Design of ℓ_1-Optimal Controllers with Robustness versus Performance Tradeoff

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Abstract

A new design methodology that allows for flexible management of the tradeoff between robustness and quality of the system performance is presented in this paper. The proposed strategy combines a recently developed quasi-robust linear programming concept with a well known ℓ_1-optimal controller synthesis approach. The efficiency of the resulting technique is further demonstrated using several examples.

I. Introduction

Standard ℓ_1-optimal controller synthesis aim at minimizing the worst case peak-to-peak gain of systems disturbed by unknown persistent signals bounded in magnitude; they apply to a large variety of control problems owing to their ability to deal efficiently with time-domain performance objectives. See [5], [10], and [11] for examples of customary ℓ_1 oriented design techniques and [6] for a comprehensive summary of results and references.

Linear programming (LP) problems are well known for their attractive computational properties, see [16]. However, their inability to handle unknown, but bounded, parameter variations have motivated the development of alternate approaches that generate solutions with better robustness properties for these perturbed LP problems, see [3], [14], and [15]. In particular, [3] proposes a quasi-robust strategy (which is inspired by previous contributions in the field of robust convex programming, see [2] and [7]) that offers an elegant procedure to balance the tradeoff between robustness to parameter variations and size of the optimal cost for a given perturbed LP problem.

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Taking advantage of the fact that ℓ_1-optimal design problems can be restated as LP problems, a new approach is developed in this paper that combines the original concept of quasi-robust LP developed in [3] with the ℓ_1-optimal controller synthesis technique of [10]. The new methodology employs a free design parameter allowing for a flexible management of the tradeoff between robustness to disturbance signals and magnitude of the worst peak-to-peak gain of the designed system. Specifically, decreasing the value of the tradeoff parameter induces a proportional reduction of the estimated worst possible peak-to-peak gain, but, simultaneously, it also increases the probability that the system fails to achieve such improved level of performance. By adequatly adjusting the tradeoff parameter, it is hence possible to obtain designs with significantly lower peak-to-peak gains (than those achieved with standard ℓ_1-optimization techniques) often at the modest cost of a small probability of failure.

The paper is structured as follows. The required notation is introduced in §II. Then, the background material concerning robust linear programming and ℓ_1-optimal controller synthesis are presented in §III. The main result relevant to the new flexible design approach is stated in §IV, followed by a thorough analysis of the probability of robustness failure in §V. Finally, the new methodology is applied to several example problems in §VI.

II. Notation

Let $\mathbb{Z}_+$ denote the sets of positive integers.

Given a real number x, the floor function $\lfloor x \rfloor$ rounds x to the nearest integer value towards minus infinity.

Given a matrix A and a set $\mathbf{A}$, let $|A|$ and $|\mathbf{A}|$ denote the entry-wise absolute value of A and the cardinality of $\mathbf{A}$, respectively.

For any matrix $A \in \mathbb{R}^{m \times n}$, $A \triangleq [A_{ij}]_{\substack{i\in\{1,\dots,m\}\\ j\in\{1,\dots,n\}}}$, where A_{ij} is the ij^{th} entry of A. Similarly, for any vector $a \in \mathbb{R}^n$, $a \triangleq [a_1 \dots a_n]^T$, where a_i is the i^{th} entry of a. This notation carries to the case of MIMO systems and vector signals.

Let ℓ_p^n denote the space of all infinite sequences $\{s(k)\}_{k=0}^{\infty}$ of vectors of length n, $s(k) \in \mathbb{R}^n$, equipped with the norm $\|s\|_p < \infty$, where

$$\|s\|_p \triangleq \sqrt[p]{\sum_{k=0}^{\infty} \sum_{i=0}^{n} |s_i(k)|^p}.$$

For $p = \infty$, also define $\|s\|_\infty \triangleq \sup_{k\geq 0} \max_{i\in\{1,\ldots,n\}} |s_i(k)|$. Given a bounded operator $S : \ell_p^n \mapsto \ell_p^m$ with $s \mapsto S(s)$, let

$$\|S\|_{p-ind} \triangleq \sup_{s\neq 0} \frac{\|S(s)\|_p}{\|s\|_p}$$

be the induced p-norm of S. Furthermore, if S is linear and causal, then $S(s)$ is determined by the convolution $(S * s)(k) \triangleq \sum_{l=0}^{k} S(k,l)s(l)$, where $S(k,l)$ denotes the kernel of S. In the case when S is also time-invariant, $S(s)$ simplifies to $(S * s)(k) \triangleq \sum_{l=0}^{k} S(l)s(k-l)$, where $\{S(l)\}_{l=0}^{\infty}$ is the impulse response of S. Then, it is known that, see [6], $\|S\|_{\infty-ind} = \|S\|_1$, where

$$\|S\|_1 \triangleq \max_{i\in\{1,\ldots,m\}} \sum_{j=1}^{n} \sum_{k=0}^{\infty} |S_{ij}(k)|. \tag{1}$$

Moreover, let $\hat{S}(z) \triangleq \sum_{k=0}^{\infty} S(k)z^{-k}$ denote the z-transform of the impulse response of S and define

$$\|\hat{S}\|_{\mathcal{A}} \triangleq \|S\|_1$$

as the $\mathcal{A}$-norm of S.

III. Background

The robust and quasi-robust LP formulations as well as the ℓ_1-optimal controller synthesis methodology are detailed below as they are instrumental in the development of the new flexible approach proposed in this paper.

A. Robust Linear Programming

Consider the following nominal linear programming (LP) problem.

Problem 3.1 (Nominal LP Problem):

$$\begin{aligned} &\max_{x\in\mathbb{R}^n} \quad c'x \\ \text{subject to} \quad &Ax \leq b \\ &l \leq x \leq u, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $l \in \mathbb{R}^n$, and $u \in \mathbb{R}^n$ and the inequalities are understood to be defined component-wise.

Many design problems are restated as LP problems since there exists an abundance of excellent LP algorithms which can provide fast and accurate solutions to high dimensional problems at small computational expense, see [6] and [16]. Still, nominal LP formulations such as that of Problem 3.1 do not admit perturbations in the entries of matrix A. This is considered to be a serious drawback as it is common to encouter applications based on models with unknown parameters that are inside a set with finite support. In the presence of such perturbations, the nominal LP solution may fail to hold for some of the constraints in $Ax \leq b$.

For this reason, the following perturbed version of Problem 3.1 has been studied in [14] and [3]. Consider the i^{th} row of matrix A and let $\mathbf{J}_i$ be the set of entries in row i that are uncertain. Each perturbed entry is modeled as a symmetric and bounded unknown variable $\tilde{A}_{ij}$, $j \in \mathbf{J}_i$, limited to the interval $\tilde{A}_{ij} \in [A_{ij} - \hat{A}_{ij}, A_{ij} + \hat{A}_{ij}]$, where A_{ij} and $\hat{A}_{ij}$ are the nominal value and the maximal deviation allowed for the ij^{th} entry of A, respectively.

Soyster, see [14], offers a robust solution for such perturbed LP problem. However, the last solution appears to be excessively conservative for many applications. Bertsimas $\&$ Sim, see [3], hence suggest a quasi-robust strategy that is more flexible in terms of tradeoffs between robustness and performance in place of the completely robust solution of [14]. The approach of [3] is based on the realistic assumption that the probability of all perturbed entries of $\tilde{A}$ simultaneously and significantly deviating from their nominal value A is usually low. The quasi-robust problem solution exhibits a higher cost $c'x$ as compared when [14] is employed. This improvement is achieved at the price of possible, albeit not frequent, violation of the constraints $\tilde{A}x \leq b$, $\tilde{A} \in [A - \hat{A}, A + \hat{A}]$. The quasi-robust problem is cited below.

