in any deterministic or probabilistic manner. In order to obtain network resource along the path π_i, user i submits a bid b_{ik} to each link e_k on the path π_i. Each link e_k sets its price per unit of commodity as follows:

$$\mu_k = \frac{\sum_{j=1}^{n} b_{jk}}{c_k}.$$

User i receives an allocation of commodity b_{ik}/μ_k from each link $e_k \in \pi_i$. However, the maximum amount of flow that user i can dispatch on path π_i is limited by the critical link to

$$\min_{k: e_k \in \pi_i} \left(\frac{b_{ik}}{\mu_k} \right).$$

With the linear utility function assumption, price-anticipating users maximize their surplus as follows:

$$\begin{aligned} \max_{b_{ik1}} \quad & s_i \min_{k: e_k \in \pi_i} \left(\frac{b_{ik}}{\mu_k} \right) - \sum_{k: e_k \in \pi_i} b_{ik} \\ \text{subject to} \quad & \mu_k = \frac{\sum_{j=1}^{n} b_{jk}}{c_k}, \quad \forall k : e_k \in \pi_i, \\ & b_{ik} \geq 0, \quad \forall k : e_k \in \pi_i. \end{aligned}$$

Theorem 6.1 (Convergence of LOE in networks): Suppose that the random utility functions $U_i(x) = S_i x$ satisfy Assumption 2.2. Then, within the described network setup, the loss of efficiency tends to 0 almost surely as the number of users tends to infinity.

Proof: (outline) The proof can be divided into four steps. In the first step, we show that for every link, the number of users who bid for that link alone tends to infinity with the total number of users. In the second step, we show that every link's price is greater than half the maximum price with high probability. In the third step, we show that users who bid for more than one link's commodity will become inactive with high probability as the number of users tends to infinity. Finally, we observe that in the large number of users limit, only users who bid on a single link can be active, hence allowing us to apply the results from Theorem 3.1. We consider only the case $\nu < \infty$ of Assumption 2.2 since Assumption 2.1 applied to linear utility functions forbids infinite slope expect at 0.

Step 1: Show that the number of users whose shortest path is the link e_k tends to infinity as $n \to \infty$ for every $e_k \in E$.

Let $M(e_k)$, $e_k \in E$, be the number of *single-link* users with source and destination pair $\{\sigma_i, \tau_i\}$ equal to e_k. For every link e_k, as the number of users n tends to infinity, we know that

$$\frac{M(e_k)}{n} \xrightarrow{\text{a.s.}} \Pr(\{\sigma_i, \tau_i\} = e_k) > 0.$$

Therefore, $M(e_k) \to \infty$ as $n \to \infty$.

Step 2: Show that $\mu_k > \nu/2$ with high probability as $n \to \infty$ for every $e_k \in E$.

With enough of these single-link users on each link e_k, the price of each link, μ_k, will be driven above $\nu/2$ with high probability (by the same argument that $\mu^{(n)} \to u$ in the proof of Theorem 3.1).

Step 3: Show that for every user i such that $|\pi_i| > 1$ becomes inactive with high probability as $n \to \infty$.

Every user whose source and destination pair are separated by shortest paths of length 2 or greater will eventually become inactive as n increases. The reason is that such a user has to split his total bid among multiple links and compete against many other users with comparable utility functions, but who bid only on a single-link.

For example, consider a user i who chooses a shortest path of length 2, say $\pi_i = (e_1, e_2)$. When user i acts as price-anticipator, he solves the following optimization problem:

$$\begin{aligned} \max_{b_{i1}, b_{i2}} \quad & s_i \min \left(\frac{b_{i1}}{\mu_1}, \frac{b_{i2}}{\mu_2} \right) - (b_{i1} + b_{i2}) \\ \text{subject to} \quad & \mu_k = \frac{\sum_{j=1}^{n} b_{jk}}{c_k}, \quad k = 1, 2, \\ & b_{ik} \geq 0, \quad k = 1, 2. \end{aligned}$$

We know that $s_i \leq u$ and that $\mu_1 > \nu/2$ and $\mu_2 > \nu/2$ with high probability when $n \to \infty$, then the objective function satisfies the following inequalities with high probability:

$$\begin{aligned} & s_i \min \left(\frac{b_{i1}}{\mu_1}, \frac{b_{i2}}{\mu_2} \right) - (b_{i1} + b_{i2}) \\ & < u \min \left(\frac{b_{i1}}{\nu/2}, \frac{b_{i2}}{\nu/2} \right) - (b_{i1} + b_{i2}) \leq 0. \end{aligned}$$

This implies that the optimal bids are $b_{i1} = b_{i2} = 0$ when $n \to \infty$, *i.e.*, user i becomes inactive with high probability. The same argument holds for shortest path of length greater than 2.

Step 4: Show that only users with $|\pi_i| = 1$ have a positive probability of being active as $n \to \infty$.