Problem 3.2 (Quasi-Robust LP Problem): For every i, let Γ_i be a real parameter that takes values in the interval $[0, |\mathbf{J}_i|]$.

$$
\begin{aligned}
&\max_{x,y\in\mathbb{R}^n} \quad c'x \\
&\text{subject to} \\
&\sum_{j=1}^{n} A_{ij}x_j + \max_{\substack{\mathbf{S}_i \subset \mathbb{Z},\, s_i \in \mathbb{Z}:\\ \mathbf{S}_i \subseteq \mathbf{J}_i,\\ |\mathbf{S}_i| = \lfloor \Gamma_i \rfloor,\\ s_i \in \mathbf{J}_i \setminus \mathbf{S}_i}} \left\{ \sum_{j\in\mathbf{S}_i} \hat{A}_{ij} y_j + (\Gamma_i - \lfloor \Gamma_i \rfloor)\, \hat{A}_{is_i} y_{s_i} \right\} \le b_i \qquad \forall i \qquad (2) \\
&- y_j \le x_j \le y_j \qquad \forall j \\
&l_j \le x_j \le u_j \qquad \forall j \\
&y_j \ge 0 \qquad \forall j,
\end{aligned}
$$

where $l \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ are arbitrary lower and upper bounds on x, respectively.

For each row i, the max term on the LHS of (2) constitutes a kind of deterministic protection against up to $|\mathbf{S}_i| + 1$ deviations from the nominal entries A_{ij}, $j \in \mathbf{J}_i$, where $|\mathbf{S}_i|$ of those perturbations are bounded by $\pm\hat{A}_{ij}$ and the remaining one is bounded by $\pm (\Gamma_i - \lfloor \Gamma_i \rfloor)\, \hat{A}_{ij}$. In the eventuality when more than $|\mathbf{S}_i|+1$ perturbations occur simultaneously, constraints violations may occur. Nevertheless, Problem 3.2 offers the possibility of modifying the tradeoff coefficient Γ_i as to guarantee that the probability of such failure is low.

The role of the parameter Γ_i is to adjust the tradeoff between robustness and size of the cost. More explicitely, large Γ_i favor stronger robustness properties, while small Γ_i yield increased cost values. The limit cases are determined by: $i)$ $\Gamma_i = |\mathbf{J}_i|\ \forall i$, for which Problem 3.2 yields a completely robust solution equivalent to the one proposed by Soyster in [14] and, conversely, $ii)$ $\Gamma_i = 0\ \forall i$, where Problem 3.2 is equivalent to the nominal LP Problem 3.1. The next theorem offers an LP solution for Problem 3.2.

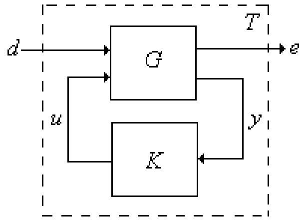

Fig. 1. Controller synthesis framework: G is the augmented plant, K is a controller, T is the closed-loop system, and d, e, u, and y are disturbance, performance, command, and measured signals, respectively.

Theorem 3.3: Problem 3.2 is equivalent to

$$\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^n, z\in\mathbb{R}^m, p\in\mathbb{R}^{m\times n}} \quad c'x$$

$$\text{subject to}$$

$$\sum_{j=1}^{n} A_{ij}x_j + z_i\Gamma_i + \sum_{j\in\mathbf{J}_i} p_{ij} \leq b_i \qquad \forall i \tag{3}$$

$$z_i + p_{ij} \geq \hat{A}_{ij}y_j \qquad \forall i, j \in \mathbf{J}_i \tag{4}$$

$$-y_j \leq x_j \leq y_j \qquad \forall j \tag{5}$$

$$l_j \leq x_j \leq u_j \qquad \forall j \tag{6}$$

$$p_{ij} \geq 0 \qquad \forall i, j \in \mathbf{J}_i \tag{7}$$

$$y_j \geq 0 \qquad \forall j \tag{8}$$

$$z_i \geq 0 \qquad \forall i, \tag{9}$$

where $l \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ are arbitrary lower and upper bounds on x, respectively.

Proof: See [3]. ■

In the worst case, i.e., when $|\mathbf{J}_i| = n$, the above LP problem contains $mn+2n+m$ variables, $mn+2n+m$ multivariable constraints (3)–(5), and $mn+3n+m$ box constraints (6)–(9).

B. ℓ_1-Optimal Controller Synthesis

There exists a few solutions to the problem of synthesizing robust ℓ_1-optimal controllers, see [5], [10], and [11] for examples of customary solutions and [6] for a comprehensive summary of results and references.

Fig.1 illustrates the standard controller synthesis framework. Here, it is assumed that the augmented plant G and the controller K are discrete, causal, LTI systems, but not necessarily stable. The resulting closed-loop system $\hat{T} = \hat{G}_{ed} + \hat{G}_{eu}\hat{K}(I - \hat{G}_{uy}\hat{K})^{-1}\hat{G}_{yd}$, where G is partitioned according to its input and ouput signals (recall that $\hat{S}$ denotes the z-transform of S). The chosen design objective is to find a controller that minimizes the worst case peak-to-peak gain between d and e. It is well known that such objective is equivalent to the minimization of $\|T\|_1$, see [6].

It is possible (and desirable) to simplify the structure of the above control problem by removing the linear fractional dependance of T on K. By virtue of the Youla parameterization of all internally stabilizing controllers, see [6], the closed-loop system transfer function can be redefined as $T = H - U * Q * V$, where H, U, Q, and V are all stable, discrete, causal, LTI sub-systems. The sub-systems H, U, and V are functions of G, while Q is the new design variable which parametrizes all internally stabilizing controllers K. The last substitution allows for a robust ℓ_1-optimal control problem where T is an affine function of Q.

Problem 3.4 (ℓ_1-Optimal Controller Synthesis Problem):

$$\min_{Q} \;\; \|H - U * Q * V\|_1.$$

An LP formulation for Problem 3.4 is given below provided that the following assumptions hold:

A.1 The transfer function Q exhibits a finite impulse response (FIR) of length τ_Q,

A.2 The transfer function $T = H - U * Q * V$ has a FIR of length $\leq \tau_T$,

A.3 The Q parameters denoted by $Q_{\alpha\beta}(\kappa)$, $Q(\kappa) \in \mathbb{R}^{m_Q \times n_Q}$, $\kappa \in \{0, ..., \tau_Q - 1\}$, are bounded by arbitrary lower and upper bounds $l_{\alpha\beta}(\kappa)$ and $u_{\alpha\beta}(\kappa)$, respectively.

Assumption A.1 permits a convex problem formulation with respect to the parameters of the impulse response of Q. Assumption A.2 allows for a finite dimensional problem formulation. In practice, T does not always exhibit a FIR, so T is chosen large enough to allow for an accurate computation of $\|T\|_1$. Assumption A.3 limits the gains of the controller.

Theorem 3.5: Let $\{Q(\kappa)\}_{k=0}^{\tau_Q-1}$, $Q(\kappa) \in \mathbb{R}^{m_Q \times n_Q}$, $\{T(k)\}_{k=0}^{\tau_T-1}$, $T(k) \in \mathbb{R}^{m_T \times n_T}$, and $\{\Theta(k)\}_{k=0}^{\tau_T-1}$, $\Theta(k) \in \mathbb{R}^{m_T \times n_T}$, characterize the impulse responses of Q, T, and Θ, respectively. Additionally, let $\gamma \in \mathbb{R}$. Under assumptions A.1, A.2, and A.3, Problem 3.4 is equivalent to

$$\min_{Q,T,\Theta,\gamma} \quad \gamma$$

$$\text{subject to}$$

$$\sum_{j=1}^{n_T} \sum_{k=0}^{\tau_T - 1} \Theta_{ij}(k) \leq \gamma \qquad \forall i \tag{10}$$

$$-\Theta_{ij}(k) \leq T_{ij}(k) \leq \Theta_{ij}(k) \qquad \forall i, j, k \tag{11}$$

$$T_{ij}(k) = H_{ij} - \sum_{\alpha=1}^{m_Q} \sum_{\beta=1}^{n_Q} \sum_{\kappa=0}^{k} (U_{i\alpha} * V_{\beta j})(k-\kappa)\ Q_{\alpha\beta}(\kappa) \qquad \forall i, j, k \tag{12}$$

$$\Theta_{ij}(k) \geq 0 \qquad \forall i, j, k \tag{13}$$

$$l_{\alpha\beta}(\kappa) \leq Q_{\alpha\beta}(\kappa) \leq u_{\alpha\beta}(\kappa) \qquad \forall \alpha, \beta, \kappa. \tag{14}$$

Proof: See [10]. ■

Let Q_r^*, T_r^*, Θ_r^*, and γ_r^* denote a solution to Theorem 3.5. The above LP problem contains $2m_T n_T \tau_T + m_Q n_Q \tau_Q + 1$ variables, $3m_T n_T \tau_T + m_T$ multivariable constraints (10)–(12), and $m_T n_T \tau_T + 2m_Q n_Q \tau_Q$ box constraints (13)–(14).

IV. A Flexible Approach to the Design of Robust ℓ_1-Optimal Controllers

Consider Fig.1 with all signals and systems defined as in §III-B. The solution of Problem 3.4 yields a controller that minimizes the worst possible peak-to-peak gain of T. However, assuming that all sub-signals d_j of the input signal d are bounded in magnitude by 1, for such worst case to occur, a very specific sequence of vector impulses of the form $d(k) = [d_1(k) \cdots d_{n_T}(k)]'$, $d_j(k) = \pm 1$, $k \in \mathbb{Z}_+$, must enter system T. This observation follows from the ℓ_1-norm definition, see [6]. Still, it is very unlikely to happen in practice (especially when T exhibits a lengthy impulse response). Such a situation may thus lead to unduly conservative designs and encourages the development of a methodology that merges the benefits of the quasi-robust LP approach of §III-A with the ℓ_1-optimal design scheme of §III-B. The next problem formulation captures the concept of a robust ℓ_1-optimal controller design with flexible management of the tradeoff between robustness to disturbance d and magnitude of the peak-to-peak gain of T. The idea is to minimize the worst case peak-to-peak gain between d and e when only a fraction of the admissible disturbance impulses are considered to be non-zero.

Problem 4.1 (Flexible Controller Synthesis Problem): Let $\mathbf{\Gamma} \triangleq \{\Gamma_1, ..., \Gamma_{n_T}\}$, $\Gamma_j \in \mathbb{R}_+$, be a set of tradeoff coefficients.

$$\min_Q \max_d \left\{ \max_{i \in \{1,...,m_T\}} \sum_{j=1}^{n_T} \|[H - U * Q * V]_{ij} * d_j\|_\infty : \|d_j\|_1 \leq \Gamma_j, \|d\|_\infty \leq 1 \right\}.$$

The above problem is a min-max optimization problem. As with Problem 3.4, Q is the design variable. However, here, the influence of the disturbance signal d is not implied in the cost function as it plays an active role in the computation of the worst possible peak-to-peak gain $\max_{1,...,m_T} \sum_{j=1}^{n_T} \|[H - U * Q * V]_{ij} * d_j\|_\infty$. The constraint $\|d\|_\infty \leq 1$ ensures that the disturbance signal is bounded, while the constraint $\|d_j\|_1 \leq \Gamma_j$ permits only $\lfloor \Gamma_j \rfloor$ disturbance impulses of the form $d_j(k) = \pm 1$ and one impulse $d_j(k) = \Gamma_j - \lfloor \Gamma_j \rfloor$ for every sub-signal d_j. Each coefficient Γ_j is associated with an input sub-signal d_j and may be considered as an upper bound for the number of impulses $d_j(k) \neq 0$ allowed to disturb T. Small tradeoff coefficients Γ_j tend to yield a smaller cost value (better peak-to-peak gain properties), but may also alter the robustness of the design if too many disturbance impulses $d_j(k)$ are neglected. This issue will be adressed in §V.

The following theorem is an LP solution to Problem 4.1 under assumptions A.1, A.2, and A.3 in §III-B.

Theorem 4.2: Let $\mathbf{\Gamma} \triangleq \{\Gamma_1, ..., \Gamma_{n_T}\}, \Gamma_j \in [0, \tau_T]$. Let $\{Q(\kappa)\}_{k=0}^{\tau_Q - 1}$, $Q(\kappa) \in \mathbb{R}^{m_Q \times n_Q}$, $\{T(k)\}_{k=0}^{\tau_T - 1}$, $T(k) \in \mathbb{R}^{m_T \times n_T}$, $\{\Theta(k)\}_{k=0}^{\tau_T - 1}$, $\Theta(k) \in \mathbb{R}^{m_T \times n_T}$, and $\{p(k)\}_{k=0}^{\tau_T - 1}$, $p \in \mathbb{R}^{m_T \times n_T}$, characterize the impulse responses of Q, T, Θ, and p respectively. Additionally, let $z \in \mathbb{R}^{n_T}$, $\nu \in \mathbb{R}^{m_T \times n_T}$, and

$\gamma \in \mathbb{R}$. Under assumptions A.1, A.2, and A.3, Problem 4.1 is equivalent to

$$\min_{Q,T,\Theta,z,p,\nu,\gamma} \quad \gamma$$

$$\text{subject to}$$

$$\sum_{j=1}^{n_T} \nu_{ij} \leq \gamma \qquad \forall i \tag{15}$$

$$z_j \Gamma_j + \sum_{k=0}^{\tau_T - 1} p_{ij}(k) \leq \nu_{ij} \qquad \forall i,j \tag{16}$$

$$z_j + p_{ij}(k) \geq \Theta_{ij}(k) \qquad \forall i,j,k \tag{17}$$

$$-\Theta_{ij}(k) \leq T_{ij}(k) \leq \Theta_{ij}(k) \qquad \forall i,j,k \tag{18}$$

$$T_{ij}(k) = H_{ij} - \sum_{\alpha=1}^{m_Q} \sum_{\beta=1}^{n_Q} \sum_{\kappa=0}^{k} (U_{i\alpha} * V_{\beta j})(k-\kappa)\ Q_{\alpha\beta}(\kappa) \qquad \forall i,j,k \tag{19}$$

$$p_{ij}(k) \geq 0 \qquad \forall i,j,k \tag{20}$$

$$z_j \geq 0 \qquad \forall j \tag{21}$$

$$\Theta_{ij}(k) \geq 0 \qquad \forall i,j,k \tag{22}$$

$$l_{\alpha\beta}(\kappa) \leq Q_{\alpha\beta}(\kappa) \leq u_{\alpha\beta}(\kappa) \qquad \forall \alpha,\beta,\kappa, \tag{23}$$

where $l_{\alpha\beta}(\kappa)$ and $u_{\alpha\beta}(\kappa)$ are arbitrary lower and upper bounds on the Q parameters, respectively.