In summary, with high probability as $n \to \infty$, all the active users play single-link games and the efficiency loss still converges to 0 by the results of Theorem 3.1. ■

VII. Relevant works in economics

The subject of games with many players, more specifically, markets with many traders, has been extensively studied in economic theory and game theory. The idea that price-taking behavior is characteristic of large markets has been around for a long time. It is often observed that non-cooperative equilibria of markets tend to be inefficient in the presence of only finitely many participants, but efficient when there are many small participants. The notion of perfect competition among a large number of producers dates back to Cournot's 1838 work [Cou38]. In this section, we situate our contributions with respect to other relevant works.

In a market with a continuum of traders, *e.g.*, represented by the interval $[0, 1]$, Dubey *et al.* [DMCS80] show that, under conditions of convexity of strategy sets, continuity of outcomes with respect to strategies, anonymity of traders, and aggregation of other traders' strategies, every non-cooperative equilibrium of continuum markets are Walrasian, and hence, efficient. However, an important question is the extent to which the continuum-of-participants assumption reflects markets with large, but finite, numbers of participants. This question is addressed in [HM72], [DMCS80], [Gre80], [Gre84], and [Car03], among other works. Hildenbrand and Mertens [HM72], Dubey *et al.* [DMCS80], and Green [Gre84] show that the limit of a sequence of equilibria of finite markets is an equilibrium in a continuum market if the equilibrium correspondence is upper hemicontinuous. Carmona [Car03] presents sequences of finite games whose Nash equilibria differ greatly from those of the corresponding continuum-of-agents games. In contrast to [DMCS80], we do not rely on the continuum of participants idealization. We consider a countable number of participants, whereas using the continuum model inherently ensures that each individual participant is strategically insignificant, as argued by Aumann [Aum64].

Green [Gre80] shows that if a sequence of equilibria for a sequence of replica markets converges, where increasingly large market are created by adjoining a copy of an original finite market, the limit is a price-taking equilibrium of the associated continuum representation. In dynamic markets, however, Green gives counter-examples where non-cooperative equilibria fail to be efficient even with a large number of participants. Our model of a random population of players offers heterogeneity as opposed to replica markets.

For finite markets, Postlewaite and Schmeidler [PS78] show that under some conditions on the initial distribution of resources, any allocation resulting from a Nash equilibrium is ϵ-efficient for a large enough economy, in the sense that the resulting allocation cannot be Pareto dominated if the initial endowment of commodities is reduced by an ϵ fraction. Our work differs from [PS78] in that we relax their restrictions on the initial distribution of resources and that we consider efficiency with respect to *aggregate* utility (and surplus).

The present work offers a new take on an old problem. The resource allocation problem that we consider is a special case of the market or pure-exchange economies described in the literature. In terms of motivation, modeling, and analysis, our work is very different. Our motivation lies mainly in modeling heterogeneity, as opposed to large populations. We adopt a probabilistic line of analysis. An important distinction is that we consider a random sampling of a countable (possibly infinite) number of participants with a continuous set of characteristics (*i.e.*, preferences or utility functions in our case). Our method of sampling random participants is similar in spirit to that of Palfrey and Srivastava [PS86]. They show that in an economy with "stochastically replicated" agents possessing random private information, the incentive to conceal one's private information decreases to zero as the number of agents increases. In contrast to our work, part of their results rely on the assumption that each agent's private information is drawn from a *finite* probability space.

VIII. Conclusion

In this paper, we studied the efficiency loss in two market mechanisms proposed by Kelly for the allocation of network resources. Not only is the loss of efficiency bounded, it also converges to zero almost surely under some assumptions on the utility functions (Section III). There are, however, cases where it provably does not converge to zero (Section IV). These results were extended to the situation where the supply of the network resource is elastic (Section V), as well as to more general multiple-link network topologies (Section VI). Analogies were drawn with works in the economics literature in Section VII.

We conclude that the lack of central regulation, or of cooperation among users, should not result in efficiency loss as long as no single user enjoys the commodity much more than the other users. Interpreted differently, we can also conclude that price-anticipating behavior is not advantageous in the presence of many users whose utility functions are roughly of the same magnitude.

This work is an initial step toward understanding the behavior of large and random competitive networks. The setup considered here requires combining tools from probability, optimization, and game theory. The methods developed here can be extended to other setups, such as modeling the *user-response* to a certain network situation, by sampling the utility functions from a dependent random process. Another natural extension of the setup discussed here is the inclusion of a population of users that are not self-optimizing. Incorporating a fraction of the users who behave in a fixed manner can be easily done using the methods developed here.

Important directions for future research include deriving convergence rates for the loss of efficiency in terms of the number of users and studying the effect of network topology on the loss of efficiency. It would be interesting to study whether some topologies exhibit a reduced loss by design. Another related problem that may, perhaps, be approached using a probabilistic model concerns games where routing is part of the strategic decision, as opposed to our assumption of fixed routing given the network topology. Finally, as we observed in experiments, even for linear utility functions that are modulated by a random variable with heavy-tail distribution, there is significant asymptotic efficiency loss. Characterizing the asymptotic distribution of this efficiency loss is a challenging problem.

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