Proof: See Appendix I. ■

For a given set $\mathbf{\Gamma}$, let $Q_f^*(\mathbf{\Gamma})$, $T_f^*(\mathbf{\Gamma})$, $\Theta_f^*(\mathbf{\Gamma})$, $z_f^*(\mathbf{\Gamma})$, $p_f^*(\mathbf{\Gamma})$, $\nu_f^*(\mathbf{\Gamma})$, and $\gamma_f^*(\mathbf{\Gamma})$ denote a solution to the LP problem of Theorem 4.2. This LP problem involves $3m_T n_T \tau_T + m_Q n_Q \tau_Q + m_T n_T + n_T + 1$ variables, $4m_T n_T \tau_T + m_T n_T + m_T$ multivariable constraints (15)–(19), and $2m_T n_T \tau_T + 2m_Q n_Q \tau_Q + n_T$ box constraints (20)–(23). The size of the LP problem of Theorem 4.2 is slightly larger than that of Theorem 3.5, but the former offers a more sophisticated design. Moreover, it is possible to merge constraints (15) and (16) as to eliminate the variable ν, but the presence of ν will prove most helpful for the robustness failure probability analysis of §V.

Note that Theorem 4.2 is equivalent to Theorem 3.5 when $\mathbf{\Gamma} = \{\tau_T, ..., \tau_T\}$. On the other hand, when $\mathbf{\Gamma} = \{0, ..., 0\}$, one recovers a nominal LP problem. In the present context, the elaboration of such nominal problem requires that $d = 0$, which is not useful.

V. Probability of Robustness Failure

In practice it is unlikely that a disturbance signal systematically and repeatedly exhibits the very sequence that induces the worst case peak-to-peak gain. This fact justifies the flexible design strategy proposed in §IV as a mean to reduce the conservativeness associated with the ℓ_1-optimal controller synthesis approach of §III-B. In Theorem 4.2, it is easy to see that $\gamma_f^*(\mathbf{\Gamma}) \rightarrow 0$ when $\max_{j \in \{1,...,n_T\}} |\Gamma_j| \rightarrow 0$. However, while a reduction of the Γ_j tradeoff coefficients decreases the associated cost $\gamma_f^*(\mathbf{\Gamma})$, it is very important to assess the impact of the magnitude of Γ_j on the robustness of the resulting solution. One way to proceed is to compute, for each ij^{th} input-output channel, the probability that

$$\|[T_f^*(\mathbf{\Gamma})]_{ij} * d_j\|_\infty \leq \nu_{ij}^*(\mathbf{\Gamma}) \tag{24}$$

fails to hold for a uniformly sampled random sequence of impulses $d_j(k) = \pm 1$, $j \in \{1, ..., n_T\}$, $k \in \{0, ..., \tau_T - 1\}$. It is easily seen from the convex structure of (24) that the probability value is maximized when using the abovementioned binomial distribution for $d_j(k)$ as compared to other arbitrary symmetric distributions with compact support set $[-1, 1]$. An advantage of using inequalities of the form (24) to assess robustness lies in their ability to indicate which input-output channel is most vulnerable to failure. In the example problems section, §VI, the failure probability of (24) is computed employing Monte Carlo simulations involving random disturbance sequences.

The failure probability of (24) cannot be computed before deriving the solution of Theorem 4.2 as it relies on $T_f^*(\mathbf{\Gamma})$ and $\nu_{ij}^*(\mathbf{\Gamma})$. Morevover, the associated computational cost, albeit not excessive, is significantly larger than that of simply solving the LP problem in Theorem 4.2. The next theorem gives an upper bound for the failure probability of (24) which depends solely on Γ_{ij} and τ_T. This new bound may be helpful for designers as it offers a mean to rapidly gain insight into the potential tradeoff opportunities.

Theorem 5.1: Let the disturbance impulses $d_j(k)$ be treated as random variables. Suppose the $d_j(k)$, $j \in \{1, ..., n_T\}$, $k \in \{0, ..., \tau_T - 1\}$, are independent identically distributed with a common probability density function that is arbitrary, but symmetric about the zero mean and with compact support set $[-1, 1]$. Then,

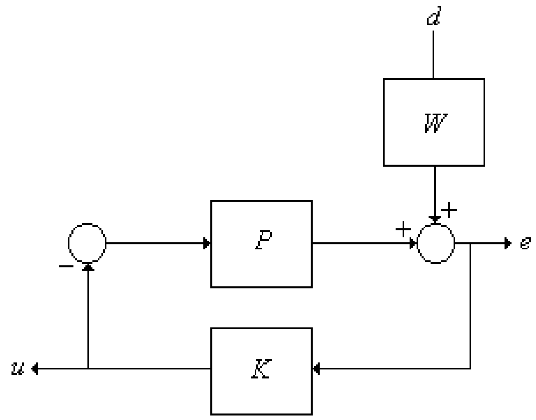

Fig. 2. Disturbance rejection block diagram: P and W are arbitrary plant and filter, respectively, K is a controller, and d, e, and u are disturbance, performance, and command signals, respectively.

$$\Pr\left(\|[T_f^*(\mathbf{\Gamma})]_{ij} * d_j\|_\infty > \nu_{ij}^*(\mathbf{\Gamma})\right) \leq \left(1 - \mathrm{mod}\left(\frac{\Gamma_{ij}+\tau_T}{2}, 1\right)\right) C\left((\tau_T, \left\lfloor \frac{\Gamma_{ij}+\tau_T}{2} \right\rfloor\right) + \sum_{l=\left\lfloor \frac{\Gamma_{ij}+\tau_T}{2} \right\rfloor + 1}^{\tau_T} C(\tau_T, l), \tag{25}$$

where $\mathrm{mod}\left(\frac{\Gamma_{ij}+\tau_T}{2}, 1\right)$ is the remainder in the division of $\frac{\Gamma_{ij}+\tau_T}{2}$ by 1, and

$$C(\tau_T, l) = \begin{cases} 2^{-\tau_T} & \text{if } l = 0 \text{ or } l = \tau_T, \\ \sqrt[2]{\frac{\tau_T}{2\pi l(\tau_T - l)}} \exp\left(\tau_T \log\left(\frac{\tau_T}{2(\tau_T - l)}\right) + l \log\left(\frac{\tau_T - l}{l}\right)\right) & \text{otherwise}. \end{cases}$$

Proof: Follows from a straightforward adaptation of Theorem 3 in [3] to Problem 4.1. ∎

VI. Example Problems

The relevance and usefulness of the methodology proposed in §IV is confirmed with three example problems.

A. Disturbance Rejection Problem with an Arbitrary System

Consider the block diagram in Fig.2, where the plant P and filter W are assumed SISO, discrete, stable, strictly-proper, and LTI. Both are of order 20 with their poles and zeros randomly determined, see Appendix II-A for precise transfer function representations.

The objective is to attenuate the influence, in the peak-to-peak sense, of the disturbance signal d on both the performance and command signals e and u, respectively. This is usually achieved by synthesizing an internally stabilizing, causal, LTI controller that minimizes the $\mathcal{A}$-norm of the following pair of transfer functions (recall that $\hat{S}$ denotes the z-transform of S).

$$\min_K \left\| \begin{array}{c} \hat{T}_{ed}(K) \\ \hat{T}_{ud}(K) \end{array} \right\|_{\mathcal{A}} = \min_K \left\| \begin{array}{c} \hat{W}(1+\hat{P}\hat{K})^{-1} \\ \hat{W}\hat{K}(1+\hat{P}\hat{K})^{-1} \end{array} \right\|_{\mathcal{A}} . \tag{26}$$

By the Youla parameterization, see [6], $\hat{Q} = \hat{K}(1+\hat{P}\hat{K})^{-1}$, so the optimization problem (26) is rearranged in the form of Problem 3.4 as follows.

$$\min_Q \left\| \begin{array}{c} W - W * P * Q \\ W * Q \end{array} \right\|_1 , \tag{27}$$

where $H = [W \quad 0]^T$, $U = [W * P \quad -W]^T$, and $V = 1$.

Note that, in this example, $m_T = 2$ and $n_T = m_Q = n_Q = 1$. The robust approach of §III-B is first employed to solve (27). The solution $\gamma_r^* = 930.33$ is obtained with $\tau_Q = 30$ and $\tau_T = 60$. The quality of this solution is confirmed by a calculation which shows that it lies within a 0.5% error bound of the ℓ_1-optimal solution involving an arbitrarily large τ_Q and an infinite τ_T. See [10] for details on how to compute a tight lower bound for such an ℓ_1-optimal solution.

The new methodology presented in §IV allows to compute other controllers of the same order (i.e., with $\tau_Q = 30$) for (27) that improve significantly the system performance at the tradeoff of a relatively small loss in robustness. For simplicity, let $\Gamma_{11} = \Gamma_{21} = \Gamma$ for the example considered. The results are depicted in Fig.3 in the form of a normalized performance cost (NPC) curve and a pair of robustness failure probability (RFP) curves. The three curves are functions of the tradeoff coefficient Γ which is limited here to $\Gamma \in [1, 30]$ as having $\Gamma \in]30, 60]$ does not allow for any noticeable tradeoff. The NPC curve refers to the ratio $\gamma_f^*(\mathbf{\Gamma})/\gamma_r^*$, while the RFP curves represent the probability that, for each of the two input-output channels, (24) fails to hold for a uniformly sampled random sequence of impulses $d(k) = \pm 1$, $k \in \{0, ..., \tau_T - 1\}$. Stated differently, each point on a RFP curve is the probability that an arbitrary bounded disturbance signal d acting on system T yields a peak-to-peak gain response which exceeds $\gamma_f^*(\mathbf{\Gamma})$ for a given input-output channel (note that $[\nu_f^*(\mathbf{\Gamma})]_{i1} = \gamma_f^*(\mathbf{\Gamma})$ for both $i \in \{1, 2\}$ in the present example). The RFP curves are computed pointwise for $\Gamma = \{1, ..., 30\}$. Each point on an RFP curve is computed according to the results of 10000 simulations involving the robustness criterion (24) for different randomly

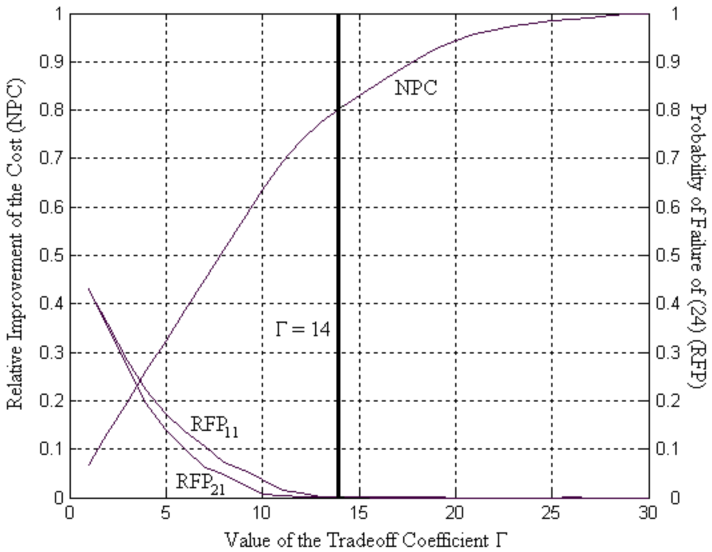

Fig. 3. Disturbance rejection tradeoff options: normalized performance cost curves and robustness failure probability curves, all with respect to the variation of the Γ coefficient.

generated sequences $d(k) = \pm 1$, $k \in \{0, ..., \tau_T - 1\}$. The NPC curve is normalized (i.e. scaled by γ_r^*) to facilitate the comparison with RFP curves towards identification of possible interesting tradeoffs.

From Fig.3, it appears that for $\Gamma \geq 13$, the probability of a robustness failure is very low. In particular, $\Gamma = 14$ offers an interesting tradeoff as it allows for an improvement in performance of 20%, while the robustness failure probability remains below 0.02% for both input-output channels.

Two bounds of the form (25), which depend only on Γ and τ_T, are shown in Fig.4. The bounds B_{60} refers to $\tau_T = 60$. As expected, it is an upper bound for both RFP curves. The distance between B_{60} and both RFP_{11} and RFP_{21} can be attributed to the fact that the impulse response of T decays towards zero at an exponential rate. This is particularly true for large values of Γ which explains why the size of the distance between these curves increases with Γ. Hence, assuming that the impulse response of T can be sufficiently well approximated by its 30 first pulses, the RFP curves are then compared with the tighter upper bound B_{30}.

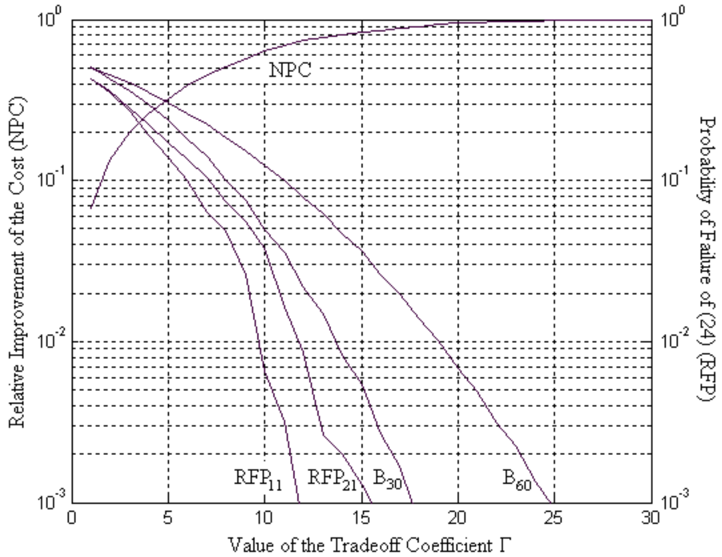

Fig. 4. Disturbance rejection tradeoff options: normalized performance cost curve, robustness failure probability curves, and independent upper bounds for the robustness failure probability curves, all with respect to the variation of the Γ coefficient.

B. Control of an Active Suspension System

Consider the relation $[e\ u]^T = T * d$, where T represents a model of an active-suspension system which maps the elevation of the road surface d into both the vertical displacement e of the passengers in the vehicle and the control effort u required by the actuators to compensate for it. Since the variation of the road level is limited, the disturbance d is modeled as a persistent signal bounded in magnitude. Similarly, the magnitude of signal e can be seen as a measure of the discomfort of the passengers. The design objective for the suspension system is hence to compensate for the admissible disturbance d as to minimize the magnitude of the output vector signal $[e\ u]^T$.

Fig.5 presents a block diagram of an augmented active suspension system, where the state-space model of G is given in Appendix II-B and the weight gains are $W_d = 0.1$, $W_e = 10$, and $W_u = 1/450$. Such active suspension model is first developed in [1]. Note that W_d ensures that the variation in road surface remains limited by $\pm 10cm$, while W_e and W_u normalize the relative importance of both output signals. By virtue of the Youla parameterization $\hat{Q} = \hat{K}(1 - \hat{G}_{32}\hat{K})^{-1}$,

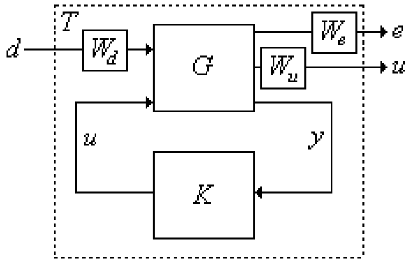

Fig. 5. Active suspension block diagram: G is the augmented suspension model, K is a controller, W_d, W_e, and W_u are constant weights, u is the command signal in Newton, and d, e, and y are disturbance, performance, and measured signals in meters, respectively.

so the control problem is defined as follows.

$$\min_Q \left\| \begin{array}{c} T_{ed}(Q) \\ T_{ud}(Q) \end{array} \right\|_1 = \min_Q \left\| \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix} + \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix} * Q * G_{31} \right\|_1, \tag{28}$$

where $H = [G_{11} \;\; G_{21}]^T$, $U = -[G_{12} * G_{31} \;\; G_{22} * G_{31}]^T$, and $V = 1$.

One way to solve (28) is to apply the ℓ_1-optimal methodology proposed in §III-B. However, as most road surfaces do not frequently exhibit abrupt variations in their elevation, the approach of §III-B may yield a solution that is too conservative for the typical conditions. Employing the flexible design methodology of §IV, on the other hand, may prove particularly advantageous.

Both approaches are implemented and their respective efficiency compared below. As with the previous problem, $m_T = 2$ and $n_T = m_Q = n_Q = 1$. With $\tau_Q = 40$ and $\tau_T = 60$, the approach of §III-B yields a cost of $\gamma_r^* = 1.25$ which is at least within 2.0% of the ℓ_1-optimal solution, see §VI-A for details. For simplicity, it is assumed that $\Gamma_{11} = \Gamma_{21} = \Gamma$. The NPC and RFP curves are depicted in Fig.6 as functions of Γ. Specifically, it is seen that the tradeoff offered by $\Gamma = 15$ is appealing as the performance objective is improved by no less than 10%, while the probability of robustness failure does not exceed 0.5% for both input-output channels.

C. Common ℓ_1 Multiblock Problem

The ℓ_1-optimal control problem presented below is used by a few authors, see [8], [10], and [17]. Although it is not particularly well suited for the proposed flexible approach (as the optimal impulse response of the closed-loop system is very short for most values of Γ), it is presented here for the sole purpose of comparison.

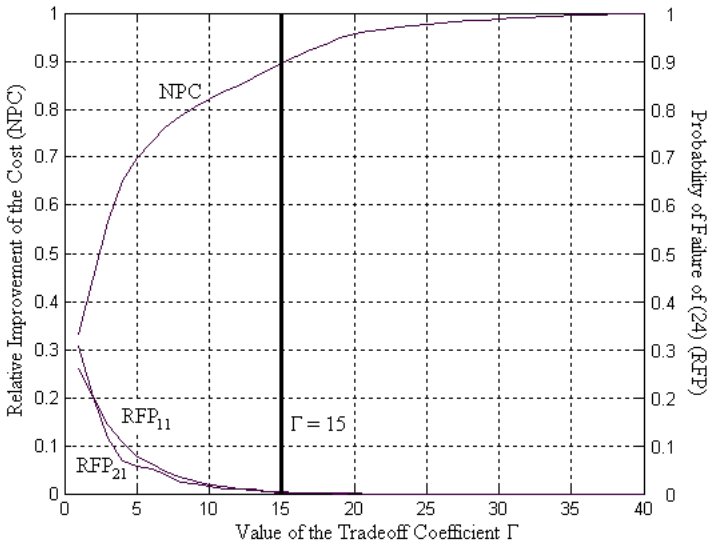

Fig. 6. Active suspension tradeoff options: normalized performance cost curve and robustness failure probability curves, all with respect to the variation of the Γ coefficient.

Consider the block diagram in Fig.7, where the transfer function of the plant P and filters W_1 and W_2 are given in Appendix II-C. The problem consists in synthesizing an internally stabilizing, causal, LTI controller that minimizes the $\mathcal{A}$-norm of the following MIMO system.

$$\min_K \left\| \begin{array}{cc} \hat{T}_{e_1d_1}(K) & \hat{T}_{e_1d_2}(K) \\ \hat{T}_{e_2d_1}(K) & \hat{T}_{e_2d_2}(K) \end{array} \right\|_{\mathcal{A}} = \min_K \left\| \begin{array}{cc} \hat{W}_1(1+\hat{P}\hat{K})^{-1} & -\hat{W}_2\hat{P}(1+\hat{P}\hat{K})^{-1} \\ -\rho\hat{W}_1\hat{K}(1+\hat{P}\hat{K})^{-1} & -\rho\hat{W}_2\hat{K}(1+\hat{P}\hat{K})^{-1} \end{array} \right\|_{\mathcal{A}} . \quad (29)$$

By the Youla parametrization, $\hat{K} = (\hat{Y} - \hat{Q}\hat{B})^{-1}(\hat{X} + \hat{Q}\hat{A})$, $\hat{P} = \hat{A}^{-1}\hat{B}$, and $\hat{X}\hat{B} + \hat{Y}\hat{A} = 1$, where A, B, X, and Y are all discrete, causal, stable, LTI systems, see Appendix II-C for detailed transfer functions. The optimization problem (29) is then rearranged in the form of Problem 3.4 as follows.

$$\min_Q \left\| \begin{array}{cc} T_{e_1d_1}(Q) & T_{e_1d_2}(Q) \\ T_{e_2d_1}(Q) & T_{e_2d_2}(Q) \end{array} \right\|_1 \quad (30)$$

$$= \min_Q \left\| \begin{array}{cc} W_1 * Y * A - W_1 * A * B * Q & -W_2 * X * B - W_2 * A * B * Q \\ -\rho W_1 * X * A - \rho W_1 * A^2 * Q & -\rho W_2 * X * A - \rho W_2 * A^2 * Q \end{array} \right\|_1 .$$

In this example, $m_T = n_T = 2$ and $m_Q = n_Q = 1$. The robust approach of §III-B yields a cost of 71.11 for (30), see [8], [10], and [17]. Such optimal solution is achieved with high

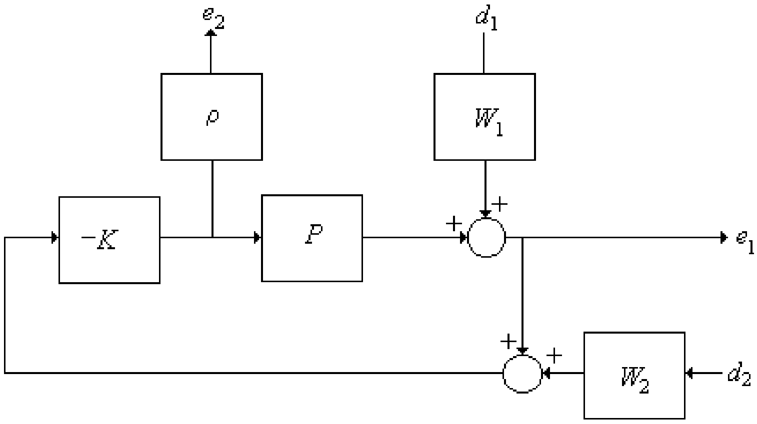

Fig. 7. Common multiblock problem block diagram: P is a plant, K is a controller, W_1 and W_2 are filters, ρ is a constant weight, and d and e are disturbance and performance signals, respectively.

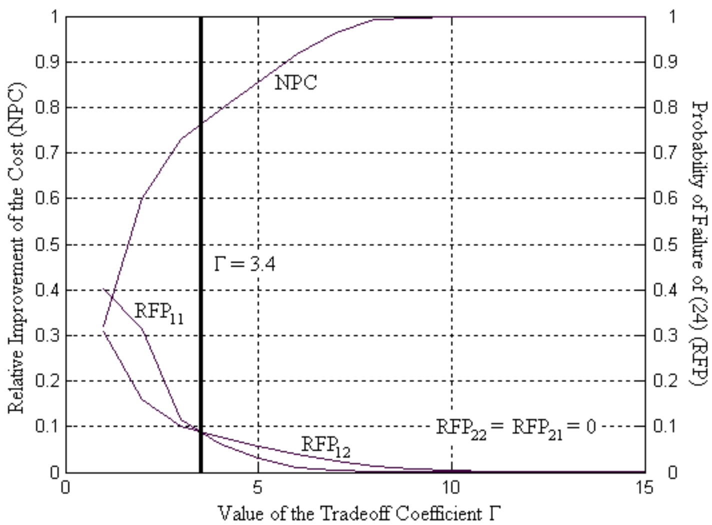

Fig. 8. Common multiblock problem tradeoff options: normalized performance cost curves and robustness failure probability curves, all with respect to the variation of the Γ coefficient.

accuracy when $\tau_Q = 15$ and $\tau_T = 30$.

Again, for simplicity, let $\Gamma_{11} = \Gamma_{12} = \Gamma_{21} = \Gamma_{22} = \Gamma$. The NPC and RFP curves are shown in Fig.8 as functions of Γ. Contrary to the RFP_{11} curve that vanishes quickly, the RFP_{12} curve exhibits a long tail which indicates that the associated input-output channel is vulnerable to robustness failure. It also stands out that $\text{RFP}_{21} = \text{RFP}_{22} = 0$. This is due to the fact that the first row is the only active row in the optimization problem because the ℓ_1 norm of the second row is always significantly smaller than the ℓ_1 norm of the first row. Consequently, the coefficients ν_{21} and ν_{22} in constraint (16) are needlessly large, but prevent against robustness failures. A way to remedy this situation is to impose $\rho = 1$, but it is not implemented here as it would obscure the comparison with previously presented results which use the original version of the problem.

Notwithstanding the negative behavior of input-output channel 12, one interesting tradeoff is offered by $\Gamma = 3.4$, where the RFP_{11} and RFP_{12} curves cross each other. At this point, the corresponding improvement in system performance and probability of robustness failure is 25% and 9% for both channels, respectively.

VII. Conclusion

A new design methodology that allows for a flexible management of the tradeoff between robustness and quality of the system performance is proposed in this paper. The new strategy merges the original concept of quasi-robust linear programming developed in [3] with the ℓ_1-optimal controller synthesis technique of [10]. It results in a linear programming problem which can be solved easily with efficient and widely-employed software packages such as CPLEX, see [4]. The new method would significantly enhance the capability of any robust control designer toolkit as it complements the well known ℓ_1-optimal controller design approach and offers an efficient mean to deal with the conservativeness inherent to standard ℓ_1 techniques. This is confirmed using several example problems.

The advantages of the new flexible approach is most visible on large problems (a similar observation is made by Bertsimas&Sim, [3], for their quasi-robust LP approach). Indeed, the larger is the problem, the less likely is the probability that the worst input sequence actually occurs. The tradeoff coefficient Γ may be seen as an estimate of the number of disturbance impulses that could be neglected while still achieving a reasonably robust design. In practice, it is often sufficient to consider small to moderate values of Γ, because the impulse response of the corresponding closed-loop system T vanishes exponentially.

Increasing the efficiency of the process behind estimating the probability of robustness failure would require a more sophisticated sampling scheme. For larger problem, and to improve the speed of the estimation process, it is possible to use Importance Sampling notions, see [13].

It is worth mentioning that the new methodology may easily incorporate any types of auxiliary linear constraints such as extra hard ℓ_1-norm constraints or specific time-domain templates or envelopes for various closed-loop system responses to known fixed input signals, see [12]. Another possibility is to consider ellipsoid-shaped constraints, instead of box constraints, to model the uncertainty. Following [2], an efficient optimization algorithm can still be derived for such problems. Note that ellipsoid constraints are particularly suitable when the parameters are obtained by sampling data.

Future research is intended to extend the proposed flexible approach to handle the presence of parametric perturbations in the closed-loop system dynamics.

APPENDIX I
PROOF OF THEOREM 4.2

From the definition of T, the convolution definition, assumption A.2, and the ℓ_1 and ℓ_∞-norm of signals definition, the optimization problem

$$\min_Q \max_d \left\{ \max_{i\in\{1,\ldots,m_T\}} \sum_{j=1}^{n_T} \|[H - U * Q * V]_{ij} * d_j\|_\infty : \|d_j\|_1 \leq \Gamma_j, \|d\|_\infty \leq 1 \right\}$$

is equivalent to

$$\begin{aligned}
&\min_{Q,T,\gamma} \quad \gamma \\
&\text{subject to} \\
&\max_{\bar{d}} \max_{i\in\{1,\ldots,m_T\}} \sum_{j=1}^{n_T} \sum_{k=0}^{\tau_T-1} |T_{ij}(k)|\bar{d}_j(k) \leq \gamma \\
&\sum_{k=0}^{\tau_T-1} \bar{d}_j(k) \leq \Gamma_j && \forall j \\
&0 \leq \bar{d}_j(k) \leq 1 && \forall j,k \\
&T_{ij}(k) = [H - U * Q * V]_{ij}(k) && \forall i,j,k,
\end{aligned}$$

with $\bar{d}_j(k) \in \mathbb{R}$, $j \in \{1, ..., n_T\}$, $k \in \{0, ..., \tau_T - 1\}$. By virtue of a strong primal-dual substitution, see [9], the above problem is furhter modified as follows

$$
\begin{aligned}
&\min_{Q,T,\gamma} \quad \gamma \\
&\text{subject to} \\
&\min_{z} \max_{\bar{d}} \left(\max_{i \in \{1,...,m_T\}} \sum_{j=1}^{n_T} \sum_{k=0}^{\tau_T - 1} |T_{ij}(k)| \bar{d}_j(k) - z_j \left(\sum_{k=0}^{\tau_T - 1} \bar{d}_j(k) - \Gamma_j \right) \right) \leq \gamma \\
&0 \leq \bar{d}_j(k) \leq 1 && \forall j, k \\
&T_{ij}(k) = [H - U * Q * V]_{ij}(k) && \forall i, j, k \\
&z_j \geq 0 && \forall j,
\end{aligned}
$$

with $z_j \in \mathbb{R}$, $j \in \{1, ..., n_T\}$, and, with respect to basic properties of the max and sum functions, it is rearranged as

$$
\begin{aligned}
&\min_{Q,T,\gamma} \quad \gamma \\
&\text{subject to} \\
&\min_{z} \max_{\bar{d}} \max_{i \in \{1,...,m_T\}} \sum_{j=1}^{n_T} \left(\sum_{k=0}^{\tau_T - 1} \left(|T_{ij}(k)| - z_j \right) \bar{d}_j(k) + z_j \Gamma_j \right) \leq \gamma \\
&0 \leq \bar{d}_j(k) \leq 1 && \forall j, k \\
&T_{ij}(k) = [H - U * Q * V]_{ij}(k) && \forall i, j, k \\
&z_j \geq 0 && \forall j.
\end{aligned}
$$

It is easy to see that $\bar{d}_j(k) = 1$ when $|T_{ij}(k)| - z_j \geq 0$ and $\bar{d}_j(k) = 0$ otherwise. For this reason, the dependance on $\bar{d}_j(k)$ can be removed as follows

$$\begin{aligned}
&\min_{Q,T,p,\gamma} \quad \gamma \\
&\text{subject to} \\
&\min_{z} \max_{i \in \{1,\ldots,m_T\}} \sum_{j=1}^{n_T} \left(\sum_{k=0}^{\tau_T - 1} p_{ij}(k) + z_j \Gamma_j \right) \leq \gamma \\
&z_j + p_{ij}(k) \geq |T_{ij}(k)| && \forall i, j, k \\
&T_{ij}(k) = [H - U * Q * V]_{ij}(k) && \forall i, j, k \\
&p_{ij}(k) \geq 0 && \forall i, j, k \\
&z_j \geq 0 && \forall j.
\end{aligned}$$

The final LP version of the optimization problem is obtained with the introduction of the elective variable ν, assumptions A.1 and A.3, and the definition of convolution.

$$\begin{aligned}
&\min_{Q,T,\Theta,z,p,\nu,\gamma} \quad \gamma \\
&\text{subject to} \\
&\sum_{j=1}^{n_T} \nu_{ij} \leq \gamma && \forall i \\
&z_j \Gamma_j + \sum_{k=0}^{\tau_T - 1} p_{ij}(k) \leq \nu_{ij} && \forall i, j \\
&-\Theta_{ij}(k) \leq T_{ij}(k) \leq \Theta_{ij}(k) && \forall i, j, k \\
&T_{ij}(k) = H_{ij} - \sum_{\alpha=1}^{m_Q} \sum_{\beta=1}^{n_Q} \sum_{\kappa=0}^{k} (U_{i\alpha} * V_{\beta j})(k - \kappa)\ Q_{\alpha\beta}(\kappa) && \forall i, j, k \\
&p_{ij}(k) \geq 0 && \forall i, j, k \\
&z_j \geq 0 && \forall j \\
&\Theta_{ij}(k) \geq 0 && \forall i, j, k \\
&l_{\alpha\beta}(\kappa) \leq Q_{\alpha\beta}(\kappa) \leq u_{\alpha\beta}(\kappa) && \forall \alpha, \beta, \kappa.
\end{aligned}$$

APPENDIX II

AUXILIARY DATA FOR THE EXAMPLE OF §VI

A. Auxiliary Data for the Example of §VI-A

The discrete transfer function of P and W are respectively given by $\hat{P} = \hat{P}_1\ \hat{P}_2\ \hat{P}_3$,

$$\hat{P}_1 = \frac{(z-0.5512)(z-0.462)(z-0.3792)(z-0.9442)(z-1.056)(z-1.237)(z+1.232)}{(z+0.2324)(z+0.2589)(z+0.1661)(z+0.1436)(z+0.1202)(z+0.3634)(z+0.5239)},$$

$$\hat{P}_2 = \frac{(z+1.1)(z+1.018)(z+0.7043)(z+0.6447)(z+0.6313)(z+0.4931)(z+0.321)}{(z+0.7253)(z-0.2581)(z-0.2678)(z-0.3019)(z-0.3166)(z-0.1837)(z-0.1103)},$$

$$\hat{P}_3 = \frac{(z+2.005)(z+2.12)(z+2.325)(z+0.1821)(z+0.1132)(z-0.08599)}{(z-0.08584)(z-0.5852)(z-0.6474)(z-0.7175)(z-0.7548)(z-0.03123)},$$

and $\hat{W} = \hat{W}_1\ \hat{W}_2\ \hat{W}_3$,

$$\hat{W}_1 = \frac{(z-1.521)(z-1.227)(z-0.9131)(z-0.8892)(z-0.6682)(z-0.5869)(z-0.5246)}{(z-0.3303)(z-0.3085)(z-0.4023)(z-0.4631)(z-0.522)(z-0.5655)(z-0.1367)},$$

$$\hat{W}_2 = \frac{(z-0.4855)(z-0.4801)(z-2.309)(z+0.6962)(z+0.7829)(z+1.107)(z+0.2512)}{(z-0.1289)(z-0.06518)(z-0.02852)(z+0.3045)(z+0.3737)(z+0.6868)(z+0.6875)},$$

$$\hat{W}_3 = \frac{(z+0.07832)(z+0.03844)(z+0.01179)(z+0.005005)(z-0.05594)(z-0.007524)}{(z+0.727)(z+0.7551)(z+0.7611)(z+0.1459)(z+0.09422)(z+0.0189)}.$$

B. Auxiliary Data for the Example of §VI-B

The state-space model of G is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1/M & K_1/M & -D/M & D/M \\ K_1/m & -(K_1+K_2)/m & D/m & -D/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1/M \\ K_2/m & -1/m \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where $M = 300kg$, $m = 50kg$, $K_1 = 3000N/m$, $K_2 = 30000N/m$, and $D = 600Ns/m$. The continuous-time model is discretized using a *Tustin* bilinear transformation with a sample-time of $1s$ (which accounts for a single variation in the road level per second).

C. Auxiliary Data for the Example of §VI-C

The transfer functions of P, W_1, W_2, A, B, X, and Y are respectively given by

$$\begin{aligned}
\hat{P} &= \frac{-0.5z+1}{0.1z^2-1.05z+0.5} \text{ (unstable plant)},\\
\hat{W}_1 &= \frac{0.4z}{z-0.6} \text{ (low pass filter)},\\
\hat{W}_2 &= \frac{z-0.75}{z-0.25} \text{ (high pass filter)},\\
\hat{A} &\approx \frac{z^2-10.5z+5}{z^2},\\
\hat{B} &\approx \frac{-5z+10}{z^2},\\
\hat{X} &\approx \frac{-26.32z+13.17}{z^2},\\
\hat{Y} &\approx \frac{z^2+10.5z-26.33}{z^2}.
\end{aligned}$$

Also note that $\rho = -0.1$.

